KOÇ UNIVERSITY

MATH 106 - CALCULUS I

Midterm II December 6, 2014

Duration of Exam: 90 minutes

INSTRUCTIONS: CALCULATORS ARE NOT ALLOWED FOR THIS EXAM. No books, no notes, no questions and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Surname, Name: KEY

Signature:

Section (Check One):

Section 1: E. Ş. Yazıcı (Mon-Wed 16:00) ___
Section 2: E. Ş. Yazıcı (Mon-Wed 13:00) ___
Section 3: Doğan Bilge (Mon-Wed 11:30) ___
Section 4: Doğan Bilge (Mon-Wed 14:30) ___
Section 5: Altan Erdoğan (Tu-Th 16:00) ___

PROBLEM	POINTS	SCORE
1	~~30~~ 40	
2	~~30~~ 20	
3	15	
4	15	
5	15	
TOTAL	**105**	

1. Compute the following integrals if they converge.

a) (~~8~~ 10 points) $\int_0^2 \frac{1}{1-x^2}\,dx =$.

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow \begin{cases} A-B=0 \\ A+B=1 \end{cases} \Rightarrow \begin{cases} A=1/2 \\ B=1/2 \end{cases}$$

$$\int \left(\frac{1}{2(1-x)} + \frac{1}{2(1+x)}\right) dx = -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| \Big|_0^1$$

$$= \lim_{b\to 1^-} -\frac{1}{2}\ln|1-b| + \frac{1}{2}\ln|1+b| \Big|_0^b$$

$$+ \lim_{b\to 1^+} -\frac{1}{2}\ln|1-b| + \frac{1}{2}\ln|1+b| \Big|_b^2$$

$$= -\infty$$

divergent.

b) (~~7~~ 10 points) $\int x(\ln x)^2\,dx =$

$$\begin{cases} u = (\ln x)^2, & dv = x\,dx \\ du = \frac{2\ln x}{x}\,dx, & v = \frac{x^2}{2} \end{cases}$$

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \int \frac{x^{\cancel{2}}}{\cancel{2}} \cdot \cancel{2}\ln x \frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \int x \cdot \ln x\,dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^4}{4} + C$$

$$\begin{cases} u = \ln x \\ du = \frac{1}{x}\,dx \\ dv = x\,dx \\ v = \frac{x^2}{2} \end{cases}$$

c) (~~8~~ 10 points) $\int \frac{e^{2x}}{e^x+1}\,dx = \int \frac{u^2}{u+1}\,\frac{du}{u} = \int \frac{u}{u+1}\,du$

$$\begin{cases} u = e^x \\ du = e^x\,dx \end{cases}$$

$$= \int \left(1 - \frac{1}{u+1}\right) du$$

$$= \int du - \int \frac{1}{u+1}\,du$$

$$= u - \ln|u+1|$$

$$= e^x - \ln(e^x+1) + C$$

d) (~~7~~ 10 points) $\int_{-1}^{1} \frac{x}{(\tan x)^2+1}\,dx =$

$f(x) = \frac{x}{(\tan x)^2+1}$, $(\tan x = -\tan x)$

$f(-x) = \frac{-x}{(-\tan(x))^2+1} = -f(x)$

Since f is an odd function, symmetric along y-axis.

Then $\int_{-1}^{0} f(x) = -\int_{0}^{1} f(x)$. Thus $\int_{-1}^{1} f(x) = 0$.

2. Compute the following limits if they exist.

a) (~~15~~ 10 points) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (1+\frac{2i}{n})^2 =$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^2 = \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(1+\frac{2i}{n}\right)^2 =$$

$$\Delta x = \frac{2}{n}$$

$$\frac{1}{2} \int_1^3 x^2\,dx = \frac{1}{2} \frac{x^3}{3} \Big|_1^3 =$$

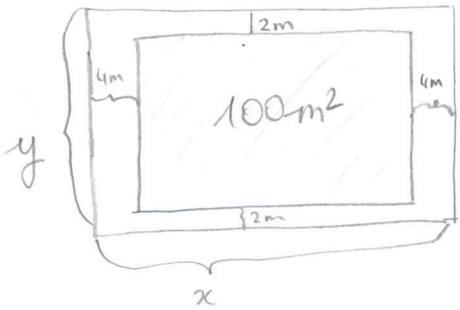
$$\frac{9}{2} - \frac{1}{6} = \frac{26}{6} = \frac{13}{3}$$

b) (~~15~~ 10 points) $\lim_{x\to 0} \frac{\int_0^{\sqrt{x}} (e^{-a^2})\,da}{\sqrt{x}} = \left[\frac{0}{0}\right]$

By l'hopital rule:

$$\lim_{x\to 0} \frac{\left(e^{-(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}\right)}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0} e^{-x} = 1.$$

3. (15 points) A billboard is to be made with $100m^2$ of $\overbrace{\text{printed area}}^{A}$ and with margins of 2m at the top and bottom and 4m on each side. Find the outside dimensions of the billboard if its total area is to be minimum.



(1) $(x-8).(y-4) = 100$ (since $A = 100m^2$)

We want total area to be minimum which is $x.y$.

From (1) $y = \frac{100}{x-8} + 4$.

Let f be the total area, $\begin{cases} f(x) = x \cdot \left(\frac{100}{(x-8)} + 4\right) \\ \text{where } x \text{ positive.} \end{cases}$

$$f'(x) = \left(\frac{100}{x-8} + 4\right) + x\left(\frac{-100}{(x-8)^2}\right)$$

$$= \frac{100(x-8) + 4(x-8)^2 - 100x}{(x-8)^2} = \frac{4(x-8)^2 - 800}{(x-8)^2}$$

CP: $f'(x) = 0 \iff x = 8 + 10\sqrt{2}$

SP: $f'(x)$ not defined $\iff (x-8)^2 = 0 \Rightarrow x = \mp 2\sqrt{2}$

$$f(8+10\sqrt{2}) = 8+10\sqrt{2}\left(\frac{100}{10\sqrt{2}} + 4\right) = 12 + 10\sqrt{2} + \frac{\sqrt{2}}{2}$$

	0		$2\sqrt{2}$		$8+10\sqrt{2}$		$+\infty$
$f'(x)$		$-$		$-$		$+$	

So $f(8+10\sqrt{2})$ is the minimum area

$y = 4 + 5\sqrt{2}$ m

$x = 8 + 10\sqrt{2}$ m

4. (15 points) Use mean value theorem to show that

$$|tan^{-1}x - tan^{-1}y| \leq |x - y|$$

$(Note: tan^{-1} = \arctan x)$

let $\arctan x = c_1$, $\arctan y = c_2$ then $x = \tan c_1$, $y = \tan c_2$

By MVT, $\exists c$ such that $(\tan(c))' = \dfrac{\tan c_1 - \tan c_2}{c_1 - c_2}$

where $c_2 < c_1$.

Since $(\tan(c))' = \dfrac{1}{\cos^2 c} \geq 1$, we have $\left|\dfrac{\tan c_1 - \tan c_2}{c_1 - c_2}\right| \geq 1$.

which is equivalent to

$$|x - y| \geq |\arctan x - \arctan y|.$$

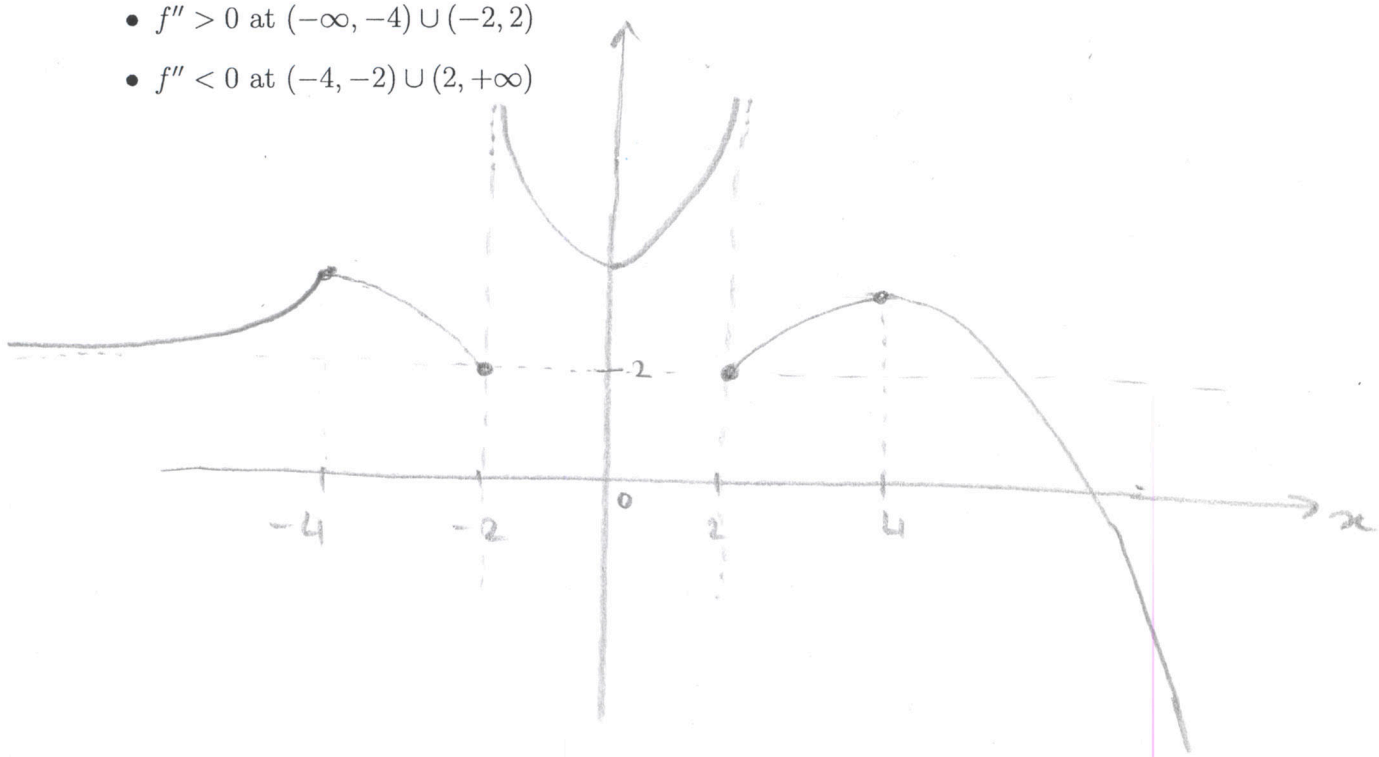
$f(x) = \tan^{-1} x \qquad f(y) = \tan^{-1} y$

$f'(x) = \dfrac{1}{1+x^2} \geq 1$

By MVT for all $x, y \in \mathbb{R}$ where $x < y$

$\exists c \in (x, y)$ such that $\dfrac{f(y) - f(x)}{y - x} \leq f'(c) \leq 1$

So $|\tan^{-1} y - \tan^{-1} x| \leq |\tan y - \tan x|$

5. (15 points) Graph a function f that satisfies the following properties:

- Domain is all real numbers.
- $f(x)$ is differentiable for $(-\infty, -4) \cup (-4, -2) \cup (-2, 2) \cup (2, +\infty)$
- $\lim_{x \to +\infty} f(x) = -\infty$, $\lim_{x \to -2^+} f(x) = \lim_{x \to 2^-} f(x) = \infty$,
- $\lim_{x \to -\infty} f(x) = \lim_{x \to -2^-} f(x) = \lim_{x \to 2^+} f(x) = f(-2) = f(2) = 2$
- $f' > 0$ at $(-\infty, -4) \cup (0, 2) \cup (2, 4)$
- $f' < 0$ at $(-4, -2) \cup (0, 2) \cup (4, +\infty)$
- $f'' > 0$ at $(-\infty, -4) \cup (-2, 2)$
- $f'' < 0$ at $(-4, -2) \cup (2, +\infty)$