SOLUTIONS

KOÇ UNIVERSITY

MATH 106 - CALCULUS

Midterm I March 18, 2005

Duration of Exam: 90 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and talking allowed. You must always **explain your answers and show your work** to receive **full credit. BONUS POINTS will be awarded for HIGH QUALITY WORK.** Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below. GOOD LUCK!**

Surname, Name: ALBU, TOMA

Signature: ____________

Attendance Sheet Number: N/A

Section (Check One):

Section 1: Prof. Toma Albu ✓
Section 2: Prof. Toma Albu ✓

PROBLEM	POINTS	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

Problem 1. Calculate the following limit or show that it does not exist:

(a) (5 pts) $\lim_{x\to 4} \frac{|8-2x|}{3x-12}(3-x)$

$\lim_{x\to 4^+} \frac{|8-2x|}{3x-12}\cdot(3-x) = \lim_{x\to 4^+} \frac{2x-8}{3x-12}\cdot(3-x) =$ 1 pt

$= \lim_{x\to 4^+} \frac{2\cancel{(x-4)}}{3\cancel{(x-4)}}\cdot(3-x) = \frac{2}{3}(3-4) = \boxed{-\frac{2}{3}}$ 1 pt

$\lim_{x\to 4^-} \frac{|8-2x|}{3x-12}\cdot(3-x) = \lim_{x\to 4^-} \frac{8-2x}{3x-12}\cdot(3-x) =$ 1 pt

$= \lim_{x\to 4^-} \frac{-2\cancel{(x-4)}}{3\cancel{(x-4)}}\cdot(3-x) = -\frac{2}{3}(3-4) = \boxed{\frac{2}{3}}$ 1 pt

Since the one sided limits are different, $\lim_{x\to 4} \frac{8-2x}{3x-12}(3-x)$ $\boxed{\text{DNE}}$

(b) (5 pts) $\lim_{x\to -\infty} \frac{-4x^3+3x^2-5}{5x^3-2x+3} = \lim_{x\to -\infty} \frac{\cancel{x^3}\left(-4+\frac{3}{x}-\frac{5}{x^3}\right)}{\cancel{x^3}\left(5-\frac{2}{x^2}+\frac{3}{x^3}\right)}$ 2pt

$= \lim_{x\to -\infty} \frac{-4+\frac{3}{x}-\frac{5}{x^3}}{5-\frac{2}{x^2}+\frac{3}{x^3}} = \frac{\lim_{x\to -\infty}\left(-4+\frac{3}{x}-\frac{5}{x^3}\right)}{\lim_{x\to -\infty}\left(5-\frac{2}{x^2}+\frac{3}{x^3}\right)} =$ 2pt

$= \frac{-4+3\lim_{x\to -\infty}\frac{1}{x}-5\lim_{x\to -\infty}\frac{1}{x^3}}{5-2\lim_{x\to -\infty}\frac{1}{x^2}+3\lim_{x\to -\infty}\frac{1}{x^3}} = \frac{-4+3\cdot 0-5\cdot 0}{5-2\cdot 0+3\cdot 0} = \boxed{-\frac{4}{5}}$ 1p

(c) (5 pts) $\lim_{x\to 0} \frac{\sin(13x)}{\tan(5x)} = \lim_{x\to 0}\left[\frac{\sin(13x)}{\sin(5x)}\cdot\cos(5x)\right] =$ 1p

$= \lim_{x\to 0}\left[\frac{\sin(13x)}{13x}\cdot\frac{5x}{\sin(5x)}\cdot\frac{13}{5}\cdot\cos(5x)\right] =$ 2p

$= \frac{13}{15}\lim_{x\to 0}\frac{\sin(13x)}{13x}\cdot\lim_{x\to 0}\frac{5x}{\sin(5x)}\cdot\lim_{x\to 0}\cos(5x) =$ 1p

$= \frac{13}{5}\cdot 1\cdot 1\cdot\cos(5\cdot 0) = \boxed{\frac{13}{5}}$ 1p

(d) (5 pts) $\lim_{x\to 5} \frac{\sqrt{1+3x}-4}{x^2-25} = \lim_{x\to 5}\frac{(\sqrt{1+3x}-4)(\sqrt{1+3x}+4)}{(x^2-25)(\sqrt{1+3x}+4)} =$ 2p

$= \lim_{x\to 5}\frac{1+3x-16}{(x-5)(x+5)(\sqrt{1+3x}+4)} =$ 1pt

$= \lim_{x\to 5}\frac{3\cancel{(x-5)}}{\cancel{(x-5)}(x+5)\cdot(\sqrt{1+3x}+4)} = \lim_{x\to 5}\frac{3}{(x+5)(\sqrt{1+3x}+4)} =$ 1p

$= \frac{3}{(5+5)\cdot(\sqrt{1+3\cdot 5}+4)} = \frac{3}{10\cdot 8} = \boxed{\frac{3}{80}}$ 1p

Problem 2. Let

$$f(x) = \begin{cases} 2x^3 - 7x^2 + 3 & \text{for} \quad x \leqslant 0 \\ 5\sin^3 x + a^2 & \text{for} \quad x > 0 \end{cases}$$

where a is a real parameter.

1. (5 pts) Show that f is continuous on $(-\infty, 0) \cup (0, \infty)$ for any value of the parameter a.

2. (5 pts) Find a such that f is discontinuous at $x = 0$.

3. (10 pts) Find a such that f is continuous on $(-\infty, \infty)$. Explain your answer.

① For $x<0$, $f(x) = 2x^3 - 7x^2+3$ is an elementary function, so it is continuous on $(-\infty, 0)$ 1 1

For $x>0$, $f(x) = 5\sin^3 x + a^2$ is an elementary function, so it is continuous on $(0, \infty)$ for any $a \in \mathbb{R}$ 1 1

$\therefore$ f is continuous on $(-\infty, 0) \cup (0, \infty)$ for any value of the parameter a. 1

③ f is continuous at $x=0 \iff \lim\limits_{x\to 0} f(x) = f(0) = 3$ 1

We have:

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (5\sin^3 x + a^2) = 5\cdot \sin^3 0 + a^2 = a^2$ 2

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (2x^3 - 7x^2+3) = 2\cdot 0^3 - 7\cdot 0^2 + 3 = 3$ 2

$\therefore$ $\lim\limits_{x\to 0} f(x)$ exists $\iff \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = 3 = f(0) \iff$ 3

$\iff$ f is continuous at $x=0 \iff a^2 = 3 \iff a = \pm\sqrt{3}$ 3

$\therefore$ f is continuous on $(-\infty, \infty) \iff f$ is continuous at $x=0$ (by ①) $\iff a = \pm\sqrt{3} \iff \boxed{a \in \{-\sqrt{3}, \sqrt{3}\}}$ 2

② f is discontinuous at $x=0 \iff f$ is not continuous at $x=0 \iff a^2 \neq 3 \iff a \neq \sqrt{3}$ and $a \neq -\sqrt{3} \iff$

$\iff \boxed{a \in \mathbb{R} \setminus \{-\sqrt{3}, \sqrt{3}\}}$

Problem 3. Find the derivative of the following function f:

(a) (5 pts) $f(x) = x + \sqrt{x} + \sqrt[3]{x}$ $= x + x^{\frac{1}{2}} + x^{\frac{1}{3}}$ 1 pt

$f'(x) = \left(x + x^{\frac{1}{2}} + x^{\frac{1}{3}}\right)' = 1 + \frac{1}{2}x^{\frac{1}{2}-1} + \frac{1}{3}x^{\frac{1}{3}-1} =$ 3 pts

$= 1 + \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} =$

$= \boxed{1 + \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}}$ 1 pt

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(b) (5 pts) $f(x) = \dfrac{1 + x - x^2}{1 - x + x^2}$

$f'(x) = \dfrac{(1+x-x^2)'(1-x+x^2) - (1+x-x^2)\cdot(1-x+x^2)'}{(1-x+x^2)^2} =$ 2 pts

$= \dfrac{(1-2x)(1-x+x^2) - (1+x-x^2)(-1+2x)}{(1-x+x^2)^2} =$ 2 pts

$= \dfrac{(1-2x)\left[(1-x+x^2) + (1+x-x^2)\right]}{(1-x+x^2)^2} = \boxed{\dfrac{2\cdot(1-2x)}{(1-x+x^2)^2}}$ 1 pt

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(c) (5 pts) $f(x) = \cos(3x)\sin^3 x$.

$f'(x) = (\cos(3x))'\sin^3 x + \cos(3x)\cdot(\sin^3 x)' =$ 1 pt

$= -3\sin(3x)\cdot\sin^3 x + 3\sin^2 x\cdot\cos x\cdot\cos(3x) =$ 3 pt

$= 3\sin^2 x\,(\cos x\cdot\cos(3x) - \sin(3x)\cdot\sin x) =$

$= 3\sin^2 x\cdot\cos(x+3x) = \boxed{3\sin^2 x\cdot\cos(4x)}$ 1 pt

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(d) (5 pts) $f(x) = \left(2 + \tan^4\left(\dfrac{x}{3}\right)\right)^5$

$f'(x) = 5\cdot\left(2+\tan^4\left(\frac{x}{3}\right)\right)^4\cdot\left(2+\tan^4\left(\frac{x}{3}\right)\right)' =$ 2 pts

$= 5\cdot\left(2+\tan^4\left(\frac{x}{3}\right)\right)^4\cdot 4\tan^3\left(\frac{x}{3}\right)\cdot\left(\tan\left(\frac{x}{3}\right)\right)' =$ 1 pt

$= 20\cdot\left(2+\tan^4\left(\frac{x}{3}\right)\right)^4\cdot\tan^3\left(\frac{x}{3}\right)\cdot\frac{1}{\cos^2\left(\frac{x}{3}\right)}\cdot\frac{1}{3} =$ pt

$= \boxed{\frac{20}{3}\left(2+\tan^4\left(\frac{x}{3}\right)\right)^4\cdot\frac{\sin^3\left(\frac{x}{3}\right)}{\cos^5\left(\frac{x}{3}\right)}}$ 2 pts

Problem 4. Let f be a function defined on an interval I, and let $c \in I$ be an interior point of I.

(a) (2 pts) What does it mean that f is continuous at c?

(b) (2 pts) What does it mean that f is differentiable at c?

(c) (10 pts) Prove that if f is differentiable at c then f is continuous at c.

(d) (6 pts) Prove or disprove that if f is continuous at c then f is differentiable at c.

ⓐ f is continuous at $c \iff \lim\limits_{x\to c} f(x)$ exists and

$\lim\limits_{x\to c} f(x) = f(c)$ 2pts

ⓑ f is differentiable at $c \iff \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists

and is finite $\iff \lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists 2pts

and is finite.

ⓒ If f is differentiable at c, $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$

exists and is finite.

We have to show that $\lim\limits_{x\to c} f(x) = f(c)$.

For any $x \neq c$, we have

$f(x) = f(c) + f(x) - f(c) =$

$= f(c) + \dfrac{f(x)-f(c)}{x-c} \cdot (x-c)$ 2pts

$\therefore \lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} \left[f(c) + \dfrac{f(x)-f(c)}{x-c} \cdot (x-c) \right] =$ 4pts 2pts

$= f(c) + \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} \cdot \lim\limits_{x\to c} (x-c) = f(c) + f'(c)\cdot(c-c) =$

$= f(c) + f'(c) \cdot 0 = f(c)$, as desired 2pts

ⓓ Consider the function $f(x) = |x|$, $x \in \mathbb{R}$. 2pts

We have $\left.\begin{array}{l} \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x = 0 \\ \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(-x) = -0 = 0 \end{array}\right\} \Rightarrow \lim\limits_{x\to 0} f(x) = 0 = f(0)$ 1pt

$\therefore f$ is continuous at $x = 0$

$f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \dfrac{|x|-0}{x-0} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1$

$f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{|x|-0}{x-0} = \lim\limits_{x\to 0} \dfrac{-x}{x} = -1$ 2pt

$\therefore f$ is not differentiable at $x=0$ but continuous at $x=0$

Thus, the statement (d) is not true in general. 1pt

Problem 5.

(a) (10 pts) Find an equation for the tangent line to the curve given by the parametric equations $x = 1 + (1/t^2)$, $y = 1 - (3/t)$, at the point on the curve corresponding to $t = 2$.

$\frac{dx}{dt} = (t^{-2})' = -2t^{-3} = \boxed{-\frac{2}{t^3}}$ 2 pts

$\frac{dy}{dt} = -3(t^{-1})' = 3t^{-2} = \boxed{\frac{3}{t^2}}$ 2 pts

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{3}{t^2}}{-\frac{2}{t^3}} = -\frac{3}{2}t$ 2 pts

$m = \frac{dy}{dx}\Big|_{t=2} = -\frac{3}{2}\cdot 2 = \boxed{-3}$ 1 pt

The considered point has coordinates $x_0 = 1 + \frac{1}{2^2} = \boxed{\frac{5}{4}}$, $y_0 = 1 - \frac{3}{2} = \boxed{-\frac{1}{2}}$ 1 p

$\therefore$ The equation of the tangent line is

$y - y_0 = m(x - x_0)$

$y + \frac{1}{2} = -3(x - \frac{5}{4})$, or $\boxed{y + 3x = \frac{13}{4}}$ 2 pts

(b) (10 pts) Use implicit differentiation to find $\frac{dy}{dx}$ if

$$y^3 \cot\left(\frac{1}{y}\right) = 5x^6 + 2x^3y^4$$

1 pt Differentiate both sides of the equation

2 pts $\frac{d}{dx}(y^3)\cdot\cot(\frac{1}{y}) + y^3\cdot\frac{d}{dx}(\cot(\frac{1}{y})) = 30x^5 + 6x^2y^4 + 2x^3\cdot 4y^3\frac{dy}{dx}$

4 pts $3y^2\cdot\frac{dy}{dx}\cdot\cot(\frac{1}{y}) + y^3\cdot(-\frac{1}{\sin^2(\frac{1}{y})})\cdot\frac{(-\frac{dy}{dx})}{y^2} = 30x^5 + 6x^2y^4 + 8x^3y^3\frac{dy}{dx}$

2 pts $\therefore \frac{dy}{dx}\cdot\left(3y^2\cdot\cot(\frac{1}{y}) + \frac{y}{\sin^2(\frac{1}{y})} - 8x^3y^3\right) = 30x^5 + 6x^2y^4$

1 pt $\therefore \boxed{\frac{dy}{dx} = \frac{30x^5 + 6x^2y^4}{3y^2\cot(\frac{1}{y}) + y\csc^2(\frac{1}{y}) - 8x^3y^3}}$