SOLUTIONS

KOÇ UNIVERSITY
MATH 106 - CALCULUS

Final Exam June 9, 2005

Duration of Exam: 135 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and talking allowed. You must always **explain your answers** and **show your work** to receive **full credit. BONUS POINTS will be awarded for HIGH QUALITY WORK**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below. GOOD LUCK!**

Surname, Name: ALBU, TOMA

Signature:

Attendance Sheet Number: N/A

Section (Check One):

Section 1: Prof. Toma Albu ✓

Section 2: Prof. Toma Albu ✓

PROBLEM	POINTS	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
6 (EXTRA BONUS)	10	
TOTAL	**110**	

Problem 1.

(a) (5 pts) Find the derivative of $f(x) = \dfrac{x}{\sqrt{3-x^2}}$

3 pts ... $f'(x) = \dfrac{\sqrt{3-x^2} - x \cdot \dfrac{-2x}{2\sqrt{3-x^2}}}{3-x^2} =$

1 pt ... $= \dfrac{3-x^2+x^2}{(3-x^2)\sqrt{3-x^2}}$

1 pt ... $= \boxed{\dfrac{3}{(3-x^2)\sqrt{3-x^2}}}$

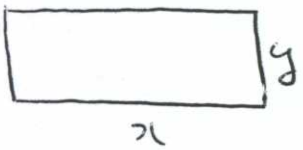
(b) (7 pts) Find the derivative of $f(x) = \ln\left(\dfrac{1-\sin x}{1+\sin x}\right)$

2 pts ... $f'(x) = \dfrac{\left(\dfrac{1-\sin x}{1+\sin x}\right)'}{\dfrac{1-\sin x}{1+\sin x}} =$

2 pts ... $= \dfrac{-\cos x(1+\sin x) - (1-\sin x)\cos x}{(1+\sin x)^2} \cdot \dfrac{1+\sin x}{1-\sin x} =$

1 pt ... $= \dfrac{-2\cos x}{1-\sin^2 x} = -\dfrac{2\cos x}{\cos^2 x} = \boxed{-\dfrac{2}{\cos x}}$

(c) (8 pts) What is the smallest perimeter possible for a rectangle whose area is 100 cm^2, and what are its dimensions?



$xy = 100$

$\therefore y = \dfrac{100}{x}$

$P = 2x + 2y = 2x + \dfrac{200}{x}$

3 pts ... $P(x) = 2x + \dfrac{200}{x}, \quad x \in (0, \infty)$

1 pt ... $P'(x) = 2 - \dfrac{200}{x^2}$

1 pt ... $P'(x) = 0 \iff x^2 = 100 \iff x = 10$ or $x = -10$

& $x > 0 \Rightarrow \boxed{x = 10}$ critical point

2 pts ... $P''(x) = 400x^{-3} \Rightarrow P''(10) > 0 \Rightarrow x = 10$ point of local, so global minimum

1 pt ... Dimensions: 10×10

Problem 2.

(a) (5 pts) Calculate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$

2 pts...... $\cos x = u$

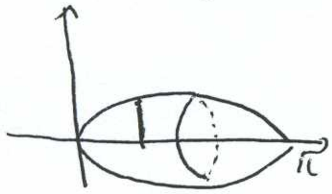
$$\int \frac{\sin x\,dx}{1+\cos^2 x} = \int \frac{-du}{1+u^2} = -\int \frac{du}{1+u^2} =$$

1 pt... $= -\arctan u + C = -\arctan(\cos x) + C$

1 pt .... $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\arctan(\cos x)\Big|_0^{\frac{\pi}{2}} = -\arctan(\cos\tfrac{\pi}{2}) + \arctan(\cos 0)$

1 pt ... $= -\arctan 0 + \arctan 1 = \boxed{\frac{\pi}{4}}$

(b) (10 pts) Find the volume of the solid generated by revolving about the x-axis the region between the curve $y = \sin x$, $0 \leqslant x \leqslant \pi$, and the x-axis.



$$V = \pi \int_0^{\pi} [R(x)]^2\,dx = \quad \ldots\ldots \quad 2\text{ pts}$$

$$= \pi \int_0^{\pi} \sin^2 x\,dx \quad \ldots\ldots \quad 1\text{ pt}$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C \quad \ldots\ldots \quad 5\text{ pts}$$

$$\therefore\ V = \pi\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Big|_0^{\pi} = \pi\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4} - 0\right) \ldots\ldots \quad 1\text{ pt}$$

$$= \boxed{\frac{\pi^2}{2}} \quad \ldots\ldots \quad 1\text{ pt}$$

(c) (5 pts) Using integration by parts, calculate $\displaystyle\int \arctan x\,dx$

$u = \arctan x \qquad dv = dx$

$du = \dfrac{dx}{1+x^2} \qquad v = x \quad \ldots\ldots \quad 2$ pts

$$\int \arctan x\,dx = x\arctan x - \int \frac{x\,dx}{1+x^2} = \ldots\ldots \quad 1\text{ pt}$$

$$= x\arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = \ldots\ldots \quad 1\text{ pt}$$

$$= \boxed{x\arctan x - \frac{1}{2}\ln(1+x^2) + C} \ldots\ldots \quad 1\text{ pt}$$

Problem 3.

(a) (8 pts) Calculate $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad \ldots\ldots\ldots \quad 2 \text{ pts}$$

$$x^2 = A(x^2+2x+1) + B(x^2-1) + C(x-1) = \quad \ldots\ldots \quad 1 \text{ pt}$$

$$= (A+B)x^2 + (2A+C)x + A - B - C \quad \ldots\ldots\ldots \quad 1 \text{ pt}$$

$$\left.\begin{array}{l} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} B+C=A \\ B+C+3A=1 \end{array}\right\} \Rightarrow 4A=1 \Rightarrow \boxed{\begin{array}{l} A=\frac{1}{4} \\ B=\frac{3}{4} \\ C=-\frac{1}{2} \end{array}} \quad \ldots\ldots \quad 2 \text{ pts}$$

$$\therefore \int \frac{x^2dx}{(x-1)(x+1)^2} = \frac{1}{4}\int\frac{dx}{x-1} + \frac{3}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{(x+1)^2} =$$

$$= \boxed{\frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)}} \quad \ldots\ldots\ldots\ldots\ldots \quad 2 \text{ pts}$$

(b) (7 pts) Evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ or state that it diverges.

$$\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{(1+x)\sqrt{x}} \quad \ldots\ldots \quad 1 \text{ pt}$$

$$\sqrt{x} = u \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{dx}{\sqrt{x}} = 2du \quad \ldots\ldots\ldots\ldots \quad 2 \text{ pts}$$

$$\int \frac{dx}{(1+x)\sqrt{x}} = \int\frac{2du}{1+u^2} = 2\arctan u + C = 2\arctan\sqrt{x} + C \ldots 2 \text{ pts}$$

$$\int_1^b \frac{dx}{(1+x)\sqrt{x}} = 2(\arctan\sqrt{b} - \arctan 1) = 2\left(\arctan\sqrt{b} - \frac{\pi}{4}\right) \ldots\ldots 1 \text{ pt}$$

$$\therefore \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{b\to\infty} 2\left(\arctan\sqrt{b} - \frac{\pi}{4}\right) = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}} \quad \ldots\ldots \quad 1 \text{ pt}$$

(c) (5 pts) Calculate $\displaystyle\lim_{x\to 0}(e^x + x)^{1/x}$

$$\lim_{x\to 0}(e^x+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln(e^x+x)^{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{\ln(e^x+x)}{x}} = \ldots 2 \text{ pts}$$

$$= e^{\lim_{x\to 0}\left(\frac{\frac{e^x+1}{e^x+x}}{1}\right)} = e^{\lim_{x\to 0}\frac{e^x+1}{e^x+x}} = \ldots\ldots \quad 2 \text{ pts}$$

$$= \boxed{e^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1 \text{ pt}$$

Problem 4

(a) (6 pts) Calculate $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{5^n}+\frac{(-1)^n}{3^{n+1}}\right) =$

$$= 3\sum_{n=0}^{\infty}\frac{1}{5^n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n = \quad \ldots\ldots\ldots \quad 2\text{pts}$$

geometric series 1pt

$$= 3\,\frac{1}{1-\frac{1}{5}} + \frac{1}{3}\,\frac{1}{1-\left(-\frac{1}{3}\right)} = \quad \ldots\ldots\ldots \quad 2\text{pts}$$

$$= 3\,\frac{1}{\frac{4}{5}} + \frac{1}{3}\,\frac{1}{\frac{4}{3}} = \frac{15}{4} + \frac{1}{4} = \boxed{4} \quad \ldots\ldots \quad 1\text{pt}$$

(b) (8 pts) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{5+3\sin n}{n^3}$ is convergent or divergent. (*Hint*: Use the *Direct Comparison Test*.)

$-1 \le \sin n \le 1 \quad \forall\, n \in \mathbb{N}$ 1pt

$5-3 \le 5+3\sin n \le 5+3$ 2pts

$0 \le \dfrac{5+3\sin n}{n^3} \le \dfrac{8}{n^3}$ 2pts

Since $\displaystyle\sum_{n=1}^{\infty}\frac{8}{n^3} = 8\cdot\sum_{n=1}^{\infty}\frac{1}{n^3}$ is CONVERGENT (3-series) ... 2pts

$\therefore \displaystyle\sum_{n=1}^{\infty}\frac{5+3\sin n}{n^3}$ is CONVERGENT by the DCT 1pt

(c) (6 pts) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+3}{2n+1}\right)$ is convergent or divergent.

$$\lim_{n\to\infty}\ln\left(\frac{n+3}{2n+1}\right) = \ln\left(\lim_{n\to\infty}\frac{n+3}{2n+1}\right) = \ldots\ldots \quad 3\text{pts}$$

$$= \ln\left(\lim_{n\to\infty}\frac{\cancel{n}\left(1+\frac{3}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)}\right) = \ln\frac{1}{2} \ldots\ldots \quad 2\text{pts}$$

$\neq 0$, so the series is divergent by the nTTFD 1pt

Problem 5.

(a) (5 pts) Prove that if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Let $s_n = \sum_{k=1}^{n} a_k,\ n \geq 1$. 1pt

$s_n = s_{n-1} + a_n$ 1pt

$a_n = s_n - s_{n-1}$ 1pt

$\sum_{n=1}^{\infty} a_n$ CONVERGENT $\iff \lim_{n\to\infty} s_n = a \in \mathbb{R}$ 1pt

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = a - a = 0$... 1pt

(b) (10 pts) Find the *radius of convergence* and the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n}$$

is (i) *absolutely convergent*, (ii) *conditionally convergent*, (iii) *convergent*, (iv) *divergent*.

$u_n = \frac{(2x-5)^n}{n}$ $\quad \left|\frac{u_{n+1}}{u_n}\right| = |2x-5| \cdot \frac{n}{n+1}$ 1pt

$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = |2x-5| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |2x-5|$ 1pt

$|2x-5| < 1 \Rightarrow \sum_{n=1}^{\infty} u_n$ is AC 1pt

$|2x-5| < 1 \iff -1 < 2x-5 < 1 \iff 4 < 2x < 6 \iff 2 < x < 3$

So, $\boxed{\sum u_n \text{ is AC} \iff x \in (2,3)}$ 2pt

$|2x-5| < 1 \iff |x - \frac{5}{2}| < \frac{1}{2} \Rightarrow \boxed{R = \frac{1}{2}}$ 1pt

$x = 2 \Rightarrow \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ HAS, so CONVERGENT, not ~~ARE~~

$\boxed{x=2} \Rightarrow \sum_{n=1}^{\infty} u_n$ is CC 2pts

$\boxed{x=3} \Rightarrow \sum u_n = \sum \frac{1}{n}$ HS, DIVERGENT

INTERVAL OF CONVERGENCE $[2,3)$ 1pt

(c) (5 pts) Find the *Maclaurin series* generated by the function

$\boxed{\text{DIVERGENT for } x \in (-\infty, 2) \cup [3, \infty)}$ 1pt

$$f(x) = x^3 \cos x + 2.$$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ 3pts

$x^3 \cos x = x^3 - \frac{x^5}{2!} + \frac{x^7}{4!} - \frac{x^9}{6!} + \cdots$ 1pt

$\boxed{x^3 \cos x + 2 = 2 + x^3 - \frac{x^5}{2!} + \frac{x^7}{4!} - \frac{x^9}{6!} + \cdots}$ 1pt

Problem 7. **(EXTRA BONUS POINTS)**

(10 pts) Find the limit of the sequence

$$a_n = \frac{(-1)^{n+1}(2n^3+5n-11)\sin(3n^4+7)}{3n^4+7}$$

or show that the limit does not exist.

$$|a_n| = \frac{2n^3+5n-11}{3n^4+7} \cdot |\sin(3n^4+7)| \leq \frac{2n^3+5n-11}{3n^4+7}$$

Put $b_n := \dfrac{2n^3+5n-11}{3n^4+7}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^3\left(2+\frac{5}{n^2}-\frac{11}{n^3}\right)}{n^4\left(3+\frac{7}{n^4}\right)} =$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\right) \cdot \lim_{n\to\infty} \frac{2+\frac{5}{n^2}-\frac{11}{n^3}}{3+\frac{7}{n^4}} =$$

$$= 0 \cdot \frac{2}{3} = 0$$

$\therefore$ $|a_n| \leq b_n$ and $\lim\limits_{n\to\infty} b_n = 0$

By a COROLLARY to SANDWICH THM, we deduce

$$\boxed{\lim_{n\to\infty} a_n = 0}$$