Question 1.

(a) (5 points) Is the function

$$f(x) = e^{\cos(x)}$$

even, odd or neither? Explain your reasoning.

$$f(x) = e^{\cos(x)} = e^{\cos(-x)} = f(-x)$$

since $\cos(x) = \cos(-x)$

$\boxed{f(x) \text{ IS EVEN}}$

(b) (5 points) Find the domain and range of the function below.

$$g(x) = \ln(x-5)$$

$$\text{Domain} = \{x : x-5 > 0\}$$
$$= (5, \infty)$$
$$\text{Range} = (-\infty, \infty)$$

(c) (5 points) Is the function

$$g(x) = \ln(x-5)$$

one-to-one? If it is one-to-one, find a formula for the inverse of $g(x)$. If it is not, explain why it is not a one-to-one function.

$$\ln(x_1 - 5) = \ln(x_2 - 5)$$
$$\Longleftrightarrow$$
$$e^{\ln(x_1-5)} = e^{\ln(x_2-5)}$$
$$\Longleftrightarrow$$
$$x_1 - 5 = x_2 - 5 \Longleftrightarrow x_1 = x_2$$

Therefore $g(x)$ IS ONE-TO-ONE

Formula for the inverse

$$y = \ln(x-5) \Longleftrightarrow x - 5 = e^y$$
$$\Longleftrightarrow x = e^y + 5$$

$\boxed{\text{INVERSE OF } g(x) \quad g^{-1}(x) = e^x + 5}$

Question 2. Evaluate the limit in each part. Show the details of your work. (Note: You cannot use L'Hospital's Rule.)

(a) (5 points) $\lim\limits_{t\to 2^+} \dfrac{t^2-4}{t^2-4t+4} = L_1$

$$L_1 = \lim_{t\to 2^+} \frac{(t-2)(t+2)}{(t-2)^2} = \lim_{t\to 2^+} \frac{\overbrace{t+2}^{>0 \text{ as } t\to 2^+}}{\underbrace{t-2}_{>0 \text{ as } t\to 2^+}} = \infty$$

(b) (5 points) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+4x} - \sqrt{x^2-3}\right) = L_2$

$$L_2 = \lim_{x\to\infty} \frac{\left(\sqrt{x^2+4x}-\sqrt{x^2-3}\right)\left(\sqrt{x^2+4x}+\sqrt{x^2-3}\right)}{\sqrt{x^2+4x}+\sqrt{x^2-3}}$$

$$= \lim_{x\to\infty} \frac{(x^2+4x)-(x^2-3)}{\sqrt{x^2+4x}+\sqrt{x^2-3}} = \lim_{x\to\infty} \frac{(4x+3)/\sqrt{x^2}}{\sqrt{\frac{x^2+4x}{x^2}}+\sqrt{\frac{x^2-3}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{4+3/x}{\sqrt{1+4/x}+\sqrt{1-3/x^2}} = \frac{4}{2} = 2$$

(c) (5 points) $\lim\limits_{x\to\pi/2} \dfrac{\cos(x)}{x-\pi/2} = L_3$

Let $h = x - \pi/2$. Note that $h \to 0$ as $x \to \pi/2$

$$L_3 = \lim_{h\to 0} \frac{\cos(h+\pi/2)}{h} = \lim_{h\to 0} \frac{-\sin h}{h} = -1$$

(d) (5 points) $\lim\limits_{x\to 0} \dfrac{\sin(\pi/4+x) - \frac{1}{\sqrt{2}}}{x} = L_4$

$$L_4 = \lim_{x\to 0} \frac{f(\pi/4+x)-f(\pi/4)}{x}$$ where $f(x) = \sin x$

$$= f'\left(\frac{\pi}{4}\right) = \cos x \Big|_{x=\pi/4} = 1/\sqrt{2}$$

Question 3.

(a) (5 points) Calculate the derivative of $h(x) = \frac{e^x}{\tan(x)}$.

By the quotient rule

$$h'(x) = \frac{(e^x)'\tan(x) - (\tan(x))' e^x}{(\tan(x))^2}$$

$$= \frac{e^x \tan(x) - \sec^2(x) e^x}{\tan^2(x)}$$

(b) (10 points) Show that if f is differentiable at 2 with $f'(2) = 5$, then

$$\lim_{h\to 0} \frac{f(2+8h) - f(2+h)}{h} = L$$

exists. Find the value of the limit.

$$L = \lim_{h\to 0} \frac{f(2+8h) - f(2) - f(2+h) + f(2)}{h}$$

$$= \lim_{h\to 0} \frac{f(2+8h) - f(2)}{h} - \underbrace{\lim_{h\to 0} \frac{f(2+h) - f(2)}{h}}_{\text{exists and equal to } f'(2)}$$

$$\overset{\left(\substack{\text{Let} \\ w=8h}\right)}{=} \underbrace{\lim_{w\to 0} \frac{f(2+w) - f(2)}{(w/8)}}_{\text{exists and equal to } 8f'(2)} - f'(2) = 7f'(2) = 35$$

(c) (10 points) Suppose $f(x)$ is a continuous function such that $f(0) = 0$. Does $xf(x)$ have to be differentiable at 0? If it has to be differentiable at 0, prove it. If it is not necessarily differentiable at 0, give a counter example.

$xf(x)$ is differentiable at 0

$\Longleftrightarrow$

$$\lim_{h\to 0} \frac{hf(h) - 0f(0)}{h} = \lim_{h\to 0} f(h)$$

exists

Since f is continuous, we have

$$\left.\frac{d(xf(x))}{dx}\right|_{x=0} = \lim_{h\to 0} f(h) = f(0) = 0$$

$\boxed{\text{Therefore } xf(x) \text{ IS DIFFERENTIABLE AT } 0}$

Question 4. (15 points) Consider the function

$$f(x) = \frac{x^3}{3} + x^2 - 2x.$$

Find an equation of a line that is tangent to $f(x)$ and parallel to the line $y = x + 5$. (Note: There may be more than one such tangent lines. It is sufficient to find an equation for one of them.)

Need to find a point (x_*, y_*) on the graph of f where the slope of the tangent line is 1.

$$f'(x_*) = x_*^2 + 2x_* - 2 = 1$$

$$\Longrightarrow$$

$$x_*^2 + 2x_* - 3 = 0$$

$$\Longrightarrow$$

$$x_* = -3 \quad \text{OR} \quad x_* = 1$$

$$\left(y_* = f(-3) = 6 \right) \qquad \left(y_* = f(1) = -2/3 \right)$$

Tangent line at $(-3, 6)$

$$y - 6 = 1\,(x - (-3))$$

$$\Longrightarrow y = x + 9$$

Tangent line at $(1, -2/3)$

$$y - (-2/3) = 1\,(x - 1)$$

$$\Longrightarrow y = x - 5/3$$

Question 5. (15 points) Show that

$$\lim_{x\to -5} x^2 - 1 = 24$$

using $\varepsilon - \delta$ definition of a limit. (Note: You cannot use the limit laws. You must apply $\varepsilon - \delta$ definition to the function $f(x) = x^2 - 1$.)

① Given any $\epsilon > 0$ show the existence of a $\delta > 0$ such that

$\boxed{\text{AIM}}$ $$\underbrace{|x-(-5)|}_{x+5} < \delta \Longrightarrow |(x^2-1)-24| < \epsilon$$

② $$|(x^2-1)-24| = |x^2-25|$$

$\boxed{\text{GUESS}}$ $$= \underbrace{|x-5|}_{<c}\,\underbrace{|x+5|}_{<\epsilon/c} < \epsilon$$

Need to use the fact that $|x-5| < c$ for x close to -5.

To be precise suppose

$$|x+5| < 1 \Longrightarrow -1 < x+5 < 1$$
$$\Longrightarrow -11 < (x-5) < -9$$
$$\Longrightarrow |x-5| < 11$$

Therefore

$|x+5| < 1$ and $|x+5| < \epsilon/11$

(implying $|x-5| < 11$)

$$\Longrightarrow \underbrace{|x-5|}_{<11}\,\underbrace{|x+5|}_{<\epsilon/11} = |(x^2-1)-24| < \epsilon$$

③ $\boxed{\text{CONCLUDE}}$ $$|x-(-5)| < \overbrace{\min(1, \epsilon/11)}^{\delta} \Longrightarrow |(x^2-1)-24| < \epsilon \quad \square$$

Question 6. (10 points) Suppose a continuous function $f(x)$ attains the values

$$f(0) = 0.1 \quad \text{and} \quad f(1) = 10.$$

Show that there exists a c such that $(f(c))^2 = f(c)$.

Consider

$$g(x) = (f(x))^2 - f(x)$$

Since $f(x)$ is continuous, so is $(f(x))^2$ and $(f(x))^2 - f(x)$.

Now

$$g(0) = (f(0))^2 - f(0)$$
$$= (0.1)^2 - 0.1 < 0$$

and

$$g(1) = (f(1))^2 - f(1)$$
$$= (10)^2 - 10 > 0$$

By the intermediate value thm there exists an $\tilde{x} \in (0,1)$ such that

$$g(\tilde{x}) = 0$$

$$\Longrightarrow (f(\tilde{x}))^2 = f(\tilde{x})$$

$\square$