1. Compute the following limits. Specify any infinite limits.

a) (5 points) $\lim_{x\to\infty} \frac{\sin(x^3+3)}{x^2-22x+3} = \lim_{x\to\infty} \frac{\sin(x^3+3)}{x^2\left(1-\frac{22}{x}+\frac{3}{x^2}\right)} =$

$$\lim_{x\to\infty} \frac{\sin(x^3+3)}{x^2} \cdot \frac{1}{\left(1-\cancel{\frac{22}{x}}+\cancel{\frac{3}{x^2}}\right)}$$

$$0 \cdot 1 = 0$$

$$-1 \le \sin(x^3+3) \le 1$$

$$\frac{-1}{x^2} \le \frac{\sin(x^3+3)}{x^2} \le \frac{1}{x^2}$$

$$\lim_{x\to\infty} -\frac{1}{x^2} \le \lim_{x\to\infty} \frac{\sin(x^3+3)}{x^2} \le \lim_{x\to\infty} \frac{1}{x^2}$$

$$0 \le \lim_{x\to\infty} \frac{\sin(x^3+3)}{x^2} \le 0$$

by sandwich theorem

$$\lim_{x\to\infty} \frac{\sin(x^3+3)}{x^2} = 0$$

b) (5 points) $\lim_{x\to-\infty} \frac{-3x^7-2x^3+15}{2x^7+3x^5-7} = \lim_{x\to-\infty} \frac{\cancel{x^7}\left(-3-\frac{2}{x^4}+\frac{15}{x^7}\right)}{\cancel{x^7}\left(2+\frac{3}{x^2}-\frac{7}{x^7}\right)} =$

$$\lim_{x\to-\infty} \frac{-3-\cancel{\frac{2}{x^4}}^{0}+\cancel{\frac{15}{x^7}}^{0}}{2+\cancel{\frac{3}{x^2}}^{0}-\cancel{\frac{7}{x^7}}^{0}} = \boxed{\frac{-3}{2}}$$

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(c) $\lim_{x \to -\infty} x^2 \sin\left(\frac{1}{x^2}\right) = \lim_{u \to 0}\left(\frac{\sin u}{u}\right) = 1$

(Let $u = \frac{1}{x^2}$ as $x \to -\infty$ $u \to 0$)

(d) $\lim_{x \to 0^+} \frac{\tan x}{\sqrt{x}} = \lim_{x \to 0^+}\left(\frac{\tan x}{x} \cdot \sqrt{x}\right)$

$= \lim_{x \to 0^+}\left(\frac{\tan x}{x}\right) \lim_{x \to 0^+} \sqrt{x}$

$= 1 \cdot 0 = 0$

Since $\sqrt{x}$ is not defined for negative values of x. $\lim_{x \to 0^-} \frac{\tan x}{\sqrt{x}}$ does not exist.

e) (5 points) $\lim_{x\to 0^+} \frac{x - \sin x}{x + \tan x} =$ $\lim_{x\to 0^+} \frac{x\left(1 - \frac{\sin x}{x}\right)}{x\left(1 + \frac{\sin x}{x \cos x}\right)} =$

$\lim_{x\to 0^+} \frac{\left(1 - \frac{\sin x}{x}\right)}{\left(1 + \frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)} = \frac{1-1}{1+1.1} = \frac{0}{3} = 0$

2. Differentiate the following functions.

a) (5 points) $f(x) = \frac{\sin(2x)}{\sin x - \cos x}$

$f'(x) = \frac{2(\cos 2x)(\sin x - \cos x) - (\cos x + \sin x) . \sin(2x}{(\sin x - \cos x)^2}$

b) (5 points) $f(x) = 13(x^2 - 4)^{-1/3}(x+5)^{1/5}$

$$f'(x) = 13\left[-\frac{1}{3}(x^2-4)^{-4/3} \cdot 2x \cdot (x+5)^{1/5} + \frac{1}{5}(x+5)^{-4/5} \cdot (x^2-4)^{-1/3}\right]$$

3. (7 points) Let $f : \mathbf{R} \to \mathbf{R}$ be a piecewise function defined as

$$f(x) = \begin{cases} \tan^2(x^2 - 1) & \text{for } x \geq 1 \\ 1 - x^2 & \text{for } x < 1 \end{cases}$$

Is f continuous at $x = 1$? Why?

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \tan^2(x^2-1) = 0 \qquad f(1) = \tan^2(1^2-1) = 0$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 - x^2 = 0$$

4. (8 points) Find the equation of the tangent line to the curve $f(x) = x^3 + 5x^2 + 3$ at the point (1,9).

$$f'(x) = 3x^2 + 10x$$

$$f'(1) = 3 \cdot 1^2 + 10 \cdot 1 = 13$$

$$y = 13x + b$$

$$9 = 13 + b$$

$$b = -4$$

$$\boxed{y = 13x - 4}$$

5. (15 points) Show that the function $f(x) = x + \cos^2(x/5) - 2$ has exactly one zero in $(-\infty, \infty)$.

$f(0) = 0 + \cos^2 0 - 2 = -1 < 0$

$f(2) = 2 + \cos^2(2/5) - 2 = \cos^2(2/5) > 0$

As both x and $\cos^2(x/5) - 2$ are continious functions $f(x)$ is continious. So by intermediate value theorem there exist $c \in [0,2]$ such that $f(c) = 0$

Now $f'(x) = 1 - 2\cos(\frac{x}{5}) \cdot \sin(\frac{x}{5}) \cdot \frac{1}{5} = 1 - \frac{1}{5}\sin(\frac{x}{5})$

$-\frac{1}{5} \leq \frac{1}{5}\sin(\frac{x}{5}) \leq \frac{1}{5}$ so $\frac{4}{5} \leq f'(x) \leq \frac{6}{5}$ then $f'(x) > 0$ for all real numbers. Hence $f(x)$ is an increasing function. Assume $f(c_1) = f(c_2) = 0$ and w.l.o.g $c_1 < c_2$. Now as f is increasing $0 = f(c_1) < f(c_2) = 0$ contradiction. Therefore there is exactly one zero of $f(x)$

6. (15 points) Determine for which values of x is the function $f(x) = |x^2 - x|$ **not** differentiable. Why?

$$f(x) = \begin{cases} x^2 - x, & x \in (-\infty, 0) \cup (1, \infty) \\ -x^2 + x, & x \in [0,1]. \end{cases}$$

$f(x)$ is differentiable for each x except of $x=0$ and $x=1$.

$$f'_-(0) = \lim_{h \to 0-} \frac{h^2 - h}{h} = \lim_{h \to 0-} (h-1) = -1$$

$$f'_+(0) = \lim_{h \to 0+} \frac{-h^2 + h}{h} = 1, \text{ So } f'_+(0) \neq f'_-(0).$$

$$f'_-(1) = \lim_{h \to 0-} \frac{-(1+h)^2 + (1+h) + 1 - 1}{h} = \lim_{h \to 0-} \frac{-\cancel{1} - 2h - h^2 + \cancel{1} + h}{h} = -1$$

$$f'_+(1) = \lim_{h \to 0+} \frac{(1+h)^2 - (1+h) + 1 - 1}{h} = \lim_{h \to 0+} \frac{1 + 2h + h^2 - 1 - h}{h} = +1$$

$f'_+(1) \neq f'_-(1)$.

7. (20 points) State and prove the product rule for differentiation. (Product Rule is the rule used for differentiating the product of two functions.)

If the functions f and g are differentiable at the point x, then the function $f \cdot g$ is differentiable at the point x, and

$$\left(f(x) g(x)\right)' = f'(x) g(x) + f(x) g'(x)$$

$$\underline{\left(f(x)g(x)\right)'} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h)\right] + \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \cdot f(x)\right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \underline{f'(x) g(x) + f(x) g'(x)}$$

$\lim_{h \to 0} g(x+h) = g(x)$ since g is continuous at the point x.