Math 200 - Multivariable Calculus and Matrix Algebra

Midterm 1 March 23, 2010

Duration: 90 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ______________________

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 12:30) ______________

Section 2: E. Ceyhan (Mon-Wed 15:30) ______________

Question	Points	Score
1	18	
2	18	
3	28	
4	20	
5	10	
6	6	
Total	**100**	

1. (a) (8 points) Are the following vectors $v_1 = (2,0,0), v_2 = (0,1,0),\ v_3 = (2,0,1)$ in $\mathbb{R}^3$ linearly independent? Explain your answer.

3 $c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$

$\Rightarrow c_1(2,0,0) + c_2(0,1,0) + c_3(2,0,1) = (0,0,0)$

3 $\Rightarrow \left.\begin{aligned} 2c_1 + \quad 2c_3 &= 0 \\ c_2 \quad &= 0 \\ c_3 &= 0 \end{aligned}\right\} \Rightarrow c_3 = c_2 = 0 \Rightarrow c_1 = 0$

2 So, the system only has the trivial solution, then v_1, v_2, v_3 are linearly independent.

(b) (6 points) Determine the span of v_1, v_2, and v_3.

Let $S = \{v_1, v_2, v_3\}$. Then

2 $\text{span}(S) = \{c_1 v_1 + c_2 v_2 + c_3 v_3 : c_1, c_2, c_3 \in \mathbb{R}\}$

2 $= \{(2c_1 + 2c_3, c_2, c_3) : c_1, c_2, c_3 \in \mathbb{R}\}$

2 $= \mathbb{R}^3$, since any vector in $\mathbb{R}^3$ can be written as a linear combination of v_1, v_2, v_3.

(c) (4 points) Find the dimension of the span of v_1, v_2, and v_3.

4 Since v_1, v_2, v_3 are lin. indep, and they span $\mathbb{R}^3$, and no fewer vectors can span $\mathbb{R}^3$, S forms a basis for the $\text{span}(S) = \mathbb{R}^3$. Hence $\dim(\text{span}(S)) = \dim(\mathbb{R}^3) = 3$.

2. (18 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an operator defined as

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_1 + 3x_2, x_1 + x_2 + x_3).$$

5/ (a) Write down the matrix A for T with respect to the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.

$T(x_1, x_2, x_3) = (x_1+x_2,\ x_1+3x_2,\ x_1+x_2+x_3)$

$\Rightarrow \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ since $T(x_1,x_2,x_3) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

9/ (b) Find the null space of T and find a basis for it.

3 / $N(T) = \{\vec{x} : T(\vec{x}) = 0\} = \{(x_1+x_2,\ x_1+3x_2,\ x_1+x_2+x_3) = (0,0,0)\}$

5 / $\Rightarrow \left.\begin{aligned} x_1+x_2 &= 0 \\ x_1+3x_2 &= 0 \\ x_1+x_2+x_3 &= 0 \end{aligned}\right\} \Rightarrow 0 + x_3 = 0 \Rightarrow x_3 = 0$

also

$(x_1+3x_2) - (x_1+x_2) = 0 - 0$

$\Rightarrow 2x_2 = 0 \Rightarrow x_2 = 0$ and $x_1 = 0$

1 / so $N(T) = \{(0,0,0)\}$

[or you can solve the above system by G-J elimination]

4/ (b) Is T one-to-one? Why or why not?

T is 1-1, because $N(T)$ contains only the $\vec{0}$ vector.

3. (a) (12 points) Solve the following system of linear equations using Gauss-Jordan elimination.

$$\left.\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ -3x_1 + 2x_2 + x_3 &= 2 \end{aligned}\right\} \Rightarrow A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -3 & 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

4 (Then

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -2 & 0 \\ -3 & 2 & 1 & 2 \end{array}\right] \xrightarrow[3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -4 & -2 \\ 0 & -1 & 4 & 5 \end{array}\right] \xrightarrow{R_2/3}$$

4 (

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -4/3 & -2/3 \\ 0 & -1 & 4 & 5 \end{array}\right] \xrightarrow[R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & -4/3 & -2/3 \\ 0 & 0 & 8/3 & 13/3 \end{array}\right] \xrightarrow{\frac{R_3}{(8/3)}} \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{13}{8} \end{array}\right]$$

$$\xrightarrow[\frac{4}{3}R_3+R_2]{\frac{1}{3}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/8 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 13/8 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/8 \\ 3/2 \\ 13/8 \end{bmatrix}$$) 4

(b) (6 points) Find the determinant of the coefficient matrix in the above system.

6 ($\det A = (3)\left(\frac{8}{3}\right) = 8$

(c) (10 points) Solve the following system of equations using Cramer's rule.

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

3 $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\det A = |A| = 1 - (-2) = 3.$

3 $x_1 = \dfrac{D_1}{|A|} = \dfrac{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}}{3} = \dfrac{1}{3}$

3 $x_2 = \dfrac{D_2}{|A|} = \dfrac{\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{3} = \dfrac{-2}{3}$

1 so, the solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}$.

4. (20 points) Given the the following system of equations

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 + a\,x_2 &= b \end{aligned}$$

For which values of a and b the system of equations

(a) has a unique solution,

8 $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & a & b \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & a+2 & b-2 \end{array}\right]$

For the system to have a unique solution, we should have $a+2 \neq 0 \Rightarrow \boxed{a \neq -2, \text{ and } b \in \mathbb{R}}$.

(b) has no solution,

6 For no solution, we should have $a+2=0$ and $b-2 \neq 0$

$\Rightarrow \boxed{a=-2 \text{ and } b \neq 2}$

(c) and has infinitely many solutions?

6 For infinitely many solutions, $a+2=0$ and $b-2=0$

$\Rightarrow \boxed{a=-2 \text{ and } b=2}$

5. (10 points) Compute the determinant of the following matrix by the cofactor expansion.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

5 (Using colum 3, $\det A = 1\,(-1)^{4+3} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$

5 (and using colum 2 gives

$\det A = (-1)\; 1\,(-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

$= (-1)\,(1-6) = \underline{\underline{5}}$

6. (6 points) Let $T : V \to W$ be a linear operator where V and W are vector spaces. Then show that the null space of T denoted $N(T)$ is a subspace of V.

1 (Clearly $N(T) \subseteq V$.

3 ((1) Let $\vec{u}, \vec{v} \in N(T)$. Then $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ (since T is linear)

$\Rightarrow T(\vec{u}+\vec{v}) = \vec{0} + \vec{0} = \vec{0}$ (since $\vec{u}, \vec{v} \in N(T)$)

hence $\vec{u}+\vec{v} \in N(T)$ also.

2 ((2) Let $\vec{u} \in N(T)$ and $\alpha \in \mathbb{R}$.

Then $T(\alpha\vec{u}) = \alpha T(\vec{u})$ (since T is linear)

$\Rightarrow T(\alpha\vec{u}) = \alpha\vec{0} = \vec{0}$ (since $\vec{u} \in N(T)$)

hence $\alpha\vec{u} \in N(T)$.

Therefore $N(T)$ is a subspace of V.