Math-106, Fall2004/ MT-1 (problems& solutions)

Problem 1 Calculate the following limit or show that it does not exist:

(a) (5 pts) $\displaystyle\lim_{x\to+\infty} \frac{3x^5-2x^2+11}{-2x^5+3x-17}$

$$=\lim_{x\to\infty}\frac{x^5\cdot\left(3-\frac{2}{x^3}-\frac{11}{x^5}\right)}{x^5\cdot\left(-2+\frac{3}{x^4}-\frac{17}{x^5}\right)}=\frac{\lim_{x\to\infty}\left(3-\frac{2}{x^3}-\frac{11}{x^5}\right)}{\lim_{x\to\infty}\left(-2+\frac{3}{x^4}-\frac{17}{x^5}\right)}=\frac{3-2\cdot 0-11\cdot 0}{-2+3\cdot 0-17\cdot 0}=-\frac{3}{2}.$$

Note: We have used the following fact: $\displaystyle\lim_{x\to\infty}\frac{1}{x^a}=\infty$ for any $a\in\Re, a>0$.

(b) (5 pts) $\displaystyle\lim_{x\to-\infty}\frac{\cos(x^2+2)}{3x^2-11x-1}$

We have $\displaystyle\frac{\cos\left(x^2+2\right)}{3x^2-11x-1}=\frac{\frac{\cos\left(x^2+2\right)}{x^2}}{3-\frac{11}{x}-\frac{1}{x^2}}$.

$-\dfrac{1}{x^2}\le\dfrac{\cos\left(x^2+2\right)}{x^2}\le\dfrac{1}{x^2}$ $\quad\forall x\neq 0$, since $-1\le\cos\left(x^2+2\right)\le 1$.By the Sandwich Theorem,

since $\displaystyle\lim_{x\to-\infty}\frac{1}{x^2}=0$, we deduce that $\displaystyle\lim_{x\to-\infty}\frac{\cos\left(x^2+2\right)}{x^2}=0$. So

$$\lim_{x\to-\infty}\frac{\cos\left(x^2+2\right)}{3x^2-11x-1}=\frac{\lim_{x\to-\infty}\frac{\cos\left(x^2+2\right)}{x^2}}{\lim_{x\to-\infty}3-\frac{11}{x}-\frac{1}{x^2}}=\frac{0}{3}=0.$$

(c) (5 pts) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x+\tan x}$

$$=\lim_{x\to 0}\frac{x\cdot\left(1-\dfrac{\sin x}{x}\right)}{x\cdot\left(1+\dfrac{\tan x}{x}\right)}=\frac{\lim\limits_{x\to 0}\left(1-\dfrac{\sin x}{x}\right)}{\lim\limits_{x\to 0}\left(1+\dfrac{\tan x}{x}\right)}=\frac{\lim\limits_{x\to 0}\left(1-\dfrac{\sin x}{x}\right)}{\lim\limits_{x\to 0}\left(1+\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}\right)}=\frac{1-1}{1+1\cdot\dfrac{1}{1}}=\frac{0}{2}=0$$

(d) (5 pts) $\lim\limits_{x\to -\infty} x\sin(\frac{1}{x})$

Denote $\dfrac{1}{x}=t$. Then $x\to -\infty \Rightarrow t\to 0$. So, $\lim\limits_{x\to -\infty} x\cdot\sin\left(\dfrac{1}{x}\right)=\lim\limits_{x\to -\infty}\dfrac{\sin\left(\dfrac{1}{x}\right)}{\dfrac{1}{x}}=\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1$.

Problem 2 Find the derivative of the following functions.

(a) (5 pts) $f(x)=15(x^2-3)^{1/3}(x+4)^{-1/5}$

$$\Rightarrow f'(x)=15\cdot\left[\left(x^2-3\right)^{\frac{1}{3}}\right]'\cdot\left(x+4\right)^{\frac{-1}{5}}+15\cdot\left(x^2-3\right)^{\frac{1}{3}}\cdot\left[\left(x+4\right)^{\frac{-1}{5}}\right]'$$

$$=15\cdot\frac{1}{3}\cdot\left(x^2-3\right)^{\frac{-2}{3}}\cdot 2x\cdot\left(x+4\right)^{\frac{-1}{5}}+15\cdot\left(x^2-3\right)^{\frac{1}{3}}\cdot\left(\frac{-1}{5}\right)\cdot\left(x+4\right)^{\frac{-6}{5}}$$

$$=10\cdot x\cdot\left(x^2-3\right)^{\frac{-3}{2}}\cdot\left(x+4\right)^{\frac{-1}{5}}-3\cdot\left(x^2-3\right)^{\frac{1}{3}}\cdot\left(x+4\right)^{\frac{-6}{5}}.$$

(b) (5 pts) $f(x) = \dfrac{\cos(2x)}{\cos x - \sin x}$

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}, \text{ so}$$

$$f'(x) = \frac{(\cos 2x)' \cdot (\cos x - \sin x) - (\cos x - \sin x)' \cdot \cos 2x}{(\cos x - \sin x)^2}$$

$$= \frac{-2 \cdot \sin 2x \cdot (\cos x - \sin x) - (-\sin x - \cos x) \cdot \cos 2x}{(\cos x - \sin x)^2}$$

$$= \frac{2 \cdot \sin 2x \cdot (\sin x - \cos x) + (\sin x + \cos x) \cdot \cos 2x}{(\cos x - \sin x)^2}$$

$$= \frac{4 \cdot \sin x \cdot \cos x \cdot (\sin x - \cos x) + (\sin x + \cos x) \cdot (\cos^2 x - \sin^2 x)}{(\cos x - \sin x)^2}$$

$$= \frac{3 \cdot \sin^2 x \cdot \cos x - 3 \cdot \sin x \cdot \cos^2 x + \cos^3 x - \sin^3 x}{(\cos x - \sin x)^2} = \frac{(\cos x - \sin x)^3}{(\cos x - \sin x)^2} = \cos x - \sin x.$$

<u>Another solution for (b)</u>

$$f(x) = \frac{\cos 2x}{\cos x - \sin x} = \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \cos x + \sin x$$

$$\Rightarrow f'(x) = -\sin x + \cos x = \cos x - \sin x.$$

Problem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 - x^2 & \text{for } x < 1 \\ \tan^2(x^2 - 1) & \text{for } x \geq 1 \end{cases}$$

(a) (8 pts) Is f continuous at $x = 1$? Why?

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \tan^2(x^2 - 1) = \tan^2(1^2 - 1) = \tan^2(0) = 0.$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(1 - x^2) = 1 - 1^2 = 0.$$

$f(1) = \tan^2(1^2 - 1) = \tan^2(0) = 0$. So f is continuous at $x = 1$.

b) (12 pts) Is f differentiable at $x = 1$? Why?

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{\tan^2(2h + h^2)}{h} = \lim_{h \to 0^+} \frac{\sin(2h + h^2)}{2h + h^2} \cdot (2 + h) \cdot \frac{\sin(2h + h^2)}{\cos^2(2h + h^2)}$$

$$= 1 \cdot 2 \cdot \frac{0}{1} = 0.$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h} = \lim_{h \to 0^-}(-2 - h) = -2.$$

So $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ does not exist. Hence $f'(1)$ does not exist. i.e. f is not differentiable at $x = 1$.

Problem 4

(a) (10 pts) State and prove the product rule for differentiation. (This is the rule that you use to find the derivative of the product of two functions.)

Let $u, v : \text{Ñ} \to \text{Ñ}$ be two continuous functions that are differentiable at $x \in \text{Ñ}$.

Then $u \cdot v$ is differentiable at x and $\frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$

<u>Proof:</u>

$$\frac{d}{dx}(u \cdot v)(x) = \lim_{h \to 0} \frac{(u \cdot v)(x+h) - (u \cdot v)(x)}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h) \cdot v(x+h) + u(x+h) \cdot v(x) - u(x+h) \cdot v(x) - u(x) \cdot v(x)}{h}$$

$$= \lim_{h \to 0} \left(u(x+h) \cdot \left[\frac{v(x+h) - v(x)}{h} \right] + \left[\frac{u(x+h) - u(x)}{h} \right] \cdot v(x) \right)$$

$$= u(x) \cdot \frac{dv(x)}{dx} + \frac{du(x)}{dx} \cdot v(x).$$

(b) (10 pts) Let $u, v : \mathbb{R} \to \mathbb{R}$ be differentiable functions at x such that $v(x) \neq 0$, and let $y = \frac{u}{v}$. Use the product rule for differentiation and the equation $v(x)y(x) = u(x)$ to show that $y'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{v(x)^2}$.

$$\frac{d}{dx}[v(x) \cdot y(x)] = \frac{du(x)}{dx} \Rightarrow v \cdot y' + v' \cdot y = u'$$

$$\Rightarrow y' = \frac{u' - v' \cdot y}{v} = \frac{u' - v' \cdot \left(\frac{u}{v}\right)}{v} = \frac{u' \cdot v - v' \cdot u}{v^2}.$$

Problem 5 (15 pts) Find the equation for the line in the x-y plane that is tangent to the curve described by

$$x(t) = \frac{t^2 - 2t}{2}, \qquad y(t) = \frac{t^3 - 3t}{3}$$

at the point corresponding to $t = 0$.

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{2t-2}{2} = t-1 \\ \frac{dy}{dt} &= \frac{3t^2-3}{3} = t^2-1 \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2-1}{t-1} = t+1 \Rightarrow \frac{dy}{dx} = t+1.$$

At $t = 0$, slope: $m = 0+1 = 1$, $x = \dfrac{0^2 - 2\cdot 0}{2} = 0$, $y = \dfrac{0^3 - 3\cdot 0}{3} = 0$. $\therefore$ The equation for the line passing through $(0,0)$ & having slope 1 is $y = x$.

Problem 6 (15 pts) Use implicit differentiation to find $\dfrac{dy}{dx}$ if

$$y^2 \sin\left(\frac{1}{y}\right) = 3x^2 + 2y^3.$$

$$\left[2y\cdot\sin\left(\frac{1}{y}\right) + \cos\left(\frac{1}{y}\right)\cdot y^2 \cdot\left(\frac{-1}{y^2}\right)\right]\cdot\frac{dy}{dx} = 6x + 6y^2\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\cdot\left[2y\cdot\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right) - 6y^2\right] = 6x \Rightarrow \frac{dy}{dx} = \frac{6x}{\left[2y\cdot\sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right) - 6y^2\right]}.$$