MATH 106 MIDTERM II SOLUTIONS:

① $x^2-5x-2=-x^2+x+6 \Rightarrow 2x^2-6x-8=0 \Rightarrow 2(x-4)(x+1)=0$ $\quad x=4,\ x=-1$

$$\text{Area}=\int_{-1}^{4} -(x^2-5x-2)+(-x^2+x+6)\,dx=\int_{-1}^{4}-2x^2+6x+8\,dx=-\frac{2x^3}{3}+3x^2+8x\Big|_{-1}^{4}$$

$$=\left(\frac{-2\cdot 64}{3}+3\cdot 16+8\cdot 4\right)-\left(\frac{-2\cdot -1}{3}+3\cdot 1+8\cdot -1\right)=\frac{-130}{3}+85=\boxed{\frac{125}{3}}$$

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② $$\int_1^2\frac{\sqrt{4-x^2}}{x}\,dx=\int_{\pi/6}^{\pi/2}\frac{\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta}{2\sin\theta}=2\int_{\pi/6}^{\pi/2}\frac{\cos^2\theta}{\sin\theta}\,d\theta=2\int_{\pi/6}^{\pi/2}\frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$x=2\sin\theta$
$dx=2\cos\theta\,d\theta$

$$=2\int_{\pi/6}^{\pi/2}\frac{1}{\sin\theta}\,d\theta-2\int_{\pi/6}^{\pi/2}\sin\theta\,d\theta=2\int_{\pi/6}^{\pi/2}\frac{\sin\theta\,d\theta}{1-\cos^2\theta}+2\cos\theta\Big|_{\pi/6}^{\pi/2}=$$

$u=\cos\theta$

$$=-\int_{\sqrt{3}/2}^{0}\frac{1}{1-u}+\frac{1}{1+u}\,du+\left(2\cdot\cos\frac{\pi}{2}-2\cos\frac{\pi}{6}\right)=\ln|1-u|-\ln|1+u|\Big|_{\sqrt{3}/2}^{0}-\sqrt{3}$$

$$=(\ln 1-\ln 1)-\left(\ln\left|1-\frac{\sqrt{3}}{2}\right|-\ln\left|1+\frac{\sqrt{3}}{2}\right|\right)-\sqrt{3}$$

$$=\ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right|-\sqrt{3}=\boxed{\ln\frac{2+\sqrt{3}}{2-\sqrt{3}}-\sqrt{3}}$$

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③ (3a) LCT with $\frac{1}{x^2}$ $\quad \lim_{x\to\infty}\frac{\frac{x^2+x+1}{x^4+1}}{\frac{1}{x^2}}=1 \Rightarrow \int_1^\infty\frac{x^2+x+1}{x^4+1}\,dx \sim \int_1^\infty\frac{1}{x^2}\,dx$ → convergent

(3b) $$\int_0^2\frac{dx}{(x-1)^{2/3}}=\lim_{t\to 2^-}\int_0^t\frac{dx}{(x-2)^{2/3}}=\lim_{t\to 2^-}\frac{(x-2)^{1/3}}{1/3}\Big|_0^t=\lim_{t\to 2^-}3(t-2)^{1/3}+3\sqrt[3]{2}$$

$$=\boxed{3\sqrt[3]{2}}$$

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④ $$\int\arctan x\,dx=x\cdot\arctan x-\int\frac{x}{1+x^2}\,dx=x\cdot\arctan x-\frac{1}{2}\int\frac{1}{z}\,dz=\boxed{x\cdot\arctan x-\frac{1}{2}\ln(1+x^2)+C}$$

$u=\arctan x \quad dv=dx$

$du=\frac{1}{1+x^2}\,dx \quad v=x$

$z=1+x^2$
$dz=2x\,dx$

(5) $\displaystyle\int \frac{x^5}{1+x^2}\,dx = \int \frac{(u-1)^2}{u}\cdot\frac{du}{2} = \frac{1}{2}\int \frac{u^2-2u+1}{u}\,du = \frac{1}{2}\int u-2\,du + \frac{1}{2}\int \frac{1}{u}\,du =$

$u = 1+x^2$

$du = 2x\,dx$

$x^2 = u-1$

$$= \frac{1}{2}\left(\frac{u^2}{2} - 2u\right) + \frac{1}{2}\ln u$$

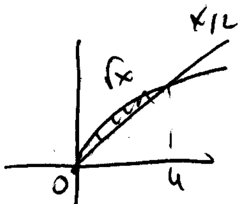
$$= \boxed{\frac{1}{4}(1+x^2)^2 - (1+x^2) + \frac{1}{2}\ln(1+x^2) + C}$$

(6) $\displaystyle\frac{2}{(x+2)x^3} = \frac{A}{(x+2)} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} \Rightarrow Ax^3 + B(x+2)x^2 + C(x+2)x + D(x+2) = 2$

$(A+B)x^3 + (2B+C)x^2 + (2C+D)x + 2D = 2$

$\Rightarrow D = 1 \quad C = -2^{-1} \quad B = 4^{-1} \quad A = -4^{-1}$

$$\int \frac{2}{x^4+2x^3}\,dx = \int \frac{-4^{-1}}{x+2} + \frac{4^{-1}}{x} + \frac{-2^{-1}}{x^2} + \frac{1}{x^3}\,dx = \boxed{-\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x| + \frac{1}{2x} - \frac{1}{2x^2} + C}$$

(7) 

a) $\displaystyle\int_0^4 \pi\left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2\right)dx = \int_0^4 \pi\left(x - \frac{x^2}{4}\right)dx = \pi\left(\frac{x^2}{2} - \frac{x^3}{12}\right)\Big|_0^4$

$$= \pi\left(\frac{16}{2} - \frac{64}{12}\right) - 0$$

$$= \boxed{\frac{8\pi}{3}}$$

(b) $\displaystyle\int_0^4 2\pi x\cdot\left(\sqrt{x} - \frac{x}{2}\right)dx = 2\pi\int_0^4 x^{3/2} - \frac{x^2}{2}\,dx = 2\pi\left(\frac{x^{5/2}}{5/2} - \frac{x^3}{6}\right)\Big|_0^4 = \frac{4\pi}{5}\cdot 32 - \frac{64\pi}{3} = \boxed{\frac{64\pi}{15}}$

(8) $\displaystyle\lim_{x\to 0^+}\left(1+\frac{3}{x}\right)^x = ?$ $\qquad y = \left(1+\frac{3}{x}\right)^x \qquad \ln y = x\cdot\ln\left(1+\frac{3}{x}\right)$

$$\lim_{x\to 0^+} x\cdot\ln\left(1+\frac{3}{x}\right) = \lim_{x\to 0^+}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} \underset{\text{L'hopital}}{=} \lim_{x\to 0^+}\frac{\frac{1}{1+3/x}\cdot\frac{-3}{x^2}}{-1/x^2} = \lim_{x\to 0^+}\frac{3}{1+\frac{3}{x}} = 0$$

$\ln y \to 0 \Rightarrow y \to e^0 = \boxed{1}$

(9) $\displaystyle\lim_{x\to 1}\frac{\int_{3+x}^{4} t^{3/2}\,dt}{1-e^{2x-2}} \underset{\text{L'hopital}}{=} \lim_{x\to 1}\frac{-(3+x)^{3/2}}{-2e^{2x-2}} = \frac{-8}{-2} = \boxed{4}$

$$\left(\int_{3+x}^{4} t^{3/2}\,dt\right)' \underset{\text{FTC}}{=} -(3+x)^{3/2}$$