Math 200 - Multivariable Calculus and Matrix Algebra

Midterm 2 May 14, 2010

Duration: 90 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: ____________________

Signature: KEY

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 12:30) ____________________
Section 2: E. Ceyhan (Mon-Wed 15:30) ____________________

Question	Points	Score
1	18	
2	20	
3	24	
4	26	
5	12	
Total	**100**	

1. (18 points) Given the sphere $S: x^2+y^2+z^2+6x-8y=0$.

(a) (6 points) Find the center and radius of the sphere S.

$$\underbrace{x^2+6x+9}-9+\underbrace{y^2-8y+16}-16+z^2=0$$

$$\Rightarrow (x-(-3))^2+(y-4)^2+z^2=25$$

$\Rightarrow$ center $=(-3,4,0)$ and radius $=5$

(b) (6 points) Find the two curves $\mathbf{r_1}(t)$ and $\mathbf{r_2}(t)$ that lie on the sphere S and are parametrized by $(5\sin t)\,\mathbf{i}+(5\cos t)\,\mathbf{j}+h(t)\mathbf{k}$.

$$(5\sin t)^2+(5\cos t)^2+h^2(t)+6(5\sin t)-8(5\cos t)=0$$

$$\Rightarrow \underbrace{25\sin^2 t+25\cos^2 t}_{25}+h^2(t)+30\sin t-40\cos t=0$$

$$\Rightarrow 25+h^2(t)+10(3\sin t-4\cos t)=0$$

$$\Rightarrow h^2(t)=10(4\cos t-3\sin t)-25=5(2(4\cos t-3\sin t)-5)$$

$$\Rightarrow h(t)=\pm\sqrt{5}\sqrt{8\cos t-6\sin t-5}$$ provided that $8\cos t-6\sin t-5\geq 0$

(c) (6 points) Let $\mathbf{r}(t)$ be any curve that lies on the sphere S. Show that $\mathbf{r}(t)\cdot\frac{d\mathbf{r}(t)}{dt}=0$.

Since $\vec{r}(t)$ lies on S, $\vec{r}(t)$ has constant length, namely $|\vec{r}(t)|=5$

so $\vec{r}(t)\cdot\vec{r}(t)=25$

$$\Rightarrow \frac{d(\vec{r}(t)\cdot\vec{r}(t))}{dt}=\frac{d}{dt}25=0$$

$$\Rightarrow \vec{r}'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}'(t)=0$$

$$\Rightarrow 2\vec{r}(t)\cdot\vec{r}'(t)=0$$

$$\Rightarrow \vec{r}(t)\cdot\vec{r}'(t)=0$$

2. (20 points) Suppose that $\mathbf{u} \neq \mathbf{0}$.

(a) (6 points) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then is $\mathbf{v} = \mathbf{w}$?

Not necessarily. Let $\vec{v}$ be a nonzero vector perpendicular to $\vec{u}$. Then $\vec{u} \cdot \vec{v} = 0$. If we let $\vec{w} = 2\vec{v}$, then $\vec{u}$ is perpendicular to $\vec{w}$ also. So $\vec{u} \cdot \vec{w} = 0$.

Hence, although $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = 0$, we have $\vec{v} \neq \vec{w}$, since $\vec{v} \neq 2\vec{v}$.

(b) (6 points) If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then is $\mathbf{v} = \mathbf{w}$?

Not necessarily. Let $\vec{v} = \vec{u}$ and $\vec{w} = 2\vec{u}$.

Then $\vec{u} \times \vec{v} = \vec{u} \times \vec{u} = 0$

and $\vec{u} \times \vec{w} = \vec{u} \times (2\vec{u}) = 2(\vec{u} \times \vec{u}) = 0$.

Hence, although $\vec{u} \times \vec{v} = \vec{u} \times \vec{w} = 0$, we have $\vec{v} \neq \vec{w}$ since $\vec{u} \neq 2\vec{u}$.

(c) (8 points) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then is $\mathbf{v} = \mathbf{w}$?

Let θ_1 be the angle between $\vec{u}$ & $\vec{v}$ and θ_2 " " " $\vec{u}$ & $\vec{w}$. So $\theta_1, \theta_2 \in (0, \pi)$.

$$\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} \Rightarrow |\vec{u}||\vec{v}| \cos\theta_1 = |\vec{u}||\vec{w}| \cos\theta_2$$
$$\Rightarrow |\vec{v}| \cos\theta_1 = |\vec{w}| \cos\theta_2 \quad (*)$$

and
$$\vec{u} \times \vec{v} = \vec{u} \times \vec{w} \Rightarrow |\vec{u} \times \vec{v}| = |\vec{u} \times \vec{w}|$$
$$\Rightarrow |\vec{u}||\vec{v}||\sin\theta_1| = |\vec{u}||\vec{w}||\sin\theta_2|$$
$$\Rightarrow |\vec{v}| \sin\theta_1 = |\vec{w}| \sin\theta_2 \quad \text{since } \theta_1, \theta_2 \in (0, \pi) \quad (**)$$

Then combining $(*)$ & $(**)$,
$$\frac{|\vec{v}|}{|\vec{w}|} = \frac{\cos\theta_2}{\cos\theta_1} = \frac{\sin\theta_2}{\sin\theta_1}$$

$$\Rightarrow \frac{\sin\theta_1}{\cos\theta_1} = \frac{\sin\theta_2}{\cos\theta_2} \Rightarrow \tan\theta_1 = \tan\theta_2$$
$$\Rightarrow \theta_1 = \theta_2 \quad \text{since } \theta_1, \theta_2 \in (0, \pi)$$

So by $(*)$, we have $|\vec{v}| = |\vec{w}|$

So $\vec{v}$ & $\vec{w}$ have the same length & direction, which implies $\vec{v} = \vec{w}$.

3. (24 points) Let $f(x,y) = xy^2 + \cos x + x \sin y$.

(a) (6 points) Find all the second order partial derivatives of $f(x,y)$.

$f_x = y^2 - \sin x + \sin y$ & $f_y = 2xy + x\cos y$

So the second order partial derivatives are

$f_{xx} = -\cos x$

$f_{yy} = 2x + x(-\sin y) = 2x - x\sin y$

$f_{xy} = f_{yx} = \dfrac{\partial f_y}{\partial x} = 2y + \cos y$

(b) (6 points) If $f(x,y) = 0$, then y is (implicitly) defined as a function of x. Find the value of $\dfrac{dy}{dx}$ at the point $(1,1)$ when $f(x,y) = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y^2 - \sin x + \sin y}{2xy + x\cos y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = -\left(\frac{1 \cancel{- \sin 1 + \sin 1}}{2 + \cos 1}\right) = \frac{-1}{2+\cos 1}$$

(c) (6 points) Find the equation of the plane that is tangent to the surface defined by $z = f(x,y)$ at the point $(0,0,1)$.

Let P_0 be $(0,0,1)$. Then $f_x(P_0) = 0$, $f_y(P_0) = 0$

The equation of the tangent plane is

$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$

$\Rightarrow 0(x-0) + 0(y-0) - (z-1) = 0 \Rightarrow \underline{z = 1}$

(d) (6 points) Find the equation of the normal line at the point $(0,0,1)$ on the surface defined by $z = f(x,y)$.

The equation of the normal line is

$x = 0 + 0t = 0$

$y = 0 + 0t = 0$

$z = 1 + (-1)t = 1 - t$

4. (26 points) Given the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(a) (6 points) Find the eigenvalues of **A**.

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda)+0+0-((1-\lambda)+0+0)=0$$

$$\Rightarrow \lambda^2(1-\lambda)-(1-\lambda)=0 \Rightarrow (1-\lambda)(\lambda^2-1)=0$$

$$\rightarrow \lambda_1=1,\ \lambda_2=1 \ \&\ \lambda_3=-1.$$

(b) (8 points) Find the eigenvectors of **A**.

For $\lambda_1=\lambda_2=1$, $(A-I)\vec{e} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -u_1+u_3=0 \\ 0=0 \\ u_1-u_3=0 \end{matrix}$

so u_2 & u_3 are free variables. Letting $u_2=s$ & $u_3=t$, the eigenvector for 1 is $\begin{bmatrix} t \\ s \\ t \end{bmatrix}$, $s,t\in\mathbb{R}$

For $\lambda_3=-1$, $(A+I)\vec{e}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} v_1+v_3=0 \\ 2v_2=0 \end{matrix}\right\} \begin{matrix} v_1=-v_3 \\ v_2=0 \end{matrix}$

so v_3 is a free variable. Letting $v_3=-q$, $\vec{e}=\begin{bmatrix} q \\ 0 \\ -q \end{bmatrix}$, $q\in\mathbb{R}$ is an eigenvector for -1.

(c) (4 points) State why **A** is diagonalizable?

A is diagonalizable, since we have 3 linearly independent eigenvectors. For example, with $t=0$, $s=1 \Rightarrow \vec{e}_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, with $t=1$ & $s=0$, $\vec{e}_2=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and with $q=1$, $\vec{e}_3=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(d) (4 points) Find the diagonal form of **A**.

Diagonal form of A is $D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(e) (4 points) Find a transformation that diagonalizes **A**.

Since A is symmetric, we have orthogonality of the eigenvectors

so $Q=\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ such that $Q^T A Q = D$.

5. (12 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) = \dfrac{1-y}{1+y-x}$ and point P be $(1, 1)$.

(a) (7 points) Find the directional derivative of f along the direction of vector $\mathbf{v} = \mathbf{i} - \mathbf{j}$.

Here $\vec{v}$ is not a unit vector, so first we normalize it as

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{i-j}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j.$$

$(D_{\vec{u}} f)_P = (\nabla f)_P \cdot \vec{u}$ where

$(\nabla f)_P = (f_x i + f_y j)_P$ with $f_x = \dfrac{0-(-1)(1-y)}{(1+y-x)^2} = \dfrac{1-y}{(1+y-x)^2}$

and $f_y = \dfrac{(-1)(1+y-x)-(1-y)}{(1+y-x)^2} = \dfrac{x-2}{(1+y-x)^2}$

So $(\nabla f)_P = (0\, i + (-1) j) = 0 i - j$

Then $(D_{\vec{u}} f)_P = (0 i - j) \cdot \left(\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j\right) = 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

(b) (5 points) Find the linearization of f at the point P.

$L(x, y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

where $f(1,1) = 0$

$f_x(1,1) = 0$

$f_y(1,1) = -1$

So $L(x,y) = 0 + 0(x-1) + (-1)(y-1)$

$= 1 - y$

Alternative Solution to Part 2-(c)

$\vec{u} \neq \vec{0}$

$\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} \Rightarrow \vec{u} \cdot (\vec{v} - \vec{w}) = 0 \Rightarrow \vec{u} \perp (\vec{v} - \vec{w})$

Also

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w} \Rightarrow \vec{u} \times (\vec{v} - \vec{w}) = \vec{0} \Rightarrow \vec{u} \parallel (\vec{v} - \vec{w})$

A nonzero vector cannot both be perpendicular and parallel to a nonzero vector, so $\vec{v} - \vec{w} = \vec{0}$

$$\Rightarrow \vec{v} = \vec{w}.$$