Spring 2011 Midterm #2

Closed book & notes; only a single-sided and handwritten A4 formula sheet and a calculator allowed; 90 minutes. No questions accepted!

Instructions: There are six pages (one cover and five pages with questions) in this exam. Please inspect the exam and make sure you have all 6 pages. You may only use your calculator and your formula sheet. Do all your work on these pages. If you use the back of a page, make sure to indicate that. **You may not exchange any kind of material with another student.** *You might get one bonus point for filling the front cover properly!*

Remember: ***You must show all your work to get proper credit.***

Academic Honesty Code: Koç University Academic Honesty Code stipulates that "copying from others or providing answers or information, written or oral, to others is cheating." By taking this exam, you are assuming full responsibility for observing the Academic Honesty Code.

NAME and SURNAME: KEY **SIGNATURE:** _____

INSTRUCTOR : _____ **LECTURE TIME :** _____

1	/20
2	/20
3	/20
4	/20
5	/20
Total:	/100

Show all your work to be eligible for partial credit. Round your **final** answers to **two decimal** places.

Question 1: A certain type of weapon (silah) has probability p=0.75 of working successfully. We test stockpile (stok yığını) of n=50 weapons. Find the probability of

(a) 37 or 38 weapons working successfully using the exact binomial distribution.

$p=0.75$, $n=50$, $X=$ # of weapons working successfully,

so $X \sim \text{Binomial}(n=50, p=0.75)$

$P(X=37 \text{ or } X=38) = \binom{50}{37}(0.75)^{37}(0.25)^{13} + \binom{50}{38}(0.75)^{38}(0.25)^{12}$

$= 0.126 + 0.1294$

$= 0.26$

(b) 37 or 38 weapons working successfully using the normal approximation to binomial distribution.

$P(X=37 \text{ or } X=38) = P(37 \le X \le 38) = P\left(\frac{36.5-37.5}{3.062} \le Z \le \frac{38.5-37.5}{3.062}\right)$

$n = 50 > 25$ ✓

$np = 50(0.75) = 37.5 > 5$ ✓

$n(1-p) = 50(0.25) = 12.5 > 5$ ✓

$\sqrt{np(1-p)} = \sqrt{50(0.75)(0.25)} = \sqrt{9.375}$

$= 3.062$

so $X \overset{\text{approx}}{\sim} N(37.5, 3.062)$

$= P(-0.33 \le Z \le 0.33)$

$= 2(0.1293)$

$= 0.26$

(c) 39 or more weapons working successfully.

$P(X \ge 39) = P\left(\frac{X-37.5}{3.062} \ge \frac{38.5-37.5}{3.062}\right)$

$= P(Z \ge 0.33) = 0.5 - 0.1293 = 0.37$

(d) The stockpile is replaced if the number of failures is at least ten. Find the probability that the stockpile is replaced.

$Y =$ # of failures, so $Y \sim \text{Binomial}(n=50, p=0.25)$

$n = 50 > 25$ ✓

$np = 50(0.25) = 12.5$

$\sqrt{np(1-p)} = \sqrt{50(0.25)(0.75)}$

$= 3.062$

$\Rightarrow Y \overset{\text{approx}}{\sim} N(12.5, 3.062)$

Then $P(Y \ge 10) =$

$= P\left(Z \ge \frac{9.5-12.5}{3.062}\right)$

$= P(Z \ge -0.98)$

$= 0.5 + 0.3365 = 0.8365$

$= 0.84$

Question 2: A manufacturer (üretici) of television sets (televizyon seti) has found that for the sets he produces, the lengths of time until the first repair can be described using a normal distribution with a mean of 4.5 years and a standard deviation of 1.5 years.

(a) For a randomly selected television set, what is the probability that the length of time until the first repair is smaller than 4 years?

X = lengths of time until first repair, $X \sim N(4.5, 1.5)$

$$P(X<4) = P\left(Z < \frac{4-4.5}{1.5}\right) = P(Z < -0.33)$$

$$= 0.5 - 0.1293$$

$$= 0.3707 \approx 0.37$$

(b) If the manufacturer sets the warrantee (garanti) so that only 10.2% of the first repairs are covered by the warrantee, how long should the warrantee last?

$$P(X < x_0) = 0.102 \Rightarrow P(Z < z_0) = 0.102$$

$$\Rightarrow P(z_0 < Z < 0) = 0.398 \Rightarrow z_0 = -1.27$$

$$\frac{x_0 - 4.5}{1.5} = -1.27 \Rightarrow x_0 = 4.5 - 1.5(1.27)$$

$$= 2.595 = 2.60 \text{ years}$$

(c) What is the median value for the lengths of time until the first repair?

Median = 50th percentile and is equal to the mean for normal distributions.

So Median = 4.5

(d) Find the interquartile range (IQR) for the lengths of time until the first repair.

$$IQR = Q_3 - Q_1 = x_{0.75} - x_{0.25}$$

$$P(X < x_{0.75}) = 0.75 \Rightarrow P(Z < z_{0.75}) = 0.75$$

$$\Rightarrow z_{0.75} \cong 0.67 \Rightarrow \frac{x_{0.75} - 4.5}{1.5} = 0.67$$

$$\Rightarrow x_{0.75} = 4.5 + 1.5(0.67) = 5.505$$

and

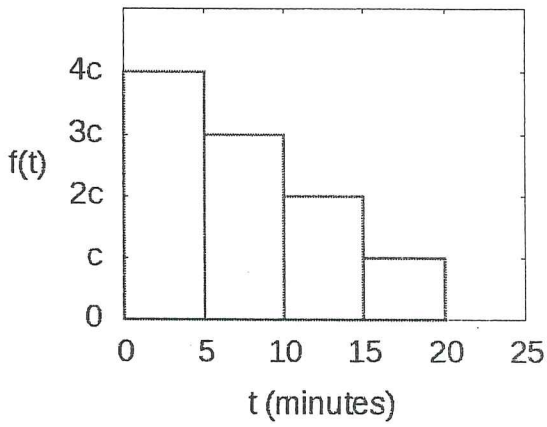
$$P(X < x_{0.25}) = 0.25 \Rightarrow P(Z < z_{0.25}) = 0.25$$

$$\Rightarrow z_{0.25} \cong -0.67 \Rightarrow \frac{x_{0.25} - 4.5}{1.5} = -0.67$$

$$\Rightarrow x_{0.25} = 4.5 - 1.5(0.67) = 3.495$$

So $IQR = 5.505 - 3.495 = \underline{2.01}$

Question 3: Ahmet's shower time is described by the continuous random variable **t** with the following probability distribution function f(t):



(a) Find the value of **c**.

$5 \times (4c + 3c + 2c + c) = 1$

$50c = 1$

$\boxed{c = 0.02}$

(b) What is the probability that Ahmet spends less than 8 minutes in the shower on a random day?

$P(t<8) = P(0<t<5) + P(5<t<8)$

$= 5 \times 0.02 \times 4 + 3 \times 0.02 \times 3$

$= 0.4 + 0.18 = \boxed{0.58}$

(c) This morning Ahmet got in the shower at 8:10 and when his mother came into his room at 8:25, he was already out of the shower. What is the probability that he spent less than 8 minutes in the shower?

$P(t<8 \mid t<15) = \dfrac{P(t<8)}{P(t<15)} = \dfrac{0.58}{5 \times 0.02 \times (4+3+2)} = \dfrac{0.58}{0.90} =$

$= 0.64$

(d) What is the probability that Ahmet spent less than 8 minutes in the shower on **more than** 20 days out of 30 last month?

X : Binomial R.V with $n=30$ $p=0.58$ $P(x>20) = ?$

check if $n \times p > 10$ and $n(1-p) > 10$

$n(1-p) = 12.6 > 10$ ✓

$np = 17.4 > 10$ ✓

Apply normal approximation to binomial.

Use $N(n \times p, \sqrt{n \times p(1-p)})$

$N(17.4, 2.703)$

Cont. Correct

$P(x>20) \simeq P\left(z > \dfrac{20+0.5-17.4}{2.703}\right)$

$P(z > 1.147)$

$= 0.5 - 0.3749$

$= \boxed{0.125}$

Question 4:

The random variable X can have the values 0, 2, or 4, and the random variable Y can have the values 0, 2, 4, or 6. The joint probability distribution of X and Y is given below:

		Y			
		0	2	4	6
X	0	**0.1**	**0.1**	**0.05**	**0.02**
	2	**0.05**	**A**	**0.1**	**0.1**
	4	**0.02**	**0.1**	**0.2**	**B**

Note: Using the values from (b) or (c) to do (a) will get you ZERO points!

(a) What is **A** and **B** if $E(X) = 2.3$? You must show your work!

$P(X=0) = 0.1 + 0.1 + 0.05 + 0.02 = 0.27$

$P(X=2) = 0.05 + A + 0.1 + 0.1 = 0.25 + A$

$P(X=4) = 0.02 + 0.1 + 0.2 + B = 0.32 + B$

$P(X=0) + P(X=2) + P(X=4) = 1$

$\Rightarrow 0.84 + A + B = 1$

$\boxed{A + B = 0.16}$

$E[X] = P(X=0).0 + P(X=2).2 + P(X=4).4 = 2.3$

$0 + 0.5 + 2A + 1.28 + 4B = 2.3$

$2A + 4B = 0.52 \Rightarrow \boxed{A + 2B = 0.26}$

$A = 0.06$

$B = 0.1$

(b) If **A** = 0.06 and **B** = 0.1, find $\text{Cov}(X,Y)$. You must show your work!

$\text{Cov}(X,Y) = \sum\sum P(X=i, Y=j).i.j - E[X].E[Y]$

$E[Y] = 0 + 0.52 + 1.4 + 1.32 = 3.24$

$\text{Cov}(X,Y) = 8.64 - (2.3 \times 3.24) = 1.188$

(c) If **A** = 0.06 and **B** = 0.1, find $\text{Var}(2X\text{-}3Y)$.

$\text{Var}(2X - 3Y) = 4\,\text{Var}(X) + 9\,\text{Var}(Y) - 12\,\text{Cov}(X,Y)$

$\sum P(x_i)\, x_i^2 = 0 + (4 \times 0.31) + (16 \times 0.42) = 7.96$

$\sum P(y_i)\, y_i^2 = 0 + (4 \times 0.26) + (16 \times 0.35) + (36 \times 0.22) = 14.5$

$\text{Var}(X) = \sum P(x_i)\, x_i^2 - (E[X])^2 = 7.96 - (2.3)^2 = 2.67$

$\text{Var}(Y) = \sum P(y_i).y_i^2 - (E[Y])^2 = 14.56 - (3.24)^2 = 4.0624$

$\text{Var}(2X - 3Y) = (4 \times 2.67) + (9 \times 4.0624) - (12 \times 1.188) = 32.9856$

Question 5:

There are 58 students who come to all the lectures, spend at least 2 hours every week doing the required reading and assigned homework, and ask their instructor to explain details that they did not understand. The probabilities for the possible grades they will get on the next midterm are given below (***Note:*** *you have to first determine which row is for grades and which is for the probabilities in the below table!*):

Grades →	3.0	3.3	3.7	4.0	→ X_i
Probabilities →	0.2	0.4	0.3	0.1	→ $P(X_i)$

(a) What is the expected average for the grades of these 58 students?

$$E[X] = (3 \times 0.2) + (3.3 \times 0.4) + (3.7 \times 0.3) + (4 \times 0.1)$$
$$= 3.43$$

$$\mu_{\bar{X}} = E[X] = 3.43$$

(b) What is the standard deviation of the sum of grades for these 58 students?

$$Var[X] = \sum x_i^2 \cdot P(x_i) - (E[X])^2 = 11.863 - (3.43)^2 = 0.0981$$
(for one student)

$$58 \cdot Var[X] = 58 \times (0.0981) = 5.6898$$ (for 58 students)

$$\sqrt{58 \cdot Var[X]} = \sqrt{5.6898} = 2.3853$$ (std for 58 students)

(c) What is the probability that the average grade for these 58 students will be greater than 3.2?

$$P(\bar{X} > 3.2) = ? \qquad \bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}})$$

$$\mu_{\bar{X}} = E[X] = 3.43$$

$$\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{\sqrt{0.0981}}{\sqrt{58}} = 0.041$$

$$P(\bar{X} > 3.2) = P\left(Z > \frac{3.2 - 3.43}{0.041}\right)$$

$$= P(Z > -5.61)$$

$$= 1$$