Math-106, Fall 2004, MT-2 (Type A) Problems and Solutions

Problem 1 **(20 pts)** Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=x^4-2x^2+3$.

(1.a) Determine the intervals in $\mathbb{R}$ where f is increasing and intervals where it is decreasing.

$f'(x)=4x^3-4x=4x\cdot(x^2-1)=4x\cdot(x-1)\cdot(x+1)$

For $x\in(-\infty,-1)$, $f'(x)<0\Rightarrow$ f is decreasing.
For $x\in(-1,0)$, $f'(x)>0\Rightarrow$ f is increasing.
For $x\in(0,1)$, $f'(x)<0\Rightarrow$ f is decreasing.
For $x\in(1,\infty)$, $f'(x)>0\Rightarrow$ f is increasing.

(1.b) Determine the local extremum values of f.

$f'(x)=0\Rightarrow x=0$ or $x=1$ or $x=-1$.
$x=-1$: f' changes sign from – to + $\Rightarrow$ It is local minimum with value $f(-1)=2$.
$x=0$: f' changes sign from + to – $\Rightarrow$ It is local maximum with value $f(0)=3$.
$x=1$: f' changes sign from – to + $\Rightarrow$ It is local minimum with value $f(1)=2$.

(1.c) Determine the intervals in $\mathbb{R}$ where f is concave up and intervals where it is concave down.

$f''(x)=12x^2-4=4\cdot(3x^2-1)$

For $x\in\left(-\infty,-\frac{1}{\sqrt{3}}\right)$, $f''(x)>0\Rightarrow$ concave up.

For $x\in\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $f''(x)<0\Rightarrow$ concave down.

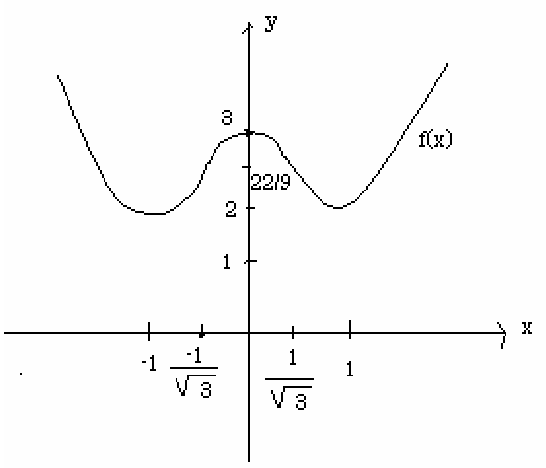
For $x\in\left(\frac{1}{\sqrt{3}},\infty\right)$, $f''(x)>0\Rightarrow$ concave up.

(1.d) Determine the inflection points of f.

$f''(x) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and when $x = \pm\frac{1}{\sqrt{3}}$, f changes sign. In other words

$\left(-\frac{1}{\sqrt{3}}, \frac{22}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{22}{9}\right)$ are both inflection points.

(1.e) Plot the graph of f.



Problem 2 (20 pts) Evaluate the following integrals.

(2.a) $\displaystyle\int \frac{7\sin x}{3+5\cos x}\,dx$

If we put $u = 3+5\cos x$, then $du = -5\sin x \cdot dx$ and

$$\int \frac{7\sin x}{3+5\cos x}\cdot dx = \int \frac{7\cdot -5\sin x}{-5\cdot(3+5\cos x)}\cdot dx = -\frac{7}{5}\int\frac{du}{u} = -\frac{7}{5}\ln|u| + C = -\frac{7}{5}\ln|3+5\cos x| + C.$$

(2.b) $\displaystyle\int \frac{dx}{x^2+4x+5}$

$\displaystyle\int \frac{dx}{x^2+4x+5} = \int\frac{dx}{1+(x+2)^2}$, if we put $u = x+2$, then $du = dx$ and

$$\int\frac{dx}{1+(x+2)^2} = \int\frac{du}{1+u^2} = Arc\tan u + C = Arc\tan(x+2) + C.$$

(2.c) $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx$

$$\int\frac{x+1}{\sqrt{1-x^2}}\cdot dx = \int\frac{x}{\sqrt{1-x^2}}\cdot dx + \int\frac{1}{\sqrt{1-x^2}}\cdot dx$$

$$\int\frac{1}{\sqrt{1-x^2}}\cdot dx = Arc\sin x + C_1.$$

To calculate $\displaystyle\int\frac{x}{\sqrt{1-x^2}}dx$, we make the substitution $u = 1-x^2$. Then $du = -2x\cdot dx$ and

$$\int\frac{x}{\sqrt{1-x^2}}dx = \int\frac{-2x}{-2\cdot\sqrt{1-x^2}}\cdot dx = \int\frac{du}{-2\sqrt{u}} = -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C_2 = -\sqrt{u} + C_2 = -\sqrt{1-x^2} + C_2.$$

$$\therefore \int\frac{x+1}{\sqrt{1-x^2}}dx = Arc\sin x - \sqrt{1-x^2} + C.$$

(2.d) $\int \tanh x \, dx$

If we put $u = e^x + e^{-x}$ then $du = \left(e^x - e^{-x}\right) \cdot dx$ and

$$\int \tanh x \cdot dx = \int \frac{du}{u} = \ln|u| + C = \ln\left(e^x + e^{-x}\right) + C = \ln(\cosh x) + C'.$$

Problem 3

(3.a) **(15 pts)** Let f be a continuous function on an interval $[a,b]$. Prove that the function $\int_a^x f(t)\,dt$ has a derivative at every $x \in [a,b]$ and

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Solution

$$F(x) = \int_a^x f(t)\cdot dt:$$

$$F'(x) = \lim_{h\to 0}\frac{1}{h}\cdot\left(F(x+h)-F(x)\right) = \lim_{h\to 0}\frac{1}{h}\cdot\left[\int_0^{x+h} f(t)\cdot dt - \int_0^x f(t)\cdot dt\right] =$$

$$= \lim_{h\to 0}\frac{1}{h}\cdot\left[\int_0^x f(t)\cdot dt + \int_x^{x+h} f(t)\cdot dt - \int_0^x f(t)\cdot dt\right] = \lim_{h\to 0}\frac{1}{h}\cdot\int_x^{x+h} f(t)\cdot dt.$$

By the Mean Value Theorem for Integrals, there exists $c \in (x, x+h)$ such that

$$\frac{1}{h}\cdot\int_x^{x+h} f(t)\cdot dt = f(c).$$

Since f is a continuous function, we have $F'(x) = \lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$.

(3.b) **(5 pts)** Evaluate

$$\frac{d}{dx}\int_{1+\sin^2 x}^{3}\frac{dt}{\ln t}.$$

$$\frac{d}{dt}\int_{1+\sin^2 x}^{3}\frac{dt}{\ln t} = \frac{-1}{\ln\left(1+\sin^2 x\right)}\cdot\left(1+\sin^2 x\right)' = \frac{-\sin(2x)}{\ln\left(1+\sin^2 x\right)}.$$

Problem 4 (15 pts) Calculate the area of the region bounded by the graphs of the functions $f(x) = x^3 - x^2$ and $g(x) = x^2 - x$.

<u>Solution:</u>

$$f(x) = g(x) \Rightarrow x^3 - x^2 = x^2 - x \Rightarrow x^3 - 2x^2 + x = 0 \Rightarrow x \cdot (x-1)^2 = 0 \Rightarrow x = 0 \quad or \quad x = 1.$$

So, the graphs of these two functions intersect at $(0,0)$ and $(1,0)$.
When $0 \le x \le 1$, $f(x) \ge g(x)$ since $f(x) - g(x) = x \cdot (x-1)^2 \ge 0$ when $x \ge 0$.
Therefore the area of the region in the question is

$$\int_0^1 (x^3 - 2x^2 + x) \cdot dx = \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right) \Bigg|_0^1 = \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{1}{12} \text{ unit sq.}$$

Problem 5 (15 pts) Calculate the length of the curve defined by

$$y(x) = \frac{2\ln x - x^2 + 3}{4}, \qquad 1 \le x \le 2.$$

Solution: We have $L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$.

$$\frac{dy}{dx} = \frac{2 \cdot \frac{1}{x} - 2x}{4} = \frac{\frac{1}{x} - x}{2} = \frac{1 - x^2}{2x}.$$

So $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1 - x^2}{2x}\right)^2} = \sqrt{1 + \frac{1 - 2x^2 + x4}{4x^2}} =$

$$= \sqrt{\frac{x^4 + 2x^2 + 1}{4x^2}} = \sqrt{\left(\frac{x^2 + 1}{2x}\right)^2} = \left|\frac{x^2 + 1}{2x}\right| = \frac{x^2 + 1}{2x}$$ since $x \in [1,2]$, so $x \ge 0$.

$$\therefore \quad L = \int_1^2 \frac{x^2 + 1}{2x} \cdot dx = \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right) \cdot dx = \frac{1}{2} \cdot \int_1^2 \left(x + \frac{1}{x}\right) \cdot dx = \frac{1}{2} \cdot \left[\frac{x^2}{2} + \ln|x|\right]_1^2 =$$

$$= \frac{1}{2} \cdot \left(\frac{4}{2} + \ln 2 - \frac{1}{2} - \ln 1\right) = \frac{1}{2} \cdot \left(\frac{3}{2} + \ln 2\right) = \frac{3}{4} + \ln\sqrt{2}.$$

Problem 6 (10 pts) Knowing that sinh is a one-to-one function, its inverse $\sinh^{-1}$ exists. Show that it satisfies

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \qquad \text{for every real number } x.$$

First solution:

$$\sinh\left[\ln\left(x+\sqrt{x^2+1}\right)\right] = \frac{e^{\ln\left(x+\sqrt{x^2+1}\right)} - e^{-\ln\left(x+\sqrt{x^2+1}\right)}}{2} = \frac{1}{2}\left[x+\sqrt{x^2+1} - \left(x+\sqrt{x^2+1}\right)^{-1}\right]$$

$$= \frac{1}{2}\cdot\left(\frac{\left(x+\sqrt{x^2+1}\right)^2 - 1}{x+\sqrt{x^2+1}}\right) = \frac{1}{2}\cdot\left(\frac{x^2+x^2+1+2x\cdot\sqrt{x^2+1}-1}{x+\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\cdot\left(\frac{2x^2+2x\cdot\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\right) = x = \sinh\left(\sinh^{-1}(x)\right)$$

$$\Rightarrow \sinh\left[\ln\left(x+\sqrt{x^2+1}\right)\right] = \sinh\left[\sinh^{-1}(x)\right]$$

$$\Rightarrow \underbrace{\sinh^{-1}\left(\sinh\left[\ln\left(x+\sqrt{x^2+1}\right)\right]\right)}_{\downarrow} = \underbrace{\sinh^{-1}\left[\sinh\left(\sinh^{-1}(x)\right)\right]}_{\downarrow}$$

$$\ln\left(x+\sqrt{x^2+1}\right) \qquad = \qquad \sinh^{-1}(x).$$

Second solution:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\left.\frac{d}{dx}\sinh y\right|_{y=\sinh^{-1}}(x)} = \left.\frac{1}{\cosh y}\right|_{y=\sinh^{-1}}(x) = \left.\frac{1}{\sqrt{1+\sinh^2 y}}\right|_{y=\sinh^{-1}(x)} = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\ln\left[x+\sqrt{x^2+1}\right] = \frac{1+\frac{2x}{2\cdot\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

$$\Rightarrow \frac{d}{dx}\sinh^{-1}(x) = \frac{d}{dx}\ln\left[x+\sqrt{x^2+1}\right] \Rightarrow \sinh^{-1}(x) = \ln\left[x+\sqrt{x^2+1}\right] + C.$$

But we know that $\sinh(0) = 0 \Rightarrow \sinh^{-1}(0) = 0$ & $\ln\left(0+\sqrt{0+1}\right) = \ln 1 = 0$.

Setting $x = 0$ in the equation $\sinh^{-1}(x) = \ln\left[x+\sqrt{x^2+1}\right] + C$, we find $C = 0$.

Hence $\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$.