Math 106 ♦ Exam # 1 ♦ Solutions

1.

a. Find $\lim\limits_{\theta\to 0}\dfrac{(\sin 3\theta)^2}{\theta^2\cos\theta}=\lim\limits_{\theta\to 0}\dfrac{9(\sin 3\theta)^2}{9\theta^2\cos\theta}=\lim\limits_{\theta\to 0}\left(9\underbrace{\left(\dfrac{\sin 3\theta}{3\theta}\right)^2}_{\to 1}\dfrac{1}{\underbrace{\cos\theta}_{\to 1}}\right)=9.$

b. Since $\lim\limits_{x\to\pm\infty}\left(\dfrac{5x}{x-4}\right)=\lim\limits_{x\to\pm\infty}\left(\dfrac{5}{1-4/x}\right)=5$, the line $y=5$ is a horizontal asymptote.

Since $\lim\limits_{x\to 4^+}\left(\dfrac{\overbrace{5x}^{20}}{\underbrace{x-4}_{0^+}}\right)=\infty$, the line $x=4$ is a vertical asymptote.

c. Using the formula $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we find

$\lim\limits_{x\to 0}\dfrac{(2+x)^3-8}{x}=\lim\limits_{x\to 0}\dfrac{8+12x+6x^2+x^3-8}{x}=\lim\limits_{x\to 0}\left(12+6x+x^2\right)=12$.

Note that this is the derivative of $f(x)=x^3$ at $x=2$, by definition.

d. Since $-1\le\sin\left(\dfrac{1}{x}\right)\le 1$ for nonzero *x*, we can make the sandwich

$-x^{2/3}\le x^{2/3}\sin\left(\dfrac{1}{x}\right)\le x^{2/3}$. Since both $x^{2/3}$ and $-x^{2/3}$ go to 0 as $x\to 0$, the

middle term will go to 0 as well. Thus $\lim\limits_{x\to 0}\left(x^{2/3}\sin\left(\dfrac{1}{x}\right)\right)=0$.

2.

a. We have

$\dfrac{d}{dx}\sqrt{1-x^2}=\dfrac{d}{dx}\left(\underbrace{1-x^2}_{u}\right)^{1/2}=\dfrac{d}{du}u^{1/2}\dfrac{d}{dx}\left(1-x^2\right)=\dfrac{1}{2}u^{-1/2}(-2x)=-\dfrac{x}{\sqrt{1-x^2}}$.

b. $\dfrac{d}{dt}\left(t^3\cos\left(\dfrac{1}{t}\right)\right)=3t^2\cos\left(\dfrac{1}{t}\right)+t^3\dfrac{d}{dt}\cos\left(\underbrace{t^{-1}}_{u}\right)=3t^2\cos\left(\dfrac{1}{t}\right)+t^3\sin\left(\dfrac{1}{t}\right)t^{-2}=$

$3t^2\cos\left(\dfrac{1}{t}\right)+t\sin\left(\dfrac{1}{t}\right)$.

c. $\dfrac{d}{dx}\left(x^2+xy-y^2\right)=\dfrac{d}{dx}(1)$. Using the product rule for $\dfrac{d}{dx}(xy)$ and the chain rule for $\dfrac{d}{dx}\left(y^2\right)$, we find $2x+\dfrac{dx}{dx}y+x\dfrac{dy}{dx}-2y\dfrac{dy}{dx}=0$, or $\dfrac{dy}{dx}=\dfrac{y+2x}{2y-x}=\dfrac{3+4}{6-2}=\dfrac{7}{4}$ at $(x,y)=(2,3)$.

d. If $y = \frac{\cos x}{x}$, then, by the quotient rule, $\frac{dy}{dx} = \frac{-(\sin x)x - (\cos x)1}{x^2}$, so $dy = \frac{-(\sin x)x - \cos x}{x^2}dx$.

3.

- Let $V(t)$ be the volume of the water in the reservoir at time *t*, and let $y(t)$ be the depth of the water. We will measure lengths in meters throughout, and time in minutes.
- The **unknown rate** is $\frac{dy}{dt}$ (when *y* is 5) , and the **given rate** is $\frac{dV}{dt} = -10$.
- We need an **equation relating *V* to *y***. The volume of a cone is $V = \frac{1}{3}\pi r^2 y$, where *r* is the radius of the water's surface. To relate *V* to *y*, we need to eliminate *r*. Considering similar triangles (draw a vertical cross section!), we find that $\frac{r}{y} = \frac{18}{6} = 3$, so that $r = 3y$ and $V = \frac{1}{3}\pi 9y^3 = 3\pi y^3$.
- To **relate the unknown rate** $\frac{dy}{dt}$ **to the given rate** $\frac{dV}{dt}$, we differentiate both sides of the equation $V = 3\pi y^3$ with respect to *t*, keeping in mind that *V* and *y* are functions of time *t*, and using the chain rule accordingly: $\frac{dV}{dt} = 9\pi y^2 \frac{dy}{dt}$.
- Now we **solve for the unknown rate and plug in** $y = 5$:
  $\frac{dy}{dt} = \frac{1}{9\pi y^2}\frac{dV}{dt} = -\frac{1}{9\pi 25}\cdot 10 = -\frac{2}{9\pi}$.

  The water level is falling at a rate of $\frac{2}{9\pi}$ meters per minute.

4.

a. The acceleration is $a = \frac{dv}{dt} \underbrace{=}_{\substack{quotient\\rule}} \frac{3\cdot\left(t^2+1\right) - 3t\cdot 2t}{\left(t^2+1\right)^2} = \frac{3-3t^2}{\left(t^2+1\right)^2}$.

b. The cockroach is running up the wall when the velocity is positive; this is the case for $0 < t \le 3$. Likewise, the cockroach is running down when $-3 \le t < 0$. Thus it reaches the lowest point on its journey at time $t = 0$.

c. The highest velocity will be attained at one of the end points, $t = 3$ and $t = -3$, or at a point where $\frac{dv}{dt} = a = \frac{3-3t^2}{\left(t^2+1\right)^2} = 0$ (meaning that the tangent to the *v* graph is horizontal). The last equation holds for $t = 1$ and $t = -1$. We evaluate *v* at these four points to see where the velocity is highest. We find $v(3) = 0.9, v(1) = 1.5$, $v(-1) = -1.5$, and $v(-3) = -0.9$. The highest velocity is 1.5, attained at $t = 1$.

d. We are looking for the times when the velocity is either 1 or -1. One approach is to consider the values of v we computed in part (c); it helps to draw a rough graph of v. Between times $t = 1$ and $t = 3$, the velocity decreases from 1.5 to 0.9, so that it will be 1 exactly once. Arguing similarly for the time intervals $[-1,1]$ and $[-3,-1]$, we see that the speed is 1 exactly 4 times.

Alternatively, we can solve the equations $v(t) = \dfrac{3t}{t^2+1} = 1$ and $v(t) = \dfrac{3t}{t^2+1} = -1$, and show that each equation has exactly two solutions in the domain, giving a total of 4 solutions.

5.

a. By definition of continuity, we want $\lim\limits_{x\to k} f(x) = f(k)$. Since we are dealing with a piecewise function, we will consider single-sided limits.
We want $\lim\limits_{x\to k^+} f(x) = \lim\limits_{x\to k^-} f(x) = f(k)$. Now
$\lim\limits_{x\to k^+} f(x) = \lim\limits_{x\to k^+} x^2 = k^2$, $\lim\limits_{x\to k^-} f(x) = \lim\limits_{x\to k^-} x^3 = k^3$, and $f(k) = k^3$. Thus it is required that $k^2 = k^3$, or $k^2 - k^3 = 0$, or $k^2(1-k) = 0$.
The function is continuous for $k = 0$ and $k = 1$.

b. Since differentiability implies continuity, our only options are $k = 0$ and $k = 1$.
In the case $k = 0$, we have to see whether $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$ exists, meaning that $\lim\limits_{h\to 0^+} \dfrac{f(h)}{h}$ and $\lim\limits_{h\to 0^-} \dfrac{f(h)}{h}$ both exist and are equal. Now $\lim\limits_{h\to 0^+} \dfrac{f(h)}{h} = \lim\limits_{h\to 0^+} \dfrac{h^2}{h} = \lim\limits_{h\to 0^+} h = 0$ and $\lim\limits_{h\to 0^-} \dfrac{f(h)}{h} = \lim\limits_{h\to 0^+} \dfrac{h^3}{h} = \lim\limits_{h\to 0^+} h^2 = 0$, so that the function is indeed differentiable for $k = 0$.

6.

a. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\left(\frac{1}{h}\left(\frac{2}{3x+3h+1} - \frac{2}{3x+1}\right)\right)$$
$$= 2\lim_{h\to 0}\left(\frac{1}{h}\left(\frac{(3x+1)-(3x+3h+1)}{(3x+3h+1)(3x+1)}\right)\right) = 2\lim_{h\to 0}\left(\frac{-3}{(3x+3h+1)(3x+1)}\right) = \frac{-6}{(3x+1)^2}.$$

b. Use the quotient rule; $\dfrac{d}{dx}\left(\dfrac{2}{3x+1}\right) = \dfrac{0\cdot(3x+1) - 2\cdot 3}{(3x+1)^2} = \dfrac{-6}{(3x+1)^2}$. Note that our answers in parts (a) and (b) are the same.

Extra Credit: We want $\left|f(x)-f\left(x_0\right)\right|<\varepsilon$, meaning that $f(x)$ is required to be on the open interval $\left(f\left(x_0\right)-\varepsilon, f\left(x_0\right)+\varepsilon\right)=(1,3)$. Since $f(x)$ is a decreasing function, it suffices to find the points *a* and *b* such that $f(a)=\frac{2}{a}=3$ and $f(b)=\frac{2}{b}=1$. We compute $a=\frac{2}{3}$ and $b=2$. If *x* is on the interval $\left(\frac{2}{3},2\right)$, then $f(x)$ will be on the interval $(1,3)$. (Illustrate these findings on the graph of $f(x)$.)

Now we can make δ the smaller of the two numbers $x_0-a=\frac{1}{3}$ and $b-x_0=1$, which is $\delta=\frac{1}{3}$. If *x* is on the interval $\left(x_0-\delta, x_0+\delta\right)=\left(\frac{2}{3},\frac{4}{3}\right)$, then $f(x)$ will be on the interval $\left(f\left(x_0\right)-\varepsilon, f\left(x_0\right)+\varepsilon\right)=(1,3)$. In other words: If $\left|x-x_0\right|<\delta$, then $\left|f(x)-f\left(x_0\right)\right|<\varepsilon$, as required.

(Note that any positive $\delta<\frac{1}{3}$ will work as well.)