Math 106 **Exam 2 Solutions** **15 December, 2005**

1. Consider $f(x) = \dfrac{x^2}{x^2-1}$.

The domain of f is $\mathbb{R} \setminus \{-1, 1\}$.

Since $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to 1^+} f(x) = +\infty$, $\lim_{x\to -1^-} f(x) = +\infty$, and $\lim_{x\to -1^+} f(x) = -\infty$, $x = 1$ and $x = -1$ are vertical asymptotes.

Since $\lim_{x\to +\infty} f(x) = 1$ and $\lim_{x\to -\infty} f(x) = 1$, $y = 1$ is the horizontal asymptote.

$$f'(x) = \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}.$$

So $f'(x) = 0$ at $x = 0$ and undefined at $x = 1$ and $x = -1$.

$$f''(x) = \frac{-2(x^2-1)^2 + 2x2(x^2-1)2x}{(x^2-1)^4} = \frac{2(x^2-1)(-x^2+1+4x^2)}{(x^2-1)^4} = \frac{2(3x^2+1)}{(x^2-1)^3}.$$

So $f''(x)$ is undefined at $x = 1$ and $x = -1$. Then

x		-1		0		1	
f'	$+++$	$\|\|$	$+++$	0	$---$	$\|\|$	$---$
f''	$+++$	$\|\|$	$---$		$---$	$\|\|$	$+++$
f	$\nearrow$	$\|\|$	$\nearrow$	0	$\searrow$	$\|\|$	$\searrow$

So we get a local maximum value $f(0) = 0$ at $x = 0$. There are no local minimum and inflection points. Combining all this information we get the following graph.

2. Let $f(x)$ be defined piecewise as follows:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ -x+2 & \text{if } 1 \le x \le 2 \end{cases}$$

a) We will find $\int_0^2 f(x)dx$ using upper Riemann sums with subintervals of equal length. But as the region can be divided into two subregions as in the above figure, we will find the upper Riemann sums for both of these subregions. Then the first upper Riemann sum, say S_1 is:

$$S_1 = \frac{1}{n}f(\frac{1}{n}) + \frac{1}{n}f(\frac{2}{n}) + ... + \frac{1}{n}f(\frac{n}{n}) = \frac{1}{n}\left[\frac{1}{n^2} + \frac{2^2}{n^2} + ... + \frac{n^2}{n^2}\right].$$

So $S_1 = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{2n^2+3n+1}{6n^2}$.

Next the second upper sum is :

$$S_2 = \frac{1}{n}f(1) + \frac{1}{n}f(1+\frac{1}{n}) + \frac{1}{n}f(1+\frac{2}{n}) + ... + \frac{1}{n}f(1+\frac{n-1}{n})$$

$$= \frac{1}{n}\left[(-1+2) + \left(-(1+\frac{1}{n})+2\right) + \left(-(1+\frac{2}{n})+2\right) + ... + \left(-(1+\frac{n-1}{n})+2\right)\right]$$

$$= \frac{1}{n}\left[1 + \left(1-\frac{1}{n}\right) + \left(1-\frac{2}{n}\right) + ... + \left(1-\frac{n-1}{n}\right)\right].$$

So $S_2 = \frac{1}{n}\left[n - \frac{1}{n}\sum_{k=1}^{n-1} k\right] = \frac{1}{n}\left[n - \frac{1}{n}\frac{(n-1)n}{2}\right] = 1 - \frac{n-1}{2n} = \frac{n+1}{2n}$.

Then $\int_0^2 f(x)dx = \lim_{n\to+\infty}(S_1+S_2) = \lim_{n\to+\infty}\left(\frac{2n^2+3n+1}{6n^2} + \frac{n+1}{2n}\right) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

b) $\int_0^2 f(x)dx = \int_0^1 x^2dx + \int_1^2 (-x+2)dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{-x^2}{2} + 2x\right]_1^2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

3. Let x and y be two nonnegative numbers whose sum is 12. We can write $y = 12 - x$ where $0 \leq x \leq 12$.

a) Let $f(x)$ be the difference of their squares that is $f(x) = x^2 - (12-x)^2 = 24x - 144$ is the function we will maximize on the interval $[0, 12]$. Since $f'(x) = 24 > 0$, f is increasing on $[0, 12]$. Therefore the maximum value of f which is 144 is obtained when $x = 12$ and $y = 0$.

b) Now the function that we will maximize is $f(x) = \sqrt{x} + \sqrt{y} = \sqrt{x} + \sqrt{12-x}$ on $[0, 12]$.

Then $f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}(12-x)^{-1/2}(-1) = \frac{1}{2}(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{12-x}})$.

Then $f'(x) = 0$ at $x = 6$. Since $f(0) = f(12) = 2\sqrt{3}$ and $f(6) = 2\sqrt{6}$ the maximum value for f which is $2\sqrt{6}$ is obtained when $x = y = 6$.

c) In this part the function we will maximize is $f(x) = \sqrt{x}(12-x) = 12x^{1/2} - x^{3/2}$ on $[0, 12]$. Then $f'(x) = 6x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{6}{\sqrt{x}} - \frac{3}{2}\sqrt{x}$.

Then $f'(x) = 0$ at $x = 4$. Since $f(0) = f(12) = 0$ and $f(4) = 16$ the maximum value of $f$which is 16 is obtained when $x = 4$ and $y = 8$.

4. a) $\int_0^1 \sqrt{x}(x+1)^2 dx = \int_0^1 x^{1/2}(x^2 + 2x + 1)dx = \int_0^1 (x^{5/2} + 2x^{3/2} + x^{1/2})dx$
$= \left[\frac{2}{7}x^{7/2} + \frac{4}{5}x^{5/2} + \frac{2}{3}x^{3/2}\right]_0^1 = 184/105.$

b) Make the substitution $u = \frac{1}{t} - 1$ and $du = -\frac{1}{t^2}dt$ in $\int \frac{1}{t^2} \sin\left(\frac{1}{t} - 1\right) dt$. Then

$\int -\sin u \, du = \cos u + c = \cos\left(\frac{1}{t} - 1\right) + c$, where c is any real number.

c) Since we have $0/0$ form in the following limit we can use l'Hopital's rule. So we get $\lim_{x \to 0^+} \frac{x^2 - x}{\sin(x^2)} = \lim_{x \to 0^+} \frac{2x - 1}{2x\cos(x^2)} = -\infty$, since as $x \to 0^+$, $2x - 1$ goes to -1 and $2x\cos(x^2)$ goes to 0 from above.

d) Since $\frac{dr}{dt} = 15\sqrt{t} + \frac{3}{\sqrt{t}}$, we can get

$r(t) = \int (15t^{1/2} + 3t^{-1/2})dt = 10t^{3/2} + 6t^{1/2} + c$. Then

$8 = r(1) = 10 + 6 + c$ implies that $c = -8$. Therefore $r(t) = 10t\sqrt{t} + 6\sqrt{t} - 8$.

5. a) The average value of $f(x)$ on $[0, 2]$ is:

$$\frac{1}{2}\int_0^2 \frac{100}{(2x+1)^3}dx = 25\int_0^2 2(2x+1)^{-3}dx = 25\left[\frac{(2x+1)^{-2}}{-2}\right]_0^2 = \frac{-25}{2}\left(\frac{1}{25}-1\right) = 12.$$

b) $$\frac{d}{dx}\left(3x^2+1+\int_1^{3x^2+1}\frac{\sin u}{u}\,du\right) = 6x + \frac{d}{dx}\int_1^{3x^2+1}\frac{\sin u}{u}\,du = 6x + \frac{\sin(3x^2+1)}{3x^2+1}(6x).$$

c) Let A be the area of the region enclosed by $y = x/2$, $y = \sqrt{8-x}$ and the x-axis. If we calculate with respect to y we obtain:

$$A = \int_0^2 (8-y^2-2y)dy = \left[8y - \frac{y^3}{3} - y^2\right]_0^2 = 16 - \frac{8}{3} - 4 = \frac{28}{3}.$$

On the other hand if we calculate with respect to x we obtain:

$$A = \int_0^4 \frac{x}{2}dx + \int_4^8 (8-x)^{1/2}dx = \left[\frac{x^2}{4}\right]_0^4 - \left[\frac{(8-x)^{3/2}}{3/2}\right]_4^8 = \frac{28}{3}.$$

Extra credit question.

Let $f(x)$ be a function such that $f''(x)$ exists for all x. Suppose that the equation $f(x) = x$ has at least three solutions. Then consider $g(x) = f(x) - x$ with three roots, say $x_1 < x_2 < x_3$. Since $g(x)$ is continuous and differentiable, by the Mean Value Theorem there is a point c_1 in (x_1, x_2), where $g'(c_1) = \dfrac{g(x_2)-g(x_1)}{x_2-x_1} = 0$. Similarly by the Mean Value Theorem there is a point c_2 in (x_2, x_3), where $g'(c_2) = \dfrac{g(x_3)-g(x_2)}{x_3-x_2} = 0$. But since $g'(x)$ is also continuous and differentiable, if we apply the Mean Value Theorem again we get there exists a point c in (c_1, c_2) with $f''(c) = g''(c) = \dfrac{g'(c_2)-g'(c_1)}{c_2-c_1} = 0$.