Math 200 - Multivariable Calculus and Matrix Algebra

Final Exam June 7, 2010

Duration: 2 hours and 15 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: ______________________

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 12:30) ______________

Section 2: E. Ceyhan (Mon-Wed 15:30) ______________

Question	Points	Score
1	14	
2	18	
3	18	
4	16	
5	18	
6	16	
Total	**100**	

1. (14 points) Consider the vector space $\mathbb{R}^3$ with the usual addition of vectors and their scalar multiplication. Let P be the plane in $\mathbb{R}^3$ defined by $3x + 2y + z = 0$. (Hint: A vector $\mathbf{u} =< u_1, u_2, u_3 >$ is in P, if $3u_1 + 2u_2 + u_3 = 0$).

(a) (8 points) Show that P is a subspace of $\mathbb{R}^3$.

Clearly, $P \subseteq \mathbb{R}^3$. Next,

(i) Let $\vec{u} = \langle u_1, u_2, u_3 \rangle \in P$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle \in P$.

Then $3u_1 + 2u_2 + u_3 = 0$ and $3v_1 + 2v_2 + v_3 = 0$.

So $(3u_1 + 2u_2 + u_3) + (3v_1 + 2v_2 + v_3) = 0$

$\Rightarrow 3(u_1 + v_1) + 2(u_2 + v_2) + (u_3 + v_3) = 0$

$\Rightarrow \vec{u} + \vec{v} \in P$. So P is closed under addition.

(ii) Let $\vec{u} = \langle u_1, u_2, u_3 \rangle \in P$ and $\alpha \in \mathbb{R}$.

Then $3u_1 + 2u_2 + u_3 = 0 \Rightarrow \alpha(3u_1 + 2u_2 + u_3) = 0$

$\Rightarrow 3(\alpha u_1) + 2(\alpha u_2) + (\alpha u_3) = 0$

$\Rightarrow \alpha \vec{u} \in P$. So P is closed under scalar multiplicat

Hence, by (i) & (ii), P is a subspace of $\mathbb{R}^3$.

(b) (6 points) Construct a basis for P.

$3u_1 + 2u_2 + u_3 = 0 \Rightarrow u_1$ is a leading variable and u_2 & u_3 are free variables.

Setting $u_2 = s$ and $u_3 = t$, we have $u_1 = \frac{-2s - t}{3}$.

So vectors in P are of the form $\langle \frac{-2s-t}{3}, s, t \rangle$, $s, t \in \mathbb{R}$

Then letting $s = 1$ & $t = 0$, we get $\vec{b}_1 = \langle \frac{-2}{3}, 1, 0 \rangle$

and " $s = 0$ & $t = 1$, " " $\vec{b}_2 = \langle \frac{-1}{3}, 0, 1 \rangle$.

So any vector in P can be written as $s\vec{b}_1 + t\vec{b}_2$, $s, t \in \mathbb{R}$

and $\vec{b}_1$ & $\vec{b}_2$ are lin. independent.

Therefore $\{\vec{b}_1, \vec{b}_2\}$ is a basis set for P.

2. (18 points) (a) (4 points) Let λ be an eigenvalue for the matrix $\mathbf{A}$ and $\mathbf{x}$ be the corresponding eigenvector. Find an eigenvalue for $\mathbf{A}^3 - \mathbf{A}$ and the corresponding eigenvector.

$Ax = \lambda x \Rightarrow A(Ax) = A^2x = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

Then $A(A^2x) = A^3x = A(\lambda^2 x) = \lambda^2(Ax) = \lambda^2(\lambda x) = \lambda^3 x$.

Hence $(A^3 - A)x = A^3x - Ax = \lambda^3 x - \lambda x = (\lambda^3 - \lambda)x$

$\Rightarrow$ $\lambda^3 - \lambda$ is an eigenvalue and x is the corresponding eigenvector for $A^3 - A$.

Use the following system of equations for parts (b)-(d).

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= -2 \\ x_2 + x_3 &= 0 \\ 2x_1 - x_3 &= 3 \end{aligned}$$

(b) (4 points) Find x_3 using Cramer's rule.

Here the coefficient matrix is $A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$.

$|A| = 1\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} & \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 7$. So $x_3 = \dfrac{\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix}}{7}$

$= \frac{\pm 7}{7} = +1$, since $\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = -0\begin{vmatrix} \cdot & \cdot \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} \end{vmatrix} = 7$.

(c) (6 points) Find the inverse of the coefficient matrix.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -4 & 3 & -2 & 0 & 1 \end{array}\right) \xrightarrow[-2R_2+R_1]{4R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 7 & -2 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{R_3/7} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2/7 & 4/7 & 1/7 \end{array}\right) \xrightarrow[-R_3+R_2]{4R_3+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1/7 & 2/7 & 4/7 \\ 0 & 1 & 0 & 2/7 & 3/7 & -1/7 \\ 0 & 0 & 1 & -2/7 & 4/7 & 1/7 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} -1/7 & 2/7 & 4/7 \\ 2/7 & 3/7 & -1/7 \\ -2/7 & 4/7 & 1/7 \end{pmatrix}$

(d) (4 points) Find the solution using the inverse in part (c).

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1/7 & 2/7 & 4/7 \\ 2/7 & 3/7 & -1/7 \\ -2/7 & 4/7 & 1/7 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

3. (18 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} y & \text{for } x \leq 1 \\ y-1 & \text{for } x > 1. \end{cases}$$

(a) (4 points) Find $\lim_{(x,y)\to(1,0)} f(x,y)$, if it exists. If not, explain why.

$\lim_{(x,y)\to(1,0)} f(x,y) = -1$ along $y=0, x>1$ and $\lim_{(x,y)\to(1,0)} f(x,y) = 0$ along $y=0, x<1$

So, by the Two-Path test, $\lim_{(x,y)\to(1,0)} f(x,y)$ does not exist.

(b) (3 points) Is f a continuous function? why or why not?

No, because, for example, at $(1,0)$ limit of $f(x,y)$ does not exist. In fact, $f(x,y)$ is discontinuous at any point on $x=1$.

(c) (4 points) Does $\frac{\partial f}{\partial x}$ exist at the point $(1,1)$? If yes, find its value. If not, explain why.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \lim_{h\to 0} \frac{f(1+h,1)-f(1,1)}{h} = \lim_{h\to 0} \frac{(1-1)-1}{h}$$

$= \lim_{h\to 0} \frac{-1}{h}$ does not exist!, so $\frac{\partial f}{\partial x}$ does not exist at $(1,1)$.

(d) (4 points) Does $\frac{\partial f}{\partial y}$ exist at the point $(1,1)$? If yes, find its value. If not, explain why.

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \lim_{h\to 0} \frac{f(1,1+h)-f(1,1)}{h} = \lim_{h\to 0} \frac{1+h-1}{h} = \lim_{h\to 0} 1 = 1.$$

(e) (3 points) Is f a differentiable function? Why?

No, it is not, because f is not even continuous at $x=1$.

4. (16 points) Let $\mathbf{f}$ and $\mathbf{g}$ be vector-valued functions defined by $\mathbf{f}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}$ and $\mathbf{g}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j} - t\mathbf{k}$ for all $t \in \mathbb{R}$. Calculate the following quantities:

(a) (6 points) $\mathbf{f}(t) \cdot \mathbf{g}(t)$

$$f(t) \cdot g(t) = \langle e^t, e^{-t}, t \rangle \cdot \langle e^{-t}, e^t, -t \rangle$$
$$= e^t(e^{-t}) + e^{-t}(e^t) + (t)(-t)$$
$$= e^0 + e^0 - t^2 = 2 - t^2$$

(b) (6 points) $\mathbf{f}(t) \times \mathbf{g}(t)$

$$f(t) \times g(t) = \begin{vmatrix} i & j & k \\ e^t & e^{-t} & t \\ e^{-t} & e^t & -t \end{vmatrix} = i \begin{vmatrix} e^{-t} & t \\ e^t & -t \end{vmatrix} - j \begin{vmatrix} e^t & t \\ e^{-t} & -t \end{vmatrix} + k \begin{vmatrix} e^t & e^{-t} \\ e^{-t} & e^t \end{vmatrix}$$
$$= i(-te^{-t} - te^t) - j(-te^t - te^{-t}) + k(e^{2t} - e^{-2t})$$
$$= -t(e^t + e^{-t})i + t(e^t + e^{-t})j + (e^{2t} - e^{-2t})k$$

(c) (4 points) $\displaystyle\int_0^1 \mathbf{f}(t)\, dt$.

$$\int_0^1 f(t)\, dt = \int_0^1 (e^t i + e^{-t} j + t k)\, dt$$
$$= \left(e^t \Big|_0^1\right) i + \left(-e^{-t}\right)\Big|_0^1 j + \left(\frac{t^2}{2}\Big|_0^1\right) k$$
$$= (e-1)\, i + (1 - e^{-1})\, j + \frac{1}{2} k$$

5. (18 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy$ where $x = r\cos t$ and $y = r\sin t$.

(a) (6 points) Compute $\frac{\partial f}{\partial r}$.

$$f(x,y) = xy \Rightarrow f(r,t) = (r\cos t)(r\sin t) = r^2 \cos t \sin t$$

$$\Rightarrow \frac{\partial f}{\partial r} = 2r\cos t \sin t$$

(b) (6 points) Use chain rule to compute $\frac{\partial f}{\partial t}$.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y \cdot r(-\sin t) + x(r\cos t)$$

$$= (r\sin t)(-r\sin t) + (r\cos t)(r\cos t)$$

$$= -r^2\sin^2 t + r^2\cos^2 t$$

$$= r^2(\cos^2 t - \sin^2 t)$$

(c) (6 points) Compute $\frac{\partial^2 f}{\partial r \partial t}$.

$$\frac{\partial^2 f}{\partial r \partial t} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial r}\left(r^2(\cos^2 t - \sin^2 t)\right)$$

$$= 2r(\cos^2 t - \sin^2 t).$$

6. (16 points)
(a) (8 points) Find the local extrema of $f(x, y) = xy - x^2 - y^2$.

$f_x = y - 2x = 0$, $f_y = x - 2y = 0 \Rightarrow x = y = 0$

so $(0,0)$ is the only critical point.

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 1$

so $\left(f_{xx} f_{yy} - f^2_{xy}\right)\Big|_{(0,0)} = (-2)(-2) - 1^2 = 3$.

Then, since $f_{xx}(0,0) = -2 < 0$ and the discriminant of f is $3 > 0$, f has a local maximum at $(0,0)$.

The value of f at $(0,0)$ is $f(0,0) = 0$.

(b) (8 points) Find the greatest and smallest values the function $f(x, y) = xy$ takes on the circle $x^2 + y^2 = 1$. We will use Lagrange multiplier Method.

$\nabla f = \lambda \nabla g$ where $g(x,y) = x^2 + y^2 - 1 = 0$

so $y\,i + x\,j = \lambda(2x\,i + 2y\,j)$

$\Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = 2\lambda(2\lambda y)$

$\Rightarrow y = 4\lambda^2 y \Rightarrow 4\lambda^2 = 1$ since $y \neq 0$ (if it were then $x = 0$ also, but $(0,0)$ is on the circle).

$\Rightarrow \lambda = \mp 1/2$

So $y = \mp x$. Then $x^2 + (\mp x)^2 = 1$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \mp \frac{1}{\sqrt{2}}$

Then we have four points to consider: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

So $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2} \leftarrow$ max.

and $f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{-1}{2} \leftarrow$ min.