Question 1. Evaluate the limit in each part. Show the details of your work.

(a) $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = L$ (Type $0/0$ Indeter.)

$$L \underset{\text{L'Hospital}}{=} \lim_{x\to 0} \frac{(\cos x - 1)'}{(x^2)'} = \lim_{x\to 0} \frac{-\sin x}{2x} \underset{\text{L'Hospital}}{=} \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

(b) $\lim_{x\to 1} \frac{(x-1)^2}{\arcsin x} = L$ (Determinate)

$$L \underset{\text{Direct substitution}}{=} \frac{(1-1)^2}{\arcsin 1} = \frac{0}{\pi/2} = 0$$

(c) $\lim_{x\to 0^+} \left(\frac{1}{\ln(x+1)} - \frac{1}{\sin x}\right) = L$ (Indeter Type $\infty - \infty$)

$$L = \lim_{x\to 0^+} \frac{\sin x - \ln(x+1)}{(\sin x)\ln(x+1)} \quad \left(\text{Indeter } 0/0\right)$$

$$(\text{L'Hospital}) = \lim_{x\to 0^+} \frac{\cos x - 1/(x+1)}{(\cos x)\ln(x+1) + (\sin x)/(x+1)}$$

$$= \lim_{x\to 0^+} \frac{(x+1)\cos x - 1}{(\cos x)\ln(x+1)(x+1) + \sin x}$$

$$(\text{L'Hospital}) = \lim_{x\to 0^+} \frac{\cos x - \sin x\,(x+1)}{\ln(x+1)(x+1) + \cos x + (\cos x)\ln(x+1) + \cos x}$$

$$\left(\text{Direct Substitution}\right) = \frac{1}{2}$$

(d) $\lim_{\theta \to 0} (\cos\theta)^{1/\theta^2} = L$ $\left(\begin{array}{l}\text{Indeterminate} \\ \text{Type } 1^\infty\end{array}\right)$

Let $y = \lim_{\theta \to 0} \ln \cos \theta^{1/\theta^2}$ $\left(= \ln \lim_{\theta \to 0} \cos \theta^{1/\theta^2} \right)$

$= \lim_{\theta \to 0} \frac{\ln \cos \theta}{\theta^2}$ $\left(\begin{array}{l}\text{Indeterminate} \\ \text{Type} \\ 0/0\end{array}\right)$

(L'Hospital) $= \lim_{\theta \to 0} \frac{1/\cos\theta \ (-\sin\theta)}{2\theta}$ $\left(\begin{array}{l}\text{Indeterminate} \\ \text{Type } 0/0\end{array}\right.$

(L'Hospital) $= \lim_{\theta \to 0} \frac{-\cos\theta}{(-\sin\theta)2\theta + 2(\cos\theta)}$

$= -1/2$

Then

$L = e^{\ln \lim_{\theta \to 0} \cos \theta^{1/\theta^2}}$

$= e^y = e^{-1/2} = \frac{1}{\sqrt{e}}$ //

Question 2.

(a) Find $\frac{dy}{dx}$ if $y = \cos^2(\ln x)$. Do not simplify your answer.

Chain Rule

$$\frac{dy}{dx} = 2\cos(\ln x)\,\frac{d(\cos(\ln x))}{dx}$$

$$= 2\cos(\ln x)\,(-\sin(\ln x))\,\frac{1}{x}$$

(b) Find $\frac{dy}{dx}$ if $y + \sec(xy) = 2x^3 + y^4$.

Use implicit differentiation

$$\frac{d(y + \sec(xy))}{dx} = \frac{d(2x^3 + y^4)}{dx}$$

$$\frac{dy}{dx} + \frac{\sin(xy)}{\cos^2(xy)}\,\underbrace{\frac{d(xy)}{dx}}_{y + x(dy/dx)} = 6x^2 + 4y^3\,\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(1 + x\frac{\sin(xy)}{\cos^2(xy)} - 4y^3\right) = 6x^2 - y\,\frac{\sin(xy)}{\cos^2(xy)} \implies \frac{dy}{dx} = \frac{6x^2 - y\left(\frac{\sin(xy)}{\cos^2(xy)}\right)}{1 + x\left(\frac{\sin(xy)}{\cos^2(xy)}\right) - 4y^3}$$

(c) Find $\frac{dy}{dx}$ at $(e, 0)$ if $x^y = \ln(x + y)$.

Apply logarithmic differ.

$$\underbrace{\ln x^y}_{y\ln x} = \ln(\ln(x+y))$$

$$\left.\frac{d(y\ln x)}{dx}\right|_{\substack{x=e\\y=0}} = \left.\frac{d(\ln(\ln(x+y)))}{dx}\right|_{\substack{x=e\\y=0}}$$

$$\left.\left(\frac{dy}{dx}\ln x + \frac{1}{x}\frac{dy}{dx}\right)\right|_{\substack{x=e\\y=0}} = \left.\frac{1}{\ln(x+y)}\,\frac{1}{(x+y)}\left(1 + \frac{dy}{dx}\right)\right|_{\substack{x=e\\y=0}}$$

$$\left.\left(\frac{dy}{dx} + \frac{1}{e}\frac{dy}{dx}\right)\right|_{\substack{x=e\\y=0}} = \left.\frac{1}{e}\left(1 + \frac{dy}{dx}\right)\right|_{\substack{x=e\\y=0}}$$

$$\implies \left.\frac{dy}{dx}\right|_{\substack{x=e\\y=0}} = \frac{1}{e}$$

Question 3. Consider the function $f(x) = x^3 + 2x^2 + x$.

(a) Find the interval(s) on which f is increasing and the interval(s) on which f is decreasing.

$$f'(x) = 3x^2 + 4x + 1 = (3x+1)(x+1)$$

	-1		$-1/3$	
$f'(x)$	$+$	$-$		$+$
$f(x)$	increasing	decreasing		increasing

$\boxed{f \text{ is increasing on } (-\tfrac{1}{3}, \infty) \text{ and } (-\infty, -1)}$

$\boxed{f \text{ is decreasing on } (-1, -1/3)}$

(b) Find the interval(s) on which f is concave up and the interval(s) on which f is concave down.

$$f''(x) = 6x + 4$$

	$-2/3$	
$f''(x)$	$-$	$+$
$f(x)$	concave-down	concave-up

$\boxed{f \text{ is concave-up on } (-2/3, \infty)}$

$\boxed{f \text{ is concave-down on } (-\infty, -2/3)}$

(c) Find the critical points of f. Classify each of these critical points as a local minimum, local maximum or neither.

From part (a)

$x = -1$ and $x = -1/3$ are the critical points.

$f''(-1) = -2 < 0$ $\quad$ $f''(-1/3) = 2 > 0$

$\boxed{f \text{ has a local max at } x = -1}$ $\quad$ $\boxed{f \text{ has a local min at } x = -1/3}$

(d) Find the points over the interval $[-2, -\frac{1}{2}]$ at which f has a global minimum and a global maximum.

The only critical point on the interval is $x = -1$

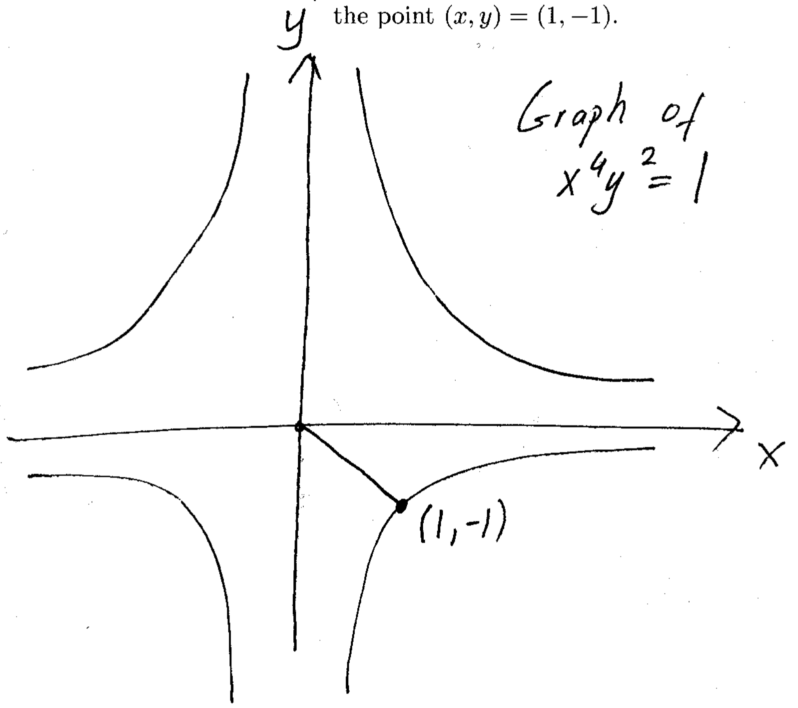
Critical Point $x = -1$ $\quad f(-1) = 0$

End points: $x = -1/2$ $\quad f(-1/2) = -1/8 + 1/2 - 1/2 = -1/8$

$x = -2$ $\quad f(-2) = -8 + 8 - 2 = -2$

$\boxed{f \text{ has a global min at } x = -2}$ $\quad$ $\boxed{f \text{ has a global max at } x = -1}$

Question 4. An object moves along the curve $x^4y^2 = 1$. If the rate of change of the x-coordinate of the object is constant and equal to -1 *units/s*, find the rate of change of the distance from the object to the origin when the object passes through the point $(x, y) = (1, -1)$.



Graph of $x^4y^2 = 1$

① Variables and equations

(x, y) (satisfying $x^4y^2 = 1$)

and

D: distance from the origin

$$D^2 = x^2 + y^2 \underset{\left(y^2 = \frac{1}{x^4}\right)}{=} x^2 + \frac{1}{x^4}$$

② Given

$$\frac{dx}{dt} = -1$$

Sought

$$\left.\frac{dD}{dt}\right|_{\substack{x=1\\y=-1}} = ?$$

③ Apply implicit differ.

$$\frac{d(D^2)}{dt} = \frac{d(x^2 + 1/x^4)}{dt}$$

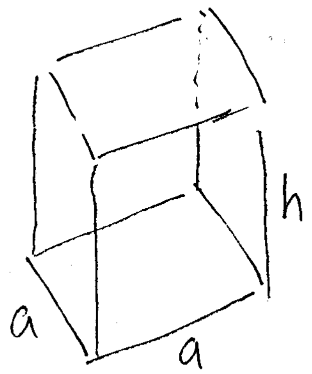
$$2D\frac{d(D)}{dt} = 2x\frac{dx}{dt} - \frac{4}{x^5}\frac{dx}{dt}$$

$$\left.\frac{dD}{dt}\right|_{\substack{x=1\\y=-1}} = \frac{1}{D}\left(x\frac{dx}{dt} - \frac{2}{x^5}\frac{dx}{dt}\right)\Bigg|_{\substack{x=1\\y=-1}}$$

(When $x = 1, y = -1$, $D = \sqrt{1^2 + (-1)^2} = \sqrt{2}$)

$$= \frac{1}{\sqrt{2}}\left(-1 - 2(-1)\right) = \frac{1}{\sqrt{2}}$$

Question 5. Consider a box with square base. In order to be sent through P.T.T., the height of the box and the perimeter of the base can add up to at most 120 cm. What is the maximum volume for such a box?



① Variables and Equations

a : base length

h : height

$$4a + h = 120 \quad (h = 120 - 4a)$$

② Function f to maximize

maximize $a^2 h$

$4a+h=120$
$\Rightarrow 4a \leq 120$
$\Rightarrow a \leq 30$

③ Eliminate h

maximize $\underbrace{a^2(120-4a)}_{f(a)}$

(Find $a \in [0,30]$ such that $f(a)$ is as large as possible)

④ Find global max of f over $[0,30]$

$$f'(a) = 240a - 12a^2$$

$f'(a) = 0 \iff a = 0$ OR $a = 20$ CRITICAL POINTS

$a = 0$	$f(0) = 0$
$a = 20$	$f(20) = 16000$
$a = 30$	$f(30) = 0$

f has a global max at $a = 20$
(maximum volume 16000)

dimensions maximizing volume
$a = 20 \quad h = 120 - 4a = 40$

Question 6. Let $f(x)$ be a twice differentiable function with $f(-1) = 1$, $f(0) = 4$ and $f(1) = 2$.

(a) Show that there exist two points $c_1 \in (-1, 0)$ and $c_2 \in (0, 1)$ such that $f'(c_1) = 3$ and $f'(c_2) = -2$.

By Mean Value Thm there exists $c_1 \in (-1,0)$ s.t.

$$f'(c_1) = \frac{f(0)-f(-1)}{0-(-1)} = 3$$

By Mean Value Thm there exists $c_2 \in (0,1)$ s.t.

$$f'(c_2) = \frac{f(1)-f(0)}{1-0} = -2$$

(b) Show that f has a critical point on $(-1, 1)$.

Since f is twice differ, f' is continuous

(From part (a)) $f'(c_1) = 3 > 0$ and $f'(c_2) = -2 < 0$

By the intermediate value thm there exists a $\tilde{c} \in (c_1, c_2)$ $(\Rightarrow c \in (-1,1))$, s

$$f'(\tilde{c}) = 0$$

(c) Show that $f''(c) < 0$ for some $c \in (-1, 1)$.

Apply the mean value thm to f' on the interval $[c_1, c_2]$

There exists a $c \in (c_1, c_2)$ such that

$$f''(c) = \frac{f'(c_2)-f'(c_1)}{c_2-c_1}$$

$$= \frac{-5}{\underbrace{c_2-c_1}_{>0}} < 0$$