SOLUTIONS

KOÇ UNIVERSITY

MATH 106 - CALCULUS

Midterm II May 6, 2005

Duration of Exam: 90 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and talking allowed. You must always **explain your answers** and **show your work** to receive **full credit. BONUS POINTS will be awarded for HIGH QUALITY WORK.** Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below. GOOD LUCK!**

Surname, Name: ALBU, TOMA

Signature: ____________________

Attendance Sheet Number: N/A

Section (Check One):

Section 1: Prof. Toma Albu ✓
Section 2: Prof. Toma Albu ✓

PROBLEM	POINTS	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

Problem 1.

Consider the function

$$f(x) = 2x^3 - 9x^2 + 12x - 5, \qquad x \in \mathbb{R}.$$

(a) (4 pts) Find all local and absolute extrema of f.

(b) (2 pts) Find all inflection points of f.

(c) (4 pts) Find the intervals where f increases and decreases.

(d) (2 pts) Find the intervals where f is concave up and is concave down.

(e) (8 pts) Sketch the graph of f.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left[x^3\left(2-\frac{9}{x}+\frac{12}{x^2}-\frac{5}{x^3}\right)\right] = \lim_{x\to-\infty} x^3 \cdot \lim\left(2-\frac{9}{x}+\frac{12}{x^2}-\frac{5}{x^3}\right) = -\infty$$

$$\lim_{x\to\infty} f(x) = \infty$$

$$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2); \quad f'(x) = 0 \iff \boxed{x = 1 \text{ or } x = 2}$$

$$f''(x) = 12x - 18; \quad f''(x) = 0 \iff x = \frac{18}{12} = \boxed{\frac{3}{2}}$$

x	$-\infty$		1		$\frac{3}{2}$		2		$+\infty$
$f'(x)$		$+$	0	$-$		$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	0	$\searrow$	$-\frac{1}{2}$	$\searrow$	-1	$\nearrow$	$+\infty$
$f''(x)$		$-$		$-$	0	$+$		$+$	
$f(x)$									

From the table above one deduces:

NO ABSOLUTE EXTREMA

$f(1) = 0$ LOCAL MAXIMUM

$f(2) = -1$ LOCAL MINIMUM

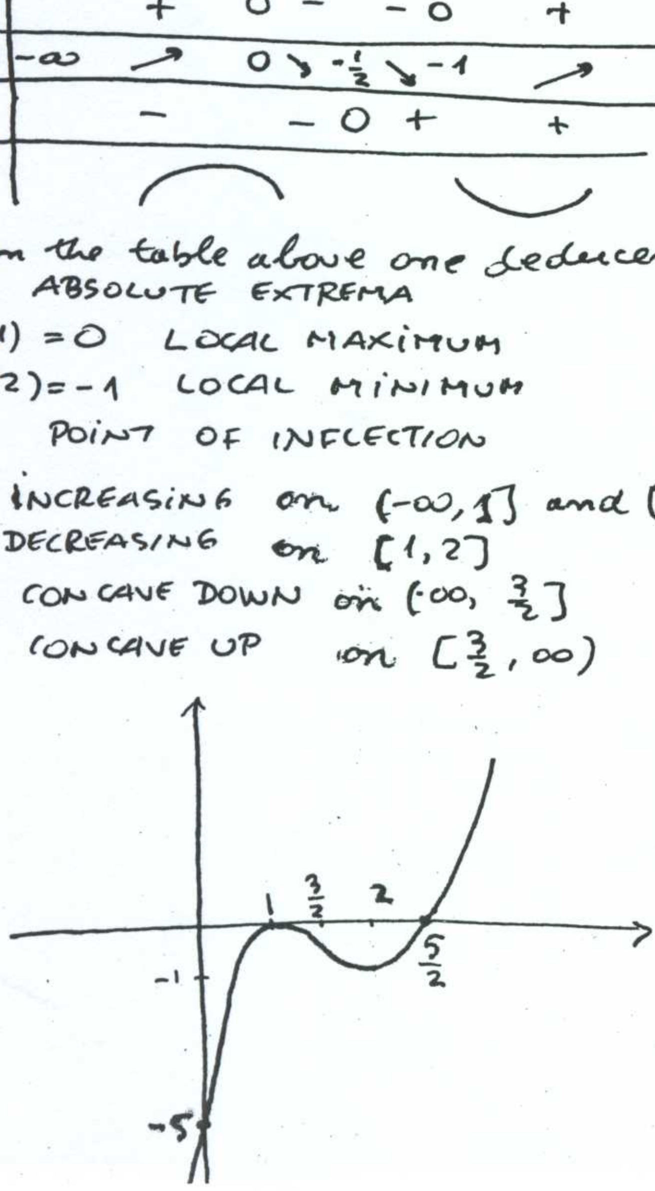
$\frac{3}{2}$ POINT OF INFLECTION

f INCREASING on $(-\infty, 1]$ and $[2, \infty)$

f DECREASING on $[1, 2]$

f CONCAVE DOWN on $(-\infty, \frac{3}{2}]$

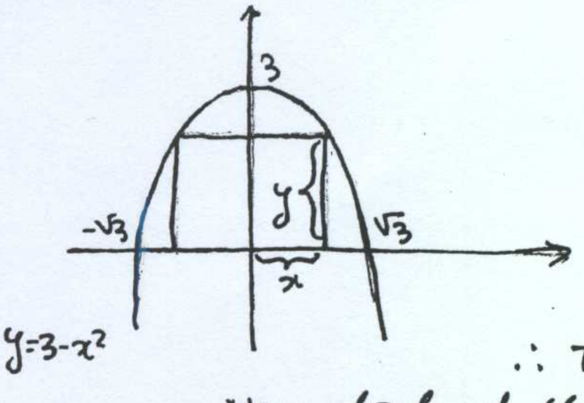
f CONCAVE UP on $[\frac{3}{2}, \infty)$



NOT ON SCALE

$f(x) = (x-1)^2(2x-5)$

Problem 2. (a) (8 pts) A rectangle with base on the x axis has its upper vertices on the curve $y = 3 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?



Area of the rectangle

$A(x) = \text{Length} \times \text{Height} = 2x \cdot y = 2x \cdot (3 - x^2)$

$A(x) = 6x - 2x^3$

Domain of A is $(0, \sqrt{3})$

$A'(x) = 6 - 6x^2$

$A'(x) = 0 \iff 6(1-x^2) = 0 \iff x = 1$ or $x = -1$

$\therefore$ The only critical point in $(0,\sqrt{3})$ is $\boxed{x=1}$

Now check whether $x=1$ is a point of MAXIMUM:

$A''(x) = -12x \Rightarrow A''(1) = -12 < 0 \Rightarrow x = 1$ is a point of local, so ABSOLUTE MAXIMUM.

$A(1) = 6 \cdot 1 - 2 \cdot 1^3 = \boxed{4}$ LARGEST AREA

DIMENSIONS of the rectangle: 2×2 (SQUARE)

(b) (2 pts) State the *Mean Value Theorem for Derivatives* (*Lagrange Theorem*).

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b). Then there exists at least one point $c \in (a,b)$ such that

$$f(b) - f(a) = (b-a) f'(c).$$

(c) (10 pts) Let $f : I \longrightarrow \mathbb{R}$, where I is an interval of real numbers, be a differentiable function on I. Prove that the derivative f' of f is zero on the interval I if and only if f is a constant function on the interval I.

„$\Leftarrow$" If f is constant on I, then $f(x) = C$, $\forall x \in I$, so clearly $f'(x) = 0$, $\forall x \in I$.

„$\Rightarrow$" We will show that for any $x_1 < x_2$ in I, we have $f(x_1) = f(x_2)$.

Since f is differentiable on I, it is differentiable at every point $x \in [x_1, x_2]$, so also continuous on $[x_1, x_2]$. Therefore, f satisfies the conditions of the LAGRANGE THEOREM on $[x_1, x_2]$. Then, $\exists\, c \in (x_1, x_2)$ with

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c).$$

But, by hypothesis, $f'(x) = 0$, $\forall x \in I$, in particular $f'(c) = 0$, and then

$$f(x_2) - f(x_1) = (x_2 - x_1) \cdot 0 = 0 \Rightarrow f(x_1) = f(x_2),$$

as desired. QED

Problem 3. Calculate the following integrals:

(a) (7 pts) $\int_{\pi/3}^{\pi/2} \sin^3 x\, dx$

First calculate $\int \sin^3 x\, dx$, and then apply the LEIBNIZ-NEWTON FORMULA.

$$\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx = \int (1-\cos^2 x)\sin x\, dx$$

Make the substitution $\cos x = u$: $du = -\sin x\, dx$, so

$$\int \sin^3 x\, dx = \int (1-u^2)(-du) = \int (u^2-1)\,du = \frac{u^3}{3} - u = \boxed{\frac{\cos^3 x}{3} - \cos x + C}$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^3 x\, dx = \left(\frac{\cos^3 x}{3} - \cos x\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\cos^3(\frac{\pi}{2})}{3} - \cos(\tfrac{\pi}{2}) - \left(\frac{\cos^3(\frac{\pi}{3})}{3} - \cos(\tfrac{\pi}{3})\right) =$$

$$= 0 - 0 - \left(\frac{1}{8\cdot 3} - \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{24} = \boxed{\frac{11}{24}}$$

(b) (7 pts) $\int \frac{dx}{3x^2+4x+5} = \int \frac{dx}{3(x^2+\frac{4}{3}x+\frac{5}{3})} = \frac{1}{3}\int \frac{dx}{(x+\frac{2}{3})^2 - \frac{4}{9} + \frac{5}{3}} =$

$$= \frac{1}{3}\int \frac{dx}{(x+\frac{2}{3})^2 + \frac{11}{9}} = \frac{1}{3}\int \frac{du}{u^2 + \left(\frac{\sqrt{11}}{3}\right)^2} = \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{11}}{3}}\cdot \arctan\left(\frac{u}{\frac{\sqrt{11}}{3}}\right) + C =$$

$\boxed{x+\frac{2}{3} = u \Rightarrow dx = du}$

$$= \frac{1}{\sqrt{11}} \arctan \frac{x+\frac{2}{3}}{\frac{\sqrt{11}}{3}} + C =$$

We used the formula:

$$= \boxed{\frac{1}{\sqrt{11}} \arctan \frac{3x+2}{\sqrt{11}} + C}$$

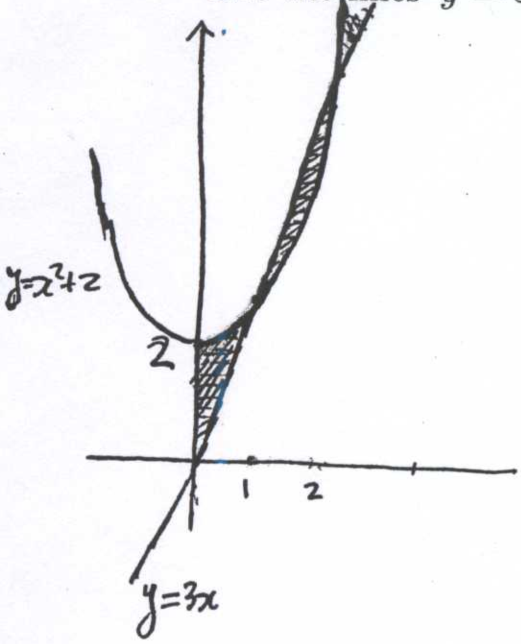
$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C.$$

(c) (6 pts) $\int \coth x\, dx = \int \frac{\cosh x}{\sinh x}\, dx = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sinh x| + C}$

$\sinh x = u \Rightarrow du = \cosh x \cdot dx$

Problem 4.

(a) (10 pts) Find the area of the region between the parabola $y = x^2 + 2$ and the lines $y = 3x$, $x = 0$, and $x = 4$.



INTERSECTION POINTS: $x^2+2=3x \Leftrightarrow x^2-3x+2=0$

$\Leftrightarrow x=1$ or $x=2$

AREA is $\int_0^4 |(x^2+2)-3x|\,dx = \int_0^4 |x^2-3x+2|\,dx =$

$= \int_0^1 (x^2+2-3x)\,dx + \int_1^2 (3x-x^2-2)\,dx + \int_2^4 (x^2+2-3x)\,dx =$

$= \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Big|_0^1 + \left(-\frac{x^3}{3} + \frac{3x^2}{2} - 2x\right)\Big|_1^2 + \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Big|_2^4$

$= \left[\left(\frac{1}{3} - \frac{3}{2} + 2\right) - 0\right] + \left[\left(-\frac{8}{3} + \frac{12}{2} - 4\right) - \left(-\frac{1}{3} + \frac{3}{2} - 2\right)\right] +$

$+ \left[\left(\frac{64}{3} - \frac{48}{2} + 8\right) - \left(\frac{8}{3} - \frac{12}{2} + 4\right)\right] =$

$= \frac{5}{6} + \left[-\frac{4}{6} + \frac{5}{6}\right] + \left[\frac{32}{6} - \frac{4}{6}\right] = \frac{5}{6} + \frac{1}{6} + \frac{28}{6} = \boxed{\frac{34}{6}} = \boxed{\frac{17}{3}}$

(b) (10 pts) Find the length of the curve given by the equation

$$x = \frac{y^3}{3} + \frac{1}{4y}$$

from $y = 1$ to $y = 3$.

$L = \int_1^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

$\frac{dx}{dy} = \frac{3y^2}{3} - \frac{1}{4y^2} = y^2 - \frac{1}{4y^2}$

$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\left(y^2 - \frac{1}{4y^2}\right)^2} = \sqrt{1+y^4-\frac{1}{2}+\frac{1}{16y^4}} = \sqrt{y^4+\frac{1}{2}+\frac{1}{16y^4}} =$

$= \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2} = y^2 + \frac{1}{4y^2}$

$\therefore L = \int_1^3 \left(y^2+\frac{1}{4y^2}\right) dy = \left(\frac{y^3}{3} - \frac{1}{4y}\right)\Big|_1^3 = \left(\frac{27}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) =$

$= \frac{107}{12} - \frac{1}{12} = \frac{106}{12} = \boxed{\frac{53}{6}}$

Problem 5.

(a) (10 pts) Given $y(x) = \int_{\ln(x^4+1)}^{3} \sin^6 t \, dt$ find $\frac{dy}{dx}$. Explain your answer.

$$y(x) = -\int_3^{\ln(x^4+1)} \sin^6 t \, dt$$

$$\therefore \frac{dy}{dx} = -\frac{d}{dx}\left[\int_3^{\ln(x^4+1)} \sin^6 t \, dt\right] = -\sin^6(\ln(x^4+1)) \cdot (\ln(x^4+1))'$$

$$= -\sin^6(\ln(x^4+1)) \cdot \frac{(x^4+1)'}{x^4+1} = \boxed{-\frac{4x^3}{x^4+1} \cdot \sin^6(\ln(x^4+1))}$$

We have used the following formula, proved in class (a consequence of the FUNDAMENTAL THM OF INTEGRAL CALCULUS Part 1)

$$\frac{d}{dx}\left[\int_a^{v(x)} f(t)\,dt\right] = f(v(x)) \cdot \frac{dv}{dx}.$$

(b) (10 pts) Given $f(x) = \operatorname{arccot}\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$ find $\frac{df}{dx}$. Simplify your answer.

Denote $u = \frac{\sin x + \cos x}{\sin x - \cos}$.

By the CHAIN-RULE,

$$\frac{df}{dx} = \frac{-u'}{1+u^2} = -\frac{\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)'}{1 + \left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)^2} =$$

$$= -\frac{\frac{(\sin x + \cos x)' \cdot (\sin x - \cos x) - (\sin x + \cos x)(\sin x - \cos x)'}{(\sin x - \cos x)^2}}{\frac{(\sin x - \cos x)^2 + (\sin x + \cos x)^2}{(\sin x - \cos x)^2}} =$$

$$= -\frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2 + (\sin x + \cos x)^2} =$$

$$= \frac{\cancel{(\sin x - \cos x)^2 + (\sin x + \cos x)^2}}{\cancel{(\sin x - \cos x)^2 + (\sin x + \cos x)^2}} = \boxed{1}$$