Question 1. Let

$$f(x) = \frac{1+\sin x}{x} \quad \text{and} \quad g(x) = \frac{2e^x}{e^x - 1}.$$

(a) Find the horizontal and vertical asymptotes of $f(x)$.

Horizontal asymptotes : $\underline{\underline{x=0}}$, as $\lim_{x\to 0^+} \frac{1+\sin x}{x} = \infty$

$\lim_{x\to 0^-} \frac{1+\sin x}{x} = -\infty$

Vertical asymptotes: To find those,

we need to calculate $\lim_{x\to\infty} \frac{1+\sin x}{x}$, and $\lim_{x\to-\infty} \frac{1+\sin x}{x}$.

$0 \leq 1+\sin x \leq 2 \Rightarrow \frac{0}{|x|} \leq \frac{1+\sin x}{x} \leq \frac{2}{|x|}$

as $x\to\mp\infty$ $\to 0$ by squeeze as $x\to\mp\infty$

$y=0$ is the only horizontal asymptote.

(b) Find a formula for the inverse of $g(x)$.

$y = \frac{2e^x}{e^x-1} \Leftrightarrow e^x \cdot y - y = 2\cdot e^x \Leftrightarrow e^x(y-2) = y$

$\Leftrightarrow e^x = \frac{y}{y-2} \Leftrightarrow x = \ln\left(\frac{y}{y-2}\right)$, so

$g^{-1}(x) = \ln\left(\frac{x}{x-2}\right)$.

(c) Differentiate the function $g(x)$.

$$g'(x) \underset{\text{quotient rule}}{=} \frac{2e^x \cdot (e^x-1) - e^x \cdot 2e^x}{(e^x-1)^2} = \frac{-2e^x}{(e^x-1)^2}.$$

Question 2. Evaluate the limit in each part. Show the details of your work. (Note: You cannot use L'Hospital's Rule.)

(a) $\lim_{x \to -2^+} \arcsin(x+1)$

$\overset{}{=} \arcsin\left(\lim_{x \to -2^+} x+1\right) = \arcsin(-1) = -\pi/2$

arcsin is continuous on its domain

(b) $\lim_{t \to \infty} \left(\sqrt{t^2+6t} - \sqrt{t^2+3}\right) = \lim_{t\to\infty} \frac{\left(\sqrt{t^2+6t} + \sqrt{t^2+3}\right)\left(\sqrt{t^2+6t} - \sqrt{t^2+3}\right)}{\sqrt{t^2+6t} + \sqrt{t^2+3}} =$

$= \lim_{t\to\infty} \frac{6t-3}{\sqrt{t^2+6t} + \sqrt{t^2+3}} = \lim_{t\to\infty} \frac{6}{\sqrt{1+\frac{6}{t}} + \sqrt{1+\frac{3}{t}}} = 3$

(c) $\lim_{\theta \to 0} \frac{\tan 2\theta}{\theta} = \lim_{\theta\to 0} \frac{\tan 2\theta}{2\theta} \cdot 2 = \lim_{\theta \to 0} \underbrace{\frac{\sin 2\theta}{2\theta}}_{\to 1} \cdot \frac{2}{\underbrace{\cos 2\theta}_{\to 1}} = 2$

(d) $\lim_{x \to 2} \frac{x^5 - 32}{x - 2} = f'(2)$, where $f(x) = x^5$, hence

$\lim_{x\to 2} \frac{x^5-32}{x-2} = 5 \cdot x^4 \Big|_{x=2} = \underline{\underline{80}}$

Question 3. Consider the functions given below.

$$f(x) = \sqrt[3]{x^2} \qquad g(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

$$h(x) = \sqrt{x+1} \qquad u(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$$

(a) Find the derivative of $h(x)$ using the limit definition of a derivative.

$$h'(x) = \lim_{h \to 0} \frac{\sqrt{x+1+h} - \sqrt{x+1}}{h} = \lim_{h \to 0} \frac{\left(\sqrt{x+1+h} + \sqrt{x+1}\right)\left(\sqrt{x+1+h} - \sqrt{x+1}\right)}{h \cdot \left(\sqrt{x+1+h} + \sqrt{x+1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h} \cdot \left(\sqrt{x+1+h} + \sqrt{x+1}\right)} = \frac{1}{2\sqrt{x+1}}$$

(b) Which of the functions $f(x)$, $g(x)$, $h(x)$ and $u(x)$ are differentiable at $x = 0$. Explain your reasoning.

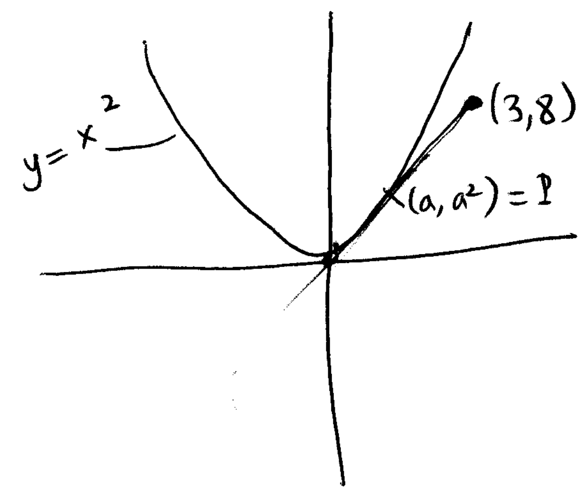
- (a) shows that $h(x)$ is diff'ble at $x=0$.
- $u(x)$ is NOT, because $\lim_{x \to 0^-} u(x) = -1 \neq 1 = \lim_{x \to 0^+} u(x)$, so u is <u>not even</u> continuous.
- $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x^2}}{x} = \lim_{x \to 0} x^{-1/3}$, which does NOT exist, hence f is NOT diff'ble at $x=0$.
- $\lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2}{x} = 0$

$\neq$

$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^-} \frac{x}{x} = 1$

$\Big\}$ $\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$ does <u>not</u> exist, hence g is <u>not</u> diff'ble.

Question 4. Find an equation of a line passing through $(3, 8)$ and tangent to the parabola $y = x^2$. (Note : There may be more than one such tangent lines. It suffices to find an equation for one of them.)



Suppose the (a) line we are seeking for passes through $P = (a, a^2)$ on the curve $y = x^2$.

– Then, this line is tangent to $y = x^2$, hence its slope at P is given by $\left.\frac{dy}{dx}\right|_{x=a} = 2a$.

– Furthermore, since this line passes through both $\underset{(a, a^2)}{\overset{}{P}}$ and $(3, 8)$, its slope also equals $\frac{(8-a^2)}{3-a}$.

$$\Longrightarrow \frac{8-a^2}{3-a} = 2a \Longleftrightarrow 8 - a^2 = 6a - 2a^2 \Longleftrightarrow a^2 - 6a + 8 = 0$$

$$\Longleftrightarrow a = 4 \quad \text{or} \quad a = 2.$$

For $a=2$, the line we look for is $\frac{y-8}{x-3} = 4$.

For $a=4$, ———— $\frac{y-8}{x-3} = 8$.

Question 5.

(a) Let $f(x)$ and $g(x)$ be two functions. Suppose for all $\varepsilon > 0$ we are given that

$$\text{if } |x-3| < \varepsilon^3, \text{ then } |f(x)-5| < \varepsilon/2$$

and that

$$\text{if } |x-3| < \varepsilon/10, \text{ then } |g(x)-10| < \varepsilon/2.$$

Find a real number $\delta > 0$ such that

$$\text{if } |x-3| < \delta, \text{then } |f(x)+g(x)-15| < \frac{1}{2}.$$

(Hint: Recall the triangular inequality $|a+b| \leq |a|+|b|$ which holds for all pair of real numbers a and b.)

By what we are given, it follows that:

if $|x-3| < \min(\varepsilon^3, \varepsilon/10)$, then $|f(x)-5| + |g(x)-10| < \varepsilon$.

By the hint, $|f(x)+g(x)-15| \leq |f(x)-5| + |g(x)-10|$, hence

if $|x-3| < \min(\varepsilon^3, \varepsilon/10)$, then $|f(x)+g(x)-15| < \varepsilon$.

So the δ

(We want is $\delta = \min\left(\frac{1}{2^3}, \frac{1}{20}\right) = \frac{1}{20}$.

(b) Show that

$$\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$$

using $\varepsilon - \delta$ definition of a limit. (Note: You cannot use the limit laws. You must apply $\varepsilon - \delta$ definition to the function $f(x) = 1/x$.)

• Given $\varepsilon > 0$, we want to find a δ such that if $|x-2| < \delta$, then $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$, in other words if $2-\delta < x < 2+\delta$, then $\frac{|x-2|}{|2x|} < \varepsilon$.

Suppose without loss of generality that $\delta \leq 1$. Then $2-\delta < x < 2+\delta \Rightarrow$
Now, if $|x-2| < \delta$, then (assuming $\delta \leq 1$) $\Rightarrow \underline{\underline{1 < x < 3}}$

$$\frac{|x-2|}{|2x|} < \delta \cdot \frac{1}{|2x|} < \frac{\delta}{2}$$

$|x| > 1$, hence $\frac{1}{|x|} < 1$

So it will suffice to choose $\delta = \min(1, 2\varepsilon)$.

• Indeed, if $|x-2| < \delta = \min(1, 2\varepsilon)$, then
(i) $2-\delta < x < 2+\delta$, and since $\delta \leq 1$, $\underline{1 < x < 3}$.

(ii) $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|} < \frac{\delta}{2\cdot|x|} \leq \frac{\cancel{2}\varepsilon}{\cancel{2}\cdot|x|} = \frac{\varepsilon}{|x|} \leq \varepsilon$, as desired.

(i)

Question 6. Is there a function f which is continuous on $(-\infty, \infty)$ and

$$f(x)f(x+2) < 0$$

for every x? Justify your answer. (Hint: Use the Intermediate Value Theorem.)

$$f(0)\, f(2) < 0 \Rightarrow \text{either } \begin{Bmatrix} f(0) < 0 \\ f(2) > 0 \end{Bmatrix} \text{ or } \begin{Bmatrix} f(2) < 0 \\ f(0) > 0 \end{Bmatrix}$$

Either case 0 is in between $f(0)$ and $f(2)$, it follows from IVT that there is a $c \in (0,2)$ such that $f(c) = 0$. But then, $0 = f(c) \cdot f(c+2) \not< 0$, hence such f cannot exist.

Question 7. (Bonus Question; No partial credit will be awarded)
Let $f(x)$ be a continuous function satisfying $|f(x)| \leq x^2$ for all $x \in [-2,2]$. Show that f is differentiable at 0 and find $f'(0)$.

First note that $|f(0)| \leq 0^2 = 0$ implies that $f(0) = 0$.

Now, $f'(0)$ "=" $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$.

if the limit exists

By the given inequality $|f(x)| \leq x^2$, it follows that $-x^2 \leq f(x) \leq x^2$, hence $-|x| \leq \frac{f(x)}{x} \leq |x|$.

$-|x| \xrightarrow{\text{as } x \to 0} 0 \xleftarrow{\text{as } x \to 0} |x|$

$\Rightarrow$ By the squeeze theorem

$\lim_{x \to 0} \frac{f(x)}{x} = 0$, hence $\underline{\underline{f'(0) = 0}}$.