Math- 106, FALL 2005
Solutions to Final Exam Problems

1-a) Using the identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, we find

$$\int_0^{\pi/4} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4}(1-\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi}{8}-\frac{1}{4}.$$

b) Using the substitution $u = 9 - x^2$, with $du = -2x\, dx$ and $x\, dx = -\frac{1}{2}du$, we find

$$\int x\left(9-x^2\right)^{1/2}dx = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\frac{2}{3}u^{3/2} + C = -\frac{1}{3}\left(9-x^2\right)^{3/2} + C.$$

c) We can break the integral up: $\int_{-1/2}^{1/2} \frac{1+2x}{1+4x^2}\,dx = \int_{-1/2}^{1/2}\frac{1}{1+4x^2}\,dx + \int_{-1/2}^{1/2}\frac{2x}{1+4x^2}\,dx$.

Since $\frac{2x}{1+4x^2}$ is an odd function, $\int_{-1/2}^{1/2}\frac{2x}{1+4x^2}\,dx = 0$.

To evaluate $\int_{-1/2}^{1/2}\frac{1}{1+4x^2}\,dx$, we make the substitution $u = 2x$, with $du = 2dx$.

Now $\int_{-1/2}^{1/2}\frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_{-1}^{1}\frac{du}{1+u^2} = \frac{1}{2}\left[\arctan u\right]_{-1}^{1} = \frac{1}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$.

Thus $\int_{-1/2}^{1/2}\frac{1+2x}{1+4x^2}\,dx = \frac{\pi}{4}$.

d) $\int_0^{\pi/2}\cos^7 t\, dt = \int_0^{\pi/2}\cos^6 t\cos t\, dt$. Now use parts, with $u = \cos^6 t$, $du = -6\cos^5 t\sin t$,

$dv = \cos t\, dt$, and $v = \sin t$. Thus $\int_0^{\pi/2}\cos^7 t\, dt = \int_0^{\pi/2}\cos^6 t\cos t\, dt$

$$= \left[\cos^6 t\sin t\right]_0^{\pi/2} + 6\int_0^{\pi/2}\cos^5 t\sin^2 t\, dt = 6\int_0^{\pi/2}\cos^5 t\left(1-\cos^2 t\right)dt$$

$= 6\int_0^{\pi/2}\cos^5 t dt - 6\int_0^{\pi/2}\cos^7 t dt$. It follows that $7\int_0^{\pi/2}\cos^7 t dt = 6\int_0^{\pi/2}\cos^5 t dt = 6\frac{8}{15}$, so

$$\int_0^{\pi/2} \cos^7 t dt = \frac{6}{7}\frac{8}{15} = \frac{48}{105}.$$

e) First we compute $\int_1^b \frac{\ln x}{x^2}dx$. We will use parts, with $u = \ln x, du = \frac{dx}{x}$,

$dv = x^{-2}dx$, and $v = -\frac{1}{x}$. Now $\int_1^b \frac{\ln x}{x^2}dx = \left[-\frac{\ln x}{x}\right]_1^b + \int_1^b \frac{dx}{x^2} = -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1$.

Then $\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 1$.

2- a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = \sum_{n=1}^{\infty}(-1)^n \cdot u_n$ where $u_n = \frac{1}{\ln(n+1)}$ for $n \geq 1$.

(1) $u_n = \frac{1}{\ln(n+1)}$ are all positive.

(2) $u_n \geq u_{n+1}$, since $\frac{1}{\ln(n+1)} \geq \frac{1}{\ln(n+2)}$ for all $n \geq 1$.

(3) $u_n = \frac{1}{\ln(n+1)} \to 0$ as $n \to \infty$.

Then by the Alternating Series Test, the series converges.

b) Here, $a_n = \frac{n^2+1}{2n^2+n-1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2+1}{2n^2+n-1} = \frac{1}{2} \neq 0$. Therefore by the n-th Term Test, the series diverges.

c) First, notice that $a_n = \frac{n+1}{n!}$ are all positive and $\frac{a_{n+1}}{a_n} = \frac{(n+2)}{(n+1)!}\cdot\frac{n!}{(n+1)} = \frac{n+2}{(n+1)^2}$.

So $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+2}{(n+1)^2} = 0 < 1$. Hence, by the Ratio Test, the series converges.

d) $0 \leq e^{-n^2} = \frac{1}{e^{n^2}} \leq \frac{1}{n^2}$ for $n \geq 1$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (this is a p-series with $p = 2 > 1$).

Therefore (by the Direct Comparison Test) $\sum_{n=1}^{\infty} e^{-n^2}$ converges, too.

3) Domain of $y =$ Ñ$\backslash\{1\}$ and $y \to 0$ only when $x \to -\infty$ but *y* is never equal to 0.

So, *x-intercept* does not exist.

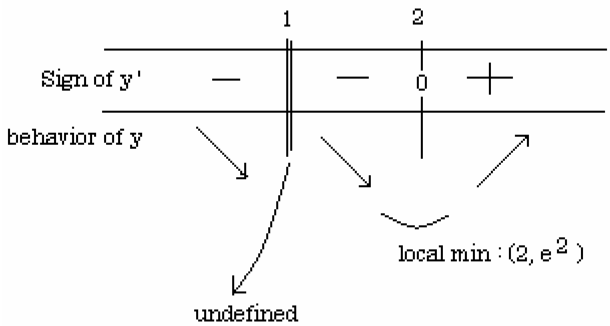
At $x = 0$, $\quad y = \dfrac{e^0}{0-1} = -1 \Rightarrow$ $(0,1)$ is the *y-intercept*.

$\lim\limits_{x\to\infty} \dfrac{e^x}{x-1} = \infty$ and $\lim\limits_{x\to-\infty} \dfrac{e^x}{x-1} = 0$. So $y = 0$ is the horizontal asymptote.

Since, $\lim\limits_{x\to1^+} \dfrac{e^x}{x-1} = \infty$ and $\lim\limits_{x\to1^-} \dfrac{e^x}{x-1} = -\infty$, $x = 1$ *is the vertical asymptote.*

$$y' = \frac{e^x \cdot (x-1) - e^x}{(x-1)^2} = \frac{e^x \cdot (x-2)}{(x-1)^2} \Rightarrow y' = 0 \text{ when } x = 2 \text{ (critical point at } x = 2\text{).}$$

<u>Note</u>: y' is undefined at $x = 1$, but 1 is not in the domain of $y = \dfrac{e^x}{x-1}$.



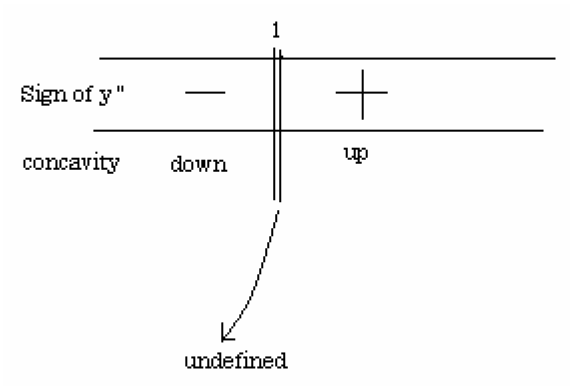
Next, $y'' = \dfrac{\left(e^x \cdot (x-2) + e^x\right) \cdot (x-1)^2 - 2 \cdot (x-1) \cdot e^x \cdot (x-2)}{(x-1)^4} = \dfrac{e^x \cdot \left((x-1)^2 - 2(x-2)\right)}{(x-1)^3}$.

$y'' = 0 \Rightarrow (x-1)^2 - 2(x-2) = 0 \Rightarrow \quad x^2 - 4x + 5 = 0$. This equation has no real roots ,

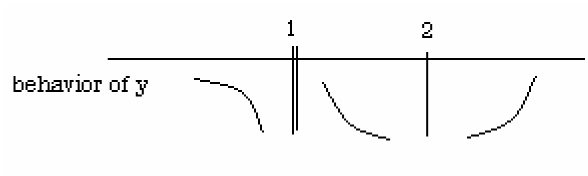
since $\Delta = 16 - 20 = -4 < 0$ (i.e. The discriminant is negative). This implies that, there is

no inflection point.

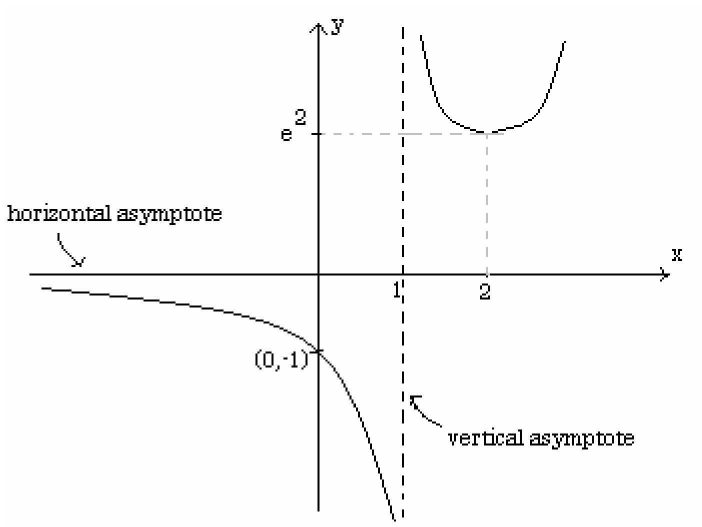
We have y'' is undefined at $x = 1$, which is not in the domain of y. So we have



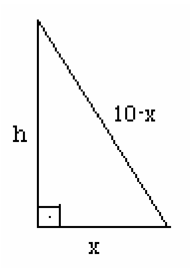
Combining our results;



So the graph of y will be as follows ;



4- a)



$$h^2 = (10-x)^2 - x^2 = 100 - 20x + x^2 - x^2 = 100 - 2x$$

$$\Rightarrow h = \sqrt{100-2x}\,.$$

So, $V(x) = \dfrac{1}{3}\cdot x^2 \cdot \sqrt{100-2x}$ where $0 \le x \le 5$.

$$h^2 + x^2 = (10-x)^2$$

b) $V'(x)=\frac{2}{3}\cdot 2x\cdot\sqrt{100-20x}+\frac{4}{3}x^2\cdot\frac{1}{2}\cdot(100-2x)^{-\frac{1}{2}}\cdot(-20)$

$$=\frac{8}{3}x\cdot\sqrt{100-20x}-\frac{40}{3}x^2\cdot\frac{1}{\sqrt{100-20x}}.$$

$$V'(x)=0\Rightarrow\frac{8}{3}x\cdot\sqrt{100-2x}=\frac{40}{3}x^2\cdot\frac{1}{\sqrt{100-20x}}$$

$$\Rightarrow 100-2x=5x\Rightarrow 100=25x\Rightarrow x=4.$$

Since $V(0)=V(5)=0$, $V(4)$ is the maximum volume. So the length side $x=4m$. and the height $h=\sqrt{100-80}=\sqrt{20}=2\cdot\sqrt{5}m$.

5-a) Use the ratio test (or the root test) to find the radius of convergence.

$\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\left(\frac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}}\frac{\sqrt{n+1}}{3^n|x|^n}\right)=3|x|\lim\limits_{n\to\infty}\sqrt{\frac{n+1}{n+2}}=3|x|$. If $3|x|<1$, or $|x|<\frac{1}{3}$, then the power series converges; if $3|x|>1$, or $|x|>\frac{1}{3}$, then the power series diverges. Thus the radius of convergence is $R=\frac{1}{3}$.

To find the interval of convergence, we need to examine the cases $x=\frac{1}{3}$ and $x=-\frac{1}{3}$. In the case $x=\frac{1}{3}$ the power series takes the form $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$; this Leibniz series converges. For $x=-\frac{1}{3}$ we have $\sum\limits_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$, a divergent p-series .

Thus the interval of convergence is $\left(-\frac{1}{3},\frac{1}{3}\right]$.

b) $e^x\sin x=\left(1+x+\frac{x^2}{2}+\ldots\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\right)=\left(x-\frac{x^3}{6}+x^2+\frac{x^3}{2}+\ldots\right)=$

$\left(x+x^2+\frac{x^3}{3}+\ldots\right)$.

c) $\dfrac{x}{1-3x+2x^2}=\dfrac{x}{(1-2x)(1-x)}=\dfrac{A}{1-2x}+\dfrac{B}{1-x}=\dfrac{A-Ax+B-2Bx}{(1-2x)(1-x)}$. We must have

$A+B=0$ and $-A-2B=1$, so $A=1$ and $B=-1$. Now $\dfrac{x}{1-3x+2x^2}$

$$=\frac{1}{1-2x}-\frac{1}{1-x}=\left(1+2x+4x^2+\ldots+2^n x^n+\ldots\right)-\left(1+x+x^2+\ldots+x^n+\ldots\right)$$

$$=x+3x^2+\ldots+\left(2^n-1\right)x^n+\ldots=\sum_{n=1}^{\infty}\left(2^n-1\right)x^n.$$

6- a) $\lim\limits_{x\to 0}(\cot x)\cdot\ln(1+x)=\lim\limits_{x\to 0}\dfrac{(\cos x)\cdot\ln(1+x)}{\sin x}\underset{\substack{\uparrow\\ L'H}}{=}\lim\limits_{x\to 0}\dfrac{(-\sin x)\cdot\ln(1+x)+(\cos x)\dfrac{1}{1+x}}{\cos x}=\dfrac{1}{1}=1.$

b) Let $y=\left(1+\dfrac{3}{x}\right)^x$, then $\ln y=x\cdot\ln\left(1+\dfrac{3}{x}\right)$ and

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}\underset{\substack{\uparrow\\ L'H}}{=}\lim_{x\to 0^+}\frac{\frac{-3/x^2}{1+\frac{3}{x}}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{-3}{x^2\cdot\left(1+\frac{3}{x}\right)}\cdot -x^2=\lim_{x\to 0^+}\frac{3x}{x+3}=0.$$

So, $\lim\limits_{x\to 0^+}y=1$.

c) $\ln y=e^{x+1}\cdot\ln(x+1)$ and by taking derivatives of both sides with respect to x, we have

$$\frac{1}{y}\cdot\frac{dy}{dx}=e^{x+1}\cdot\ln(x+1)+e^{x+1}\cdot\frac{1}{x+1}\text{ and }\frac{dy}{dx}=y\cdot e^{x+1}\cdot\ln(x+1)+e^{x+1}\cdot\frac{1}{x+1}$$

So, $\dfrac{dy}{dx}=(x+1)^{e^{x+1}}\cdot e^{x+1}\left(\ln(x+1)+\dfrac{1}{x+1}\right)$.

d) $f'(\theta)=\sqrt{2}\cdot(\cos\theta)^{\sqrt{2}-1}\cdot(-\sin\theta)+\left(\sqrt{2}\right)^{\cos\theta}\cdot(-\sin\theta)\cdot\ln\sqrt{2}$.

e) $\cosh(\ln x)=\dfrac{e^{\ln x}+e^{-\ln x}}{2}=\dfrac{x+x^{-1}}{2}=\dfrac{x+\frac{1}{x}}{2}$, so

$$\frac{d}{dx}\left(\left(x^2+1\right)\cdot\cosh(\ln x)\right)=\frac{d}{dx}\left(\left(x^2+1\right)\cdot\frac{x+\frac{1}{x}}{2}\right)=\frac{d}{dx}\left(\left(x^2+1\right)\cdot\frac{x^2+1}{2x}\right)=\frac{d}{dx}\left(\frac{\left(x^2+1\right)^2}{2x}\right)$$

$$=\frac{2\left(x^2+1\right)\cdot(2x)^2-\left(x^2+1\right)^2\cdot 2}{4x^2}=2\left(x^2+1\right)-\frac{\left(x^2+1\right)^2}{2x^2}=\frac{4x^4+4x^2-x^4-2x^2-1}{2x^2}$$

$$=\frac{3x^4+2x^2-1}{2x^2}=\frac{1}{2}\left(3x^2+2-\frac{1}{x^2}\right).$$