Question 1. Determine dy/dx where

(a) (5 points) $y = e^{-x}\tan e^{x}$

$$\frac{dy}{dx} = -e^{-x}\tan e^{x} + e^{-x}\cdot \sec^2 e^{x} \cdot e^{x}$$

$$= -e^{-x}\tan e^{x} + \sec^2 e^{x}$$

(b) (5 points) $\sqrt{xy} = x^2 + 2y^2$

$$\frac{1}{2}(xy)^{-1/2}\cdot\left(y + x\frac{dy}{dx}\right) = 2x + 4y\frac{dy}{dx}$$

$$\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} = 2x + 4y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{x}{2\sqrt{xy}} - 4y\right) = 2x - \frac{y}{2\sqrt{xy}}$$

$$\frac{dy}{dx} = \frac{2x - \frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}} - 4y} = \frac{4x\sqrt{xy} - y}{x - 8y\sqrt{xy}}.$$

(c) (5 points) $x^y = y^x$

$$y\ln x = x\ln y$$

$$\frac{dy}{dx}\ln x + y\frac{1}{x} = \ln y + x\cdot\frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}.$$

Question 2. Let $f(x)$ be a twice differentiable function.

(a) (10 points) Suppose that there are three real numbers x_1, x_2, x_3 such that $x_1 < x_2 < x_3$ and $f(x_1) = f(x_2) = f(x_3)$. Show that there exists an $\tilde{x} \in (x_1, x_3)$ such that $f''(\tilde{x}) = 0$.

Since f is twice differentiable, it is continuous and differentiable. Therefore by the mean value thm (MVT) there exist $a \in (x_1, x_2)$ and $b \in (x_2, x_3)$ s.t.

$$f'(a) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad \text{and} \quad f'(b) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = 0$$

f' is also continuous and differentiable. By MVT there exist $\tilde{x} \in (a, b) \subseteq (x_1, x_2)$ s.t.

$$f''(\tilde{x}) = \frac{f'(b) - f'(a)}{b - a} = 0 \qquad \square$$

(b) (5 points) Does the existence of an $\tilde{x}$ such that $f''(\tilde{x}) = 0$ imply that $\tilde{x}$ is an inflection point? If $\tilde{x}$ is an inflection point explain why, if not give a counter example.

No, it does not imply, that is

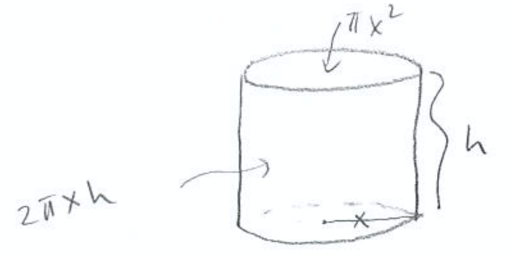
$$f''(\tilde{x}) = 0 \not\Longrightarrow \tilde{x} \text{ is an inflection point}$$

This is because even if $f''(\tilde{x}) = 0$, f'' may possibly not change sign at $\tilde{x}$.

COUNTER EXAMPLE: $f(x) = x^4$ at 0 $\quad f''(x) = 12x^2$ BUT $f''(x) \geq 0$, $f''(0) = 0$

(0,f(0)) is not an inflection point

Question 4. (15 points) You would like to design a cylindrical tank. But your total budget is limited to 1000 TL. Each m^2 of the side of the tank costs 100 TL, and each m^2 of the top and bottom has a cost of 50 TL. Find the base radius of the cylindrical tank so that its volume is large as possible.



$$50(2\pi x^2) + 100(2\pi x h) = 1000$$

$$100\pi x^2 + 200\pi x h = 1000$$

$$\pi x^2 + 2\pi x h = 10$$

$$h = \frac{10 - \pi x^2}{2\pi x}$$

$$V(x) = \pi x^2 \frac{(10 - \pi x^2)}{2\pi x} = \frac{10x - \pi x^3}{2} = 5x - \frac{\pi}{2} x^3$$

$$V'(x) = 5 - \frac{3\pi}{2} x^2 = 0 \Rightarrow \frac{3\pi}{2} x^2 = 5 \Rightarrow x^2 = \frac{10}{3\pi}$$

$$\Rightarrow x = \sqrt{\frac{10}{3\pi}}$$

$$V''(x) = -3\pi x$$

$$V''\left(\sqrt{\frac{10}{3\pi}}\right) = -3\pi \sqrt{\frac{10}{3\pi}} < 0$$

So $x = \sqrt{\frac{10}{3\pi}}$ maximizes the volume.

Question 6. Evaluate the limit in each part. Show your work to get full credit.

(a) (7 points) $\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\sin x}$

Let $\left(\frac{1}{x}\right)^{\sin x} = L \Rightarrow -\sin x \ln x = \ln L.$

Then $\lim_{x\to 0^+} L = \lim_{x\to 0^+} e^{\ln L} \underset{\text{since } e^x \text{ is continuous.}}{=} e^{\lim_{x\to 0^+} \ln L}$

$\lim_{x\to 0^+} \ln L = \lim_{x\to 0^+} -\sin x \ln x$ (it is of the form $0.\infty$).

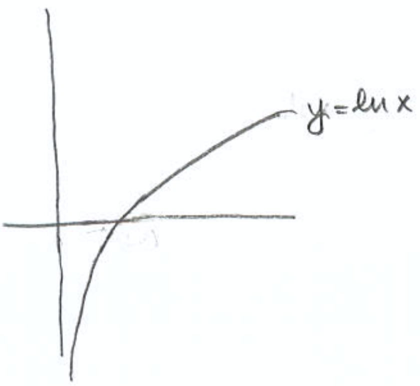
$$= \lim_{x\to 0^+} \frac{-\ln x}{\frac{1}{\sin x}} \underset{\substack{L.H.\\ \left(\frac{\infty}{\infty}\right)}}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\frac{1}{\sin^2 x}\cos x} = \lim_{x\to 0^+} \frac{\sin^2 x}{x \cdot \cos x} \underset{\substack{LH.\\ \left(\frac{0}{0}\right)}}{=} \lim_{x\to 0^+} \frac{2\sin x \cos x}{\cos x - x\sin x} \underset{\substack{\text{Direct}\\ \text{subs.}}}{=} \frac{0}{1} = 0.$$

So $\lim_{x\to 0^+} L = e^0 = 1.$ //

(b) (7 points) $\lim_{x\to 1^-} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^-} \left(\frac{x-1-\ln x}{\ln x\,(x-1)}\right)$ (it is of the form $\frac{0}{0}$).

$$\underset{\substack{L.H.\\ \left(\frac{0}{0}\right)}}{=} \lim_{x\to 1^-} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} = \lim_{x\to 1^-} \frac{x-1}{(x-1)+x\ln x} \underset{\substack{L.H\\ \left(\frac{0}{0}\right)}}{=} \lim_{x\to 1^-} \frac{1}{1+\ln x+1} \underset{\substack{\text{Direct}\\ \text{subs.}}}{=} \frac{1}{2}$$ //

(c) (6 points) $\lim_{x\to 1^-} \left(\frac{1}{\ln x} - \frac{1}{x+1}\right) = \lim_{x\to 1^-} \left(\frac{x+1-\ln x}{\ln x \cdot (x+1)}\right) \underset{\substack{\text{Direct}\\ \text{subs.}}}{=} \frac{2}{0^-} = -\infty$



as $x \to 1^-$ $\ln x \to 0^-$ so $\frac{1}{\ln x} \to -\infty$

Question 3. Let $f(x) = \dfrac{4x-4}{x^2}$.

(a) (5 points) Find the intervals on which f is increasing or decreasing.

$$f'(x) = \frac{4 \cdot x^2 - 2x(4x-4)}{x^4}$$

$$= \frac{8x - 4x^2}{x^4}$$

	0		2	
$-$		$+$		$-$
$\searrow$		$\nearrow$		$\searrow$

$f'(x) = 0$ at $x = 2$ } critical points
$f'(x)$ DNE at $x = 0$

$f(x)$ is decreasing on $(-\infty, 0)$ and $(2, \infty)$

$f(x)$ is increasing on $(0, 2)$

(b) (5 points) Find the x values where f has a local minimum and a local maximum, if they exist.

critical points are 0 & 2. $f(x)$ changes from -ve to +ve at 0. But since $f(0)$ is undefined there is no local minimum.

at 2 $f'(x)$ changes sign from +ve to -ve so $f(x)$ has a local max at $x = 2$.

(c) (5 points) Find the intervals on which f is concave up and concave down. Find also the inflection point(s), if they exist.

$$f''(x) = \frac{(8 - 8x)x^4 - 4x^3(8x - 4x^2)}{x^8}$$

$$= \frac{8x^5 - 24x^4}{x^8}$$

	0		3	
$-$		$-$		$+$
$\cap$		$\cap$		$\cup$

$f''(x) = 0$ at $x = 3 \longrightarrow (3, \frac{8}{9})$ is the inflection point

$f''(x)$ DNE at $x = 0$

concave down on $(-\infty, 0)$ $(0, 3)$

and concave up on $(3, +\infty)$

(d) (5 points) Find the absolute maximum and absolute minimum values of f on the interval $[1/2, 4]$.

$f(\frac{1}{2}) = \dfrac{2-4}{1/4} = -8 \longrightarrow$ absolute minimum

$f(2) = 1 \longrightarrow$ absolute maximum

$f(4) = 12/16 = 3/4$

Question 5. (15 points) Find the area under the curve $y = x^2 - x^3$ from $x = 0$ to $x = 1$ using the limit definition of the area. (Note: You cannot use the Fundamental Theorem of Calculus – Part II.)

Area under the curve is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

since for $0 \le x \le 1$, $x^2 \geq x^3$.

$$\Delta x = \frac{b-a}{n} = \frac{1}{n}$$

$$x_i : 0+\frac{1}{n},\ \frac{1}{n}+\frac{1}{n},\ \dots,\ \frac{i}{n},\ \dots,\ 1$$

$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right)^3\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 - \sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\right]$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 - \frac{1}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n^2}\,\frac{2n^3+3n^2+n}{6} - \frac{1}{n^3}\,\frac{n^4+2n^3+n^2}{4}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{6}\cdot\frac{2n^3+3n^2+n}{n^3} - \frac{1}{4}\cdot\frac{n^4+2n^3+n^2}{n^4}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\cdot\left(2+\overset{0}{\cancel{\frac{3}{n}}}+\overset{0}{\cancel{\frac{1}{n^2}}}\right) - \frac{1}{4}\cdot\left(1+\overset{0}{\cancel{\frac{2}{n}}}+\overset{0}{\cancel{\frac{1}{n^2}}}\right)$$

$$= \frac{1}{6} - \frac{1}{4} = \frac{1}{12}$$