Solutions to Math-106(Fall/2004) Final Exam Problems

1- a) $D = Ñ - \{-1\}$.

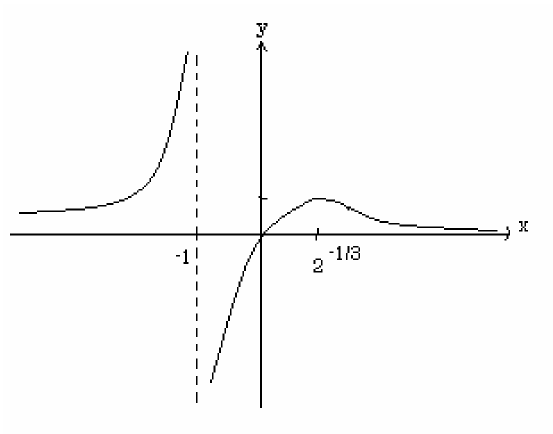
b) $\lim\limits_{x\to 1^{+1}} f(x) = -\infty$, $\lim\limits_{x\to 1^{-}} f(x) = +\infty$, $\lim\limits_{x\to +\infty} f(x) = 0$, $\lim\limits_{x\to -\infty} f(x) = 0$.

c) $f'(x) = \dfrac{x^3+1-3x^2\cdot x}{(x^3+1)^2} = \dfrac{1-2x^3}{(x^3+1)^2} = 0 \Rightarrow x = \dfrac{1}{\sqrt[3]{2}}$.

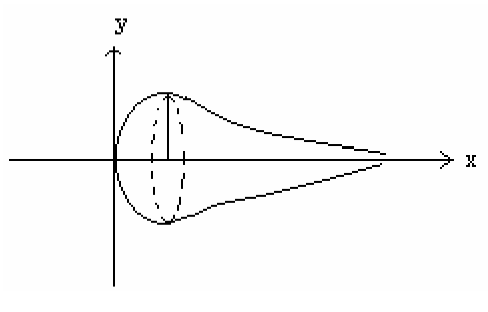
$f'(x) < 0$ for $1-2x^3 < 0 \Rightarrow \dfrac{1}{2} < x^3 \Rightarrow \dfrac{1}{\sqrt[3]{2}} < x$. So f is decreasing on $\left(\dfrac{1}{\sqrt[3]{2}}, \infty\right)$.

$f'(x) > 0 \Rightarrow \dfrac{1}{2} > x^3 \Rightarrow x < \dfrac{1}{\sqrt[3]{2}}$. So f is increasing on $(-\infty,-1)$ and $\left(-1, \dfrac{1}{\sqrt[3]{2}}\right)$.

d)



e)



$$V = \int_0^\infty \pi\cdot y^2\cdot dx = \pi\cdot\int_0^\infty \frac{x^2}{(x^3+1)^2}\cdot dx$$

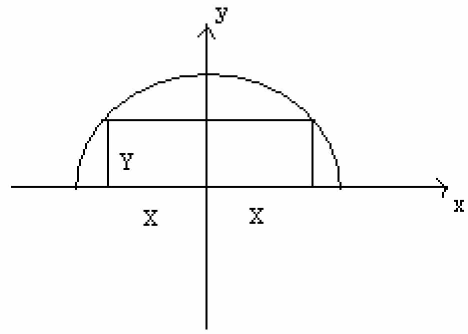
$$u = x^3+1 \Rightarrow du = 3x^2\cdot dx \Rightarrow x^2\cdot dx = \frac{1}{3}\cdot du \Rightarrow V = \frac{\pi}{3}\cdot\int_1^\infty \frac{du}{u^2} = \lim_{b\to\infty}\left[-\frac{\pi}{3}\cdot\frac{1}{u}\Big|_1^b\right]$$

$$= -\frac{\pi}{3}\cdot\lim_{b\to\infty}\left(\frac{1}{b}\right) + \frac{\pi}{3} = \frac{\pi}{3} \text{ cubic units.}$$

2)

Side lengths are $2x$ and y.

So Area; $A = 2x \cdot y$



$$x^2 + y^2 = 3^2 = 9 \Rightarrow y = \sqrt{9 - x^2}$$

$$\Rightarrow A = 2x \cdot \sqrt{9 - x^2}$$

$$\frac{dA}{dx} = 2 \cdot \left[\sqrt{9 - x^2} + x \cdot \left(\frac{-2x}{2 \cdot \sqrt{9 - x^2}} \right) \right]$$

$$= 2 \cdot \sqrt{9 - x^2} \cdot \left(1 - \frac{x^2}{9 - x^2} \right) = 0 \Rightarrow \frac{9 - x^2 - x^2}{9 - x^2} = 0 \Rightarrow 9 = 2x^2 \Rightarrow x = \frac{3}{\sqrt{2}}$$

and $y = \sqrt{9 - \left(\frac{3}{\sqrt{2}} \right)^2} = \frac{3}{\sqrt{2}}$.

Hence side lengths are $\left(3 \cdot \sqrt{2}, \frac{3}{\sqrt{2}} \right)$ and largest area is $2 \cdot \frac{3}{\sqrt{2}} \cdot \frac{3}{\sqrt{2}} = 9$ sq.units.

3-a) $df = f' \cdot dx$ and $dg = g' \cdot dx$.

$$\frac{d}{dx}(f \cdot g) = f' \cdot g + g' \cdot f \Rightarrow f \cdot g = \int \frac{d}{dx}(f \cdot g) \cdot dx = \int (f' \cdot g + g' \cdot f) \cdot dx$$

$$= \int g \cdot df + \int f \cdot dg \Rightarrow \int f \cdot dg = f \cdot g - \int g \cdot df .$$

b) If we put $u = -\sqrt{x}$, then we have $du = -\frac{dx}{2 \cdot \sqrt{x}}$ and $dx = 2u \cdot du$.

$I = \int e^{-\sqrt{x}} \cdot dx = 2 \cdot \int e^u \cdot u \cdot du$. In order to evaluate this integral, we will use integration by parts; Let $f = u$ & $g = e^u$. Then

$$I = 2 \cdot \left[u \cdot e^u - \int e^u \cdot du \right] = 2 \cdot \left(u \cdot e^u - e^u - C_1 \right) = 2 \cdot (u - 1) \cdot e^u + C = -2 \cdot \left(1 + \sqrt{x} \right) \cdot e^{-\sqrt{x}} + C$$

where C is a constant.

c) $\int_0^\infty e^{-\sqrt{x}} \cdot dx = \lim_{b \to \infty} \left[-2 \cdot \left(1 + \sqrt{x} \right) \cdot e^{-\sqrt{x}} \right]_a^b = \lim_{b \to \infty} \left[-2 \cdot \left(1 + \sqrt{b} \right) \cdot e^{-b} + 2 \right]$

Since $\lim_{b\to\infty} e^{-\sqrt{b}} = 0$ and $\lim_{b\to\infty}\left(1+\sqrt{b}\right)\cdot e^{-\sqrt{b}} = \lim_{b\to\infty}\frac{1+\sqrt{b}}{e^{\sqrt{b}}} \underset{\underset{L'H}{\uparrow}}{=} \lim_{b\to\infty}\frac{\frac{1}{2\cdot\sqrt{b}}}{e^{\sqrt{b}}\cdot\frac{1}{2\cdot\sqrt{b}}} = 0$,

$\int_0^\infty e^{-\sqrt{x}}\cdot dx = 2$.

4) We will use the method of partial fractions. Since $x^4-1=(x-1)\cdot(x+1)\cdot\left(x^2+1\right)$, we have

$\frac{2}{x^4-1}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{cx+d}{x^2+1}$. This implies

$2=a\cdot(x+1)\cdot\left(x^2+1\right)+b\cdot(x-1)\cdot\left(x^2+1\right)+(cx+d)\cdot(x-1)\cdot(x+1)$.

$x=-1 \quad\Rightarrow\quad 2=-4b\Rightarrow b=-\frac{1}{2}$,

$x=1 \quad\Rightarrow\quad 2=4a\Rightarrow a=\frac{1}{2}$,

$x=0 \quad\Rightarrow\quad 2=a-b-d\Rightarrow d=-1$ and

$x=2 \quad\Rightarrow\quad 2=15a+5b+(2c+d)\cdot 3\Rightarrow 2c+d=-1\Rightarrow c=0$. So

$$\int\frac{2}{x^4-1}\cdot dx=\int\left(\frac{\frac{1}{2}}{x-1}+\frac{\frac{-1}{2}}{x+1}+\frac{-1}{x^2+1}\right)\cdot dx=\int\left(\frac{1}{2}\cdot\frac{1}{x-1}\right)\cdot dx-\int\left(\frac{1}{2}\cdot\frac{1}{x+1}\right)\cdot dx-\int\frac{1}{x^2+1}\cdot dx$$

$$=\frac{1}{2}\cdot\ln|x-1|-\frac{1}{2}\cdot\ln|x+1|-Arc\tan x+C.$$

5-a) Let $s_n=\sum_{k=0}^{n-1}a_k \;\; for \quad n\ge 1$, be the partial sum of the first n terms of the series $\sum_{n=0}^{\infty}a_n$.

We have

$$s_n=a_0+a_1+....+a_{n-2}+a_{n-1}=\left(a_0+a_1+....+a_{n-2}\right)+a_{n-1}=s_{n-1}+a_{n-1}$$

$\therefore \; a_{n-1}=s_n-s_{n-1}$.

By definition, the series $\sum_{n=0}^{\infty}a_n$ is convergent $\Leftrightarrow$ the sequence $\{s_n\}_{n\ge 0}$ has a finite limit, say a: $\lim_{n\to\infty} s_n = a$. Then, $\lim_{n\to\infty} s_{n+1} = a$, and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_{n+1} - s_n) = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n$

$$= a - a = 0.$$

b) If $\lim_{n\to\infty} a_n = 0$, we can not say that the series $\sum_{n=0}^{\infty} a_n$ is convergent. To see that, consider the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$. Here $a_n = \frac{1}{n}$ and $\lim_{n\to\infty} a_n = 0$, but one knows that the Harmonic Series is divergent.

6-a) $\lim_{n\to\infty}\left|\dfrac{(x+2)^{n+1}/\left(3^{n+1}\cdot\sqrt{n+1}\right)}{(x+2)^n/\left(3^n\cdot\sqrt{n}\right)}\right| = \lim_{n\to\infty}|x+2|\cdot\dfrac{\sqrt{n}}{3\cdot\sqrt{n+1}} = \dfrac{|x+2|}{3}$.

Therefore, the series converges if $\frac{|x+2|}{3} < 1 \quad (|x+2| < 3)$ and diverges if $\frac{|x+2|}{3} > 1 \quad (|x+2| > 3)$. Hence the radius of convergence is: $R = 3$.

b) Endpoints of the interval of convergence are -2 ∓ 3 :-5 and 1.

When $x = -5$: $\quad \sum_{n=1}^{\infty} \dfrac{(-5+2)^n}{3^n\cdot\sqrt{n}} = \sum_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$. This new series is convergent by the Alternating series Test, because $\frac{1}{\sqrt{n}} > 0, \left\{\frac{1}{\sqrt{n}}\right\}$ is decreasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$.

When $x = 1$: $\quad \sum_{n=1}^{\infty} \dfrac{(1+2)^n}{3^n\cdot\sqrt{n}} = \sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$. This series is a divergent p-series$\left(p = \frac{1}{2} < 1\right)$.

Therefore the interval of convergence is $[-5,1)$.

7) Taylor series for a function $f(x)$ about $x = 1$ is $f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot f^{(n)}(1)\cdot(x-1)^n$. So

$$\ln x = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot \ln^{(n)}(1)\cdot(x-1)^n.$$

$$\ln(1) = 0, \qquad \ln'(x) = \frac{1}{x} \Rightarrow \ln'(1) = 1$$

$$\ln''(x) = -\frac{1}{x^2} \Rightarrow \ln''(1) = -1$$

$$\ln'''(x) = \frac{2}{x^3} \Rightarrow \ln'''(1) = 2$$

$$\ln^{(4)}(x) = -\frac{3!}{x^4} \Rightarrow \ln^{(4)}(1) = -3!$$

$\ln^{(5)}(x) = \frac{4!}{x^5} \Rightarrow \ln^{(5)}(1) = 4!$, (inductively) $\ln^{(n)}(x) = (-1)^{n+1} \cdot (n-1)!$, $\quad \forall n \notin \text{Ù}^{+}$ and

$$\ln^{(n)}(1) = (-1)^{n+1} \cdot (n-1)! \Rightarrow \frac{\ln^{(n)}(1)}{n!} = \frac{(-1)^{n+1}}{n}, \quad \forall n \in \text{Ù}^{+}.$$

Hence $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \pm \ldots\ldots$

<u>Alternative solution:</u>

Let $y = x - 1$.

$$y = x - 1 \Rightarrow \ln(x) = \ln(1+y) = \int_0^y \frac{du}{1+u} = \int_0^y \sum_{n=0}^{\infty} (-1)^n \cdot u^n \cdot du = \sum_{n=0}^{\infty} (-1)^n \cdot \left.\frac{u^{n+1}}{n+1}\right|_0^y$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot y^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x-1)^{n+1}}{n+1} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \cdot (x-1)^m}{m}.$$