MATH 106 - Fall 2008 - MT2
ANSWER KEY

1. Evaluate the following integrals.

(a) (5 points) $\int \left(3\sqrt{t} + \frac{4}{t^2}\right)\, dt = \int \left(3t^{\frac{1}{2}} + 4t^{-2}\right) dt$

$$= \frac{3t^{3/2}}{3/2} + 4\frac{t^{-1}}{-1} + c$$

$$= 2t^{3/2} - \frac{4}{t} + c$$

(b) (5 points) $\int \frac{x+2}{\sqrt{x^2+4x}}\, dx =$ let $u = x^2 + 4x$

$$\Rightarrow du = (2x+4)\,dx$$

$$\Rightarrow \frac{du}{2} = (x+2)\,dx$$

So

$$= \int \frac{1}{2\sqrt{u}}\, du = \frac{1}{2}\frac{u^{1/2}}{1/2} + c = \sqrt{u} + c$$

$$= \sqrt{x^2+4x} + c$$

(c) (5 points) $\int_{-\pi}^{\pi} \frac{\sin(\pi^2 x)}{1+\pi^2 x^2}\, dx = 0$ since $\frac{\sin(\pi^2 x)}{1+\pi^2 x^2}$ is an odd function and the region of integration is symmetric

(d) (5 points) $\int_0^1 \frac{\arctan(x)}{3+3x^2}\,dx =$ let $u = \arctan x$

$\Rightarrow du = \frac{1}{1+x^2} dx$

$$= \int_0^{\pi/4} \frac{u}{3} du = \frac{1}{3}\frac{u^2}{2} = \frac{u^2}{6}\Big|_0^{\pi/4} = \frac{\pi^2}{96}$$

2. Find the following limits.

(a) (5 points) $\lim_{x\to 1} \frac{\ln(x)}{\sin(\pi x)} =$ $\frac{0}{0}$ (so we use L'Hospital's Rule)

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x} = \frac{1}{\pi(-1)} = \frac{-1}{\pi}$$

(b) (5 points) $\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) =$

$$\lim_{x\to\infty} \frac{\left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)\left(\sqrt{x^2+x+1} + \sqrt{x^2-x}\right)}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} =$$

$$= \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}}$$

$$= 1$$

3. (10 points) Use implicit differentiation to find the equation of the tangent line to the hyperbola $x^2 + 2xy - y^2 + x = 2$ at $(1, 2)$.

$$2x + 2y + 2xy' - 2yy' + 1 = 0$$

$$y'(2x - 2y) = -1 - 2x - 2y$$

$$y' = \frac{1 + 2x + 2y}{2y - 2x}$$

$$\frac{dy}{dx}(1,2) = \frac{1 + 2 \cdot 1 + 2 \cdot 2}{2 \cdot 2 - 2 \cdot 1} = \frac{7}{2}$$

$$(y - 2) = \frac{7}{2}(x - 1)$$

$$\boxed{y = \frac{7}{2}x - \frac{3}{2}}$$

4. Let $f(x) = 2x^3 - 3x^2 - 12x$.

(a) (5 points) Find the open intervals on which $f(x)$ increases or decreases.

$$f'(x) = 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \text{ or } x = -1$$

		-1		2	
sign of f'	+	0	−	0	+
behaviour of f	↗	⌒	↘	⌣	↗

f is increasing on $(-\infty, -1) \cup (2, \infty)$ and decreasing on $(-1, 2)$

(b)(5 points) Find the local minimum and maximum values of $f(x)$.

local min. is $f(2) = -20$

" max is $f(-1) = 7$

} by FDT and partcall or by SDT (find the second derivatives and check their signal)

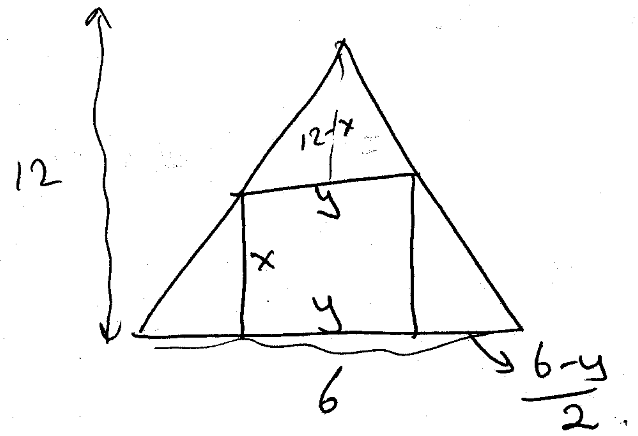
(c) Find the intervals where $f(x)$ is concave up or down and find also the inflection point(s).

$f''(x) = 12x - 6 = 0 \Rightarrow x = 1/2$

		1/2	
sign of f''	$-$		$+$
concavity	down		up

inflection point is at $x = 1/2$

5. (15 points) An isosceles triangle (ikizkenar üçgen) has a base (taban) of 6 units and a height of 12 units. Find the maximum possible area of a rectangle that can be placed inside this triangle with one side resting on the base of the triangle. Find also the dimensions of this rectangle with the maximum area.



Max $x \cdot y$

w.r.t $\frac{y}{6} = \frac{12-x}{12}$

$2y = 12 - x$

$x = 12 - 2y$

Max $f(y) = (12-2y) \cdot y = 12y - 2y^2$

$f'(y) = 12 - 4y = 0 \quad y = 3$

$x = 12 - 2 \cdot 3 = 6$

$f'(y) = 12 - 4y$ $-\infty$ | $+$ | 3 | $-$ | $+\infty$

$f(y) = 12 \cdot 3 - 2 \cdot 3^2 = \boxed{18}$

y is a critical point and $f'(y) > 0$ when $y < 3$, and $f'(y) < 0$ when $y > 3$. Hence $(3, 18)$ is the absolute max for $f(y)$.

6. (10 points) Given the function $f(x) = \int_0^x e^{-t^2}\,dt$. Show that $x = 0$ is an inflection point of $f(x)$.

$f'(x) = e^{-x^2}$ by FTC

$f''(x) = -2x e^{-x^2} = 0 \Rightarrow x = 0$

for $x < 0$, $f''(x) < 0$

" $x > 0$, $f''(x) > 0$

so $x = 0$ is an inflection point of $f(x)$.

7. (14 points) Given a differentiable function $f(x)$ on an interval (a, b). Show that:
(a) if $f'(x) = 0$ on (a, b), then $f(x)$ is a constant function.

let $x_1, x_2 \in (a, b)$ be arbitrary points. By mean value theorem there exists $c \in (x_1, x_2)$ s.t.

$f(x_1) - f(x_2) = f'(c)(x_1 - x_2) = 0$

$\Rightarrow f(x_1) = f(x_2) \Longrightarrow f = \text{const}$

(b) if $f'(x) > 0$ on (a, b), then $f(x)$ is increasing on (a, b).

$\forall x_1, x_2 \in (a, b)$ if $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$

By mean value theorem there exists c between x_1, x_2 s.t.

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

Since $f'(c) > 0$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

8. (6 points) Suppose that f and g are continuous on $[a, b]$ and f' and g' are continuous on (a, b). Assume that $f(a) = g(a)$ and $f(b) = g(b)$. Prove that there exists a number $c \in (a, b)$ such that the line tangent to the graph of f at $(c, f(c))$ is parallel to the line tangent to the graph of g at $(c, g(c))$.

Consider the function

$h(x) = f(x) - g(x)$;

$h(a) = f(a) - g(a) = 0$, $h(b) = f(b) - g(b) = 0$.

h is continuous on $[a, b]$ and differentiable on (a, b). Therefore by Rolle's theorem there exists a point $c \in (a, b)$ such that

$h'(c) = 0 \Rightarrow$

$f'(c) = g'(c)$.

Bonus Question: (5 points) Let f be continuous on $[0, 1]$ and twice differentiable on $(0, 1)$, $f(0) = f(1) = 0$ and $xf''(x) + 2\,f'(x) \geq 0$ for all $x \in (0, 1)$. Show that $f(x) \leq 0$ on $(0, 1)$.

$x f''(x) + 2f'(x) \geqslant 0 \Rightarrow (xf)'' \geqslant 0$

Hence $\varphi(x) = x f(x)$ is concave upward on $[0, 1]$.

$\varphi(0) = \varphi(1) = 0 \Rightarrow \varphi(x) \leqslant 0$

$\Rightarrow x f(x) \leq 0$, $\forall x \in [0, 1]$

$\Rightarrow f(x) \leq 0$, $\forall x \in [0, 1]$