KOÇ UNIVERSITY

MATH 106

FINAL EXAM JANUARY 10, 2008

Duration of Exam: 150 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, no questions, and talking allowed. You must always explain your answers and show your work to receive full credit. Use the back of these pages if necessary. Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: ____________________

Student ID no: ____________________

Signature: ____________________

(Check One):

(Barış Çoskunüzer – 11:30-13:20) : ___
(Barış Çoskunüzer – 14:30-16:20) : ___
(Burak Özbağcı – 14:30-16:20): ___
(Demircan Canadinç 14:30-16:20): ___
(Burak Özbağcı – 10:30-12:20): ___

PROBLEM	1	2	3	4	5	6	7	8	9	10	Total
POINTS	10	10	10	10	10	10	10	10	10	10	100
SCORE											

Problem 1 (10 pts) Let $f(x) = x.e^{-x}$.

a) Find all the critical points, and the intervals on which f is increasing & decreasing.

b) Find the inflection points, and the intervals on which f is concave up & concave down.

c) Find the asymptotes, if exist.

e) Sketch the graph of f.

Problem 2 (10 pts) Find the following limit.

$$\lim_{x \to 0^+} (\sin x)^{\tan x}$$

Problem 3 (10 pts) Find the following definite integral.

$$\int_1^e (\ln x)^2 \, dx$$

Problem 4a (7 pts) Find the MacLaurin series of the following function at $x = 0$.

$$f(x) = \ln(1 + x)$$

4b) (3 pts) Compute the following sum.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n.2^n} = \frac{1}{2} - \frac{1}{2.2^2} + \frac{1}{3.2^3} - \frac{1}{4.2^4} + ...$$

Problem 5 (10 pts) Compute the following improper integral.

$$\int_0^\infty x^2.e^{-x}\ dx$$

Problem 6 Let R be the region between $y = \frac{3}{\sqrt{x^2+9}}$, $x = 3$, x-axis, and y-axis. Find the volume of the solid obtained by rotating the region R about the following axes.

6a) (5 pts) x-axis.

6b) (5 pts) y-axis.

Problem 7 (10 pts) Find the dimensions of the closed cylindrical container that has volume 8π and features the smallest surface area.

Problem 8 (10 pts) Determine whether the series given below are convergent or divergent.

8a) (5 pts)

$$\sum_{n=3}^{\infty} \frac{1}{n.(\ln n)^2}$$

8b) (5 pts)

$$\sum_{n=1}^{\infty} \cos \frac{1}{n}$$

Problem 9 (10 pts) Determine whether the series given below are convergent or divergent.

9a) (5 pts)

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

9b) (5 pts)

$$\sum_{n=1}^{\infty} \frac{n + \sin n}{n^3 + 1}$$

Problem 10a (5 pts) Find the radius of convergence for the power series given below:

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{4^n.\sqrt{n}}$$

10b) (5 pts) Find the interval of convergence of the series above.

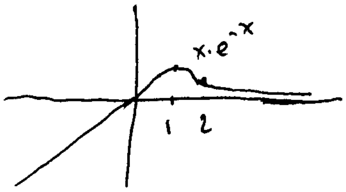
SOLUTIONS of MATH 106 FINAL (~~FALL~~ 2007)

1. a.) $f(x) = x\cdot e^{-x} \Rightarrow f'(x) = (1-x)e^{-x} \Rightarrow \quad \underset{1}{+ \mid -}$ $\quad (-\infty,1)\ f(x) \nearrow,\ (1,\infty)\ f(x) \searrow \quad x=1$ max.

b.) $f''(x) = (x-2)\cdot e^{-x} \quad \underset{2}{- \cdot +}$ $\quad (-\infty,2)$ concave down, $(2,\infty)$ concave up $\quad \mid \quad x=2$ inflection pt.

c.) $\lim\limits_{x\to\infty} x\cdot e^{-x} = \lim\limits_{x\to\infty} \dfrac{x}{e^x} \underset{\text{L'hopital}}{=} \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0 \quad \Big| \quad \lim\limits_{x\to-\infty} x\cdot e^{-x} = \lim\limits_{x\to-\infty} \underset{-\infty}{x}\cdot\underset{\infty}{e^{-x}} = -\infty \quad \Big|$ $x=0$ horizontal asymptote, no vertical asymptote

d.



2. $y = (\sin x)^{\tan x} \Rightarrow \ln y = \ln(\sin x)^{\tan x} = \tan x\cdot\ln(\sin x)$. $\quad \ln y \to 0$

$\lim\limits_{x\to0^+} \ln y = \lim\limits_{x\to0^+} \tan x\cdot\ln(\sin x) = \lim\limits_{x\to0^+} \dfrac{\ln\sin x}{\cot x} \underset{\text{L'hopital}}{=} \lim\limits_{x\to0^+} \dfrac{\frac{1}{\sin x}\cos x}{\frac{-1}{\sin^2 x}} = \lim\limits_{x\to0^+} -\sin x\cdot\cos x = 0 \Rightarrow \quad y \to e^0 = 1$ //

3. $\int(\ln x)^2dx = x\cdot(\ln x)^2 - \int 2\ln x\, dx = x(\ln x)^2 - 2\left[x\cdot\ln x - \int dx\right] = x\cdot(\ln x)^2 - 2x\ln x + 2x$

$u = (\ln x)^2 \quad dv = dx$
$du = 2\ln x\cdot\frac{1}{x}dx \quad v = x$

$u = \ln x \quad dv = dx$
$du = \frac{1}{x}dx \quad v = x$

$\Rightarrow \int_1^e (\ln x)^2 dx = x\cdot(\ln x)^2 - 2x\cdot\ln x + 2x \Big|_1^e$

$= [(e\cdot1 - \cancel{2e} + \cancel{2e}) - (0 - 0 + 2)]$

$= e - 2$ //

4. a.) $\ln(1+x) = \int\frac{dx}{1+x} = \int\sum\limits_{n=0}^{\infty}(-x)^n dx = \int\sum\limits_{n=0}^{\infty}(-1)^n\cdot x^n dx = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n\cdot x^{n+1}}{n+1} + C \quad \left[\begin{array}{l} x=0 \Rightarrow \ln(1+x)=0 \\ C=0 \end{array}\right]$

$\ln(1+x) = \sum\limits_{n=0}^{\infty}(-1)^n\cdot\dfrac{x^{n+1}}{n+1} \underset{n-1 \leftrightarrow n}{=} \sum\limits_{n=1}^{\infty}(-1)^{n-1}\cdot\dfrac{x^n}{n} \quad |x|<1$

b.) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n\cdot 2^n} = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\cdot\dfrac{x^n}{n}$ at $x = \frac{1}{2} \Rightarrow \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n\cdot2^n} = \ln(1+\frac{1}{2}) = \ln\frac{3}{2}$

5. $\int x^2\cdot e^{-x}dx = -x^2\cdot e^{-x} + \int 2x\cdot e^{-x}dx = -x^2\cdot e^{-x} + 2\left[-x\cdot e^{-x} + \int e^{-x}dx\right] = -x^2\cdot e^{-x} - 2xe^{-x} - 2e^{-x}$

$= -(x^2+2x+2)e^{-x}$

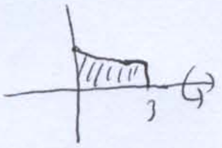
$u = x^2 \quad dv = e^{-x}dx$
$du = 2xdx \quad v = -e^{-x}$

$u = x \quad dv = e^{-x}$
$du = dx \quad v = -e^{-x}$

$\int_0^\infty x^2\cdot e^{-x}dx = \lim\limits_{t\to\infty}\int_0^t x^2\cdot e^{-x}dx = \lim\limits_{t\to\infty} -(x^2+2x+2)\cdot e^{-x}\Big|_0^t = \lim\limits_{t\to\infty} -(t^2+2t+2)\cdot e^{-t} - (-2) = 2$ //

0 (L'hopital)

6. a.)

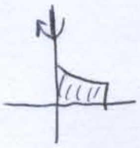


$$V=\int_0^3 \pi\left(\frac{3}{\sqrt{x^2+9}}\right)^2 dx=\int_0^3 \frac{9\pi}{x^2+9}\,dx=\pi\int_0^3 \frac{dx}{1+\frac{x^2}{9}}=\pi\int_0^1 \frac{3\,du}{1+u^2}$$

$u=\frac{x}{3}$

$$=3\pi\arctan u\Big|_0^1=3\pi(\arctan 1-\arctan 0)=3\pi\left(\frac{\pi}{4}-0\right)=\frac{3\pi^2}{4}\ //$$

b.)



$$V=\int_0^3 2\pi x\cdot\left(\frac{3}{\sqrt{x^2+9}}\right)dx=6\pi\int_0^3 \frac{x}{\sqrt{x^2+9}}\,dx=3\pi\int_9^{18}\frac{du}{\sqrt{u}}=6\pi\sqrt{u}\Big|_9^{18}=6\pi(3\sqrt{2}-3)\ //$$

$u=x^2+9$

$du=2x$

7. $V=\pi r^2h=8\pi \Rightarrow h=\frac{8}{r^2}$

$A=2\pi r^2+2\pi r\cdot h$

$=2\pi r(r+h)$

$A(r)=2\pi r^2+2\pi r\cdot\frac{8}{r^2}=2\pi r^2+\frac{16\pi}{r}$

$A'(r)=4\pi r-\frac{16\pi}{r^2} \Rightarrow A'(r)=0 \Rightarrow r^3=4 \Rightarrow r=\sqrt[3]{4}$ (max)

$h=\frac{8}{\sqrt[3]{16}}$ //

8. a. $\sum_{n=3}^{\infty}\frac{1}{n\cdot(\ln n)^2}=$ INTEGRAL TEST $\sum_{n=3}^{\infty}\frac{1}{n\cdot(\ln n)^2}\sim\int_3^{\infty}\frac{1}{x\cdot(\ln x)^2}\,dx$ convergent!

$f(x)=\frac{1}{x\cdot(\ln x)^2}$ cts, +, ↘ ✓ ($f'<0$)

$\int_3^{\infty}\frac{dx}{x\cdot(\ln x)^2}=\int_{\ln 3}^{\infty}\frac{du}{u^2}\rightarrow$ convergent by p-test

$u=\ln x$

b. $\sum_{n=1}^{\infty}\cos\frac{1}{n}$ $\lim_{n\to\infty}\cos\left(\frac{1}{n}\right)=1\neq 0$ divergent by Divergence Test

9. a. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}=$ RATIO TEST $\lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}=\lim_{n\to\infty}\frac{(n+1)^2\cdot 2^n}{n^2\cdot 2^{n+1}}=\frac{1}{2}<1$ convergent.

b. $\sum_{n=1}^{\infty}\frac{n+\sin n}{n^3+1}=$ LCT with $\sum\frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{\frac{n+\sin n}{n^3+1}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^3+n^2\sin n}{n^3+1}$$

$$=\lim_{n\to\infty}\frac{n^3\left(1+\cancel{\frac{\sin n}{n}}^{\,0}\right)}{n^3\left(1+\cancel{\frac{1}{n^3}}\right)}=1\ //$$

by LCT $\sum\frac{n+\sin n}{n^3+1}\sim\sum\frac{1}{n^2}$ convergent

10. a.) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{4^n \cdot \sqrt{n}}$

RATIO TEST: $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(x+3)^{n+1}}{4^{n+1}\sqrt{n+1}}}{\frac{(x+3)^n}{4^n \cdot \sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(x+3)^{n+1} \cdot 4^n \cdot \sqrt{n}}{(x+3)^n \cdot 4^{n+1} \cdot \sqrt{n+1}} \right|$

$\displaystyle = \frac{|x+3|}{4} < 1 \Rightarrow |x+3| < 4$

$R = 4$ //

b.) $R = 4 \Rightarrow (-7, 1)$?

$\displaystyle x = -7 \Rightarrow \sum_{n=1}^{\infty} \frac{(-7+3)^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} =$

$\frac{1}{\sqrt{n}} \to 0$

$\frac{1}{\sqrt{n}} \searrow$

by Alternating Series Test $\displaystyle\sum \frac{(-1)^n}{\sqrt{n}}$ convergent

$\displaystyle x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(1+3)^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{4^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} =$ divergent by p-test

Interval of convergence: $[-7, 1)$ //