FALL 2012 Midterm II Solutions

Question 1:

An ambulance is moving fast along the emergency lane (emniyet şeridi) on the E6 to deliver an injured person to the hospital. For each 50 meters of distance that the ambulance moves, it will encounter one car or zero cars illegally using the emergency lane. The probability that the ambulance will encounter a car illegally using the emergency lane is 0.32 for the next 50 meters, and this probability is independent of any cars that the ambulance has encountered before.

a) What is the probability that the ambulance will encounter between 15 and 20 cars (15 and 20 are included) on the emergency lane in the next 3550 meters? Use the normal approximation to do this problem. Hint: 3550 meters equal to 71 times 50 meters.

b) The expression for the exact probability in part (a) is given below:

$$\sum_{i=a}^{20} C(b,c)(0.68)^d(e)^f$$

Find a,b,c,d,e,f. (Note: $C(b,c)$ is combinations of c out of b)

c) Calculate the probability that the ambulance will encounter exactly 20 cars in the next 3550 meters using the normal approximation and write the formula for the exact solution.

$n = 71$ trials for observing X (= number of cars illegally using the emergency lane), $\left(n = \frac{3550}{50}\right)$

$X \sim Bin(71, 0.32) \Rightarrow E(X) = (71)(0.32) = 22.72$

$Var(X) = (71)(0.32)(0.68) \cong 15.45$

a) $P(15 \le X \le 20)$

$$\cong P\left(\frac{15-0.5-22.72}{\sqrt{15.45}} \le Z \le \frac{20+0.5-22.72}{\sqrt{15.45}}\right)$$

$$= P(-2.09 \le Z \le -0.56) = 0.4817 - 0.2123 = 0.2694$$

b) Add Binomial probabilities $C(71,i)(0.32)^i(0.68)^{71-i}$

$$\sum_{i=15}^{20} C(71,i)(0.32)^i(0.68)^{71-i}$$

$a = 15$, $b = 71$, $c = i$, $d = 71-i$, $e = 0.32$, $f = i$

c) $$P(X=20) \cong P\left(\frac{19.5-22.72}{\sqrt{15.45}} \le Z \le \frac{20.5-22.72}{\sqrt{15.45}}\right)$$

$$= P(-0.82 \le Z \le -0.56) = 0.2939 - 0.2123 = 0.0816$$

$P(X=20) = \binom{71}{20}(0.35)^{20}(0.68)^{51}$ in exact form.

Question 2.

The speed limit in Sariyer tunnel is 70 km/h. The speed X of cars passing through the tunnel is normally distributed with a mean of 67 km/h and a standard deviation of 2.5 km/h.

a) What is the probability that a randomly selected car passing through the tunnel will be over the speed limit?

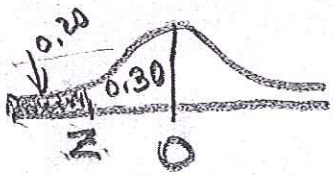
b) Find the speed x_0 such that $P(X<x_0) = 0.20$.

c) The time Y that it takes for a car to pass through the tunnel is also normally distributed, with mean μ (minutes) and standard deviation 1.3 minutes. If $P(Y>3)=0.6879$, what is μ?

a) $P(X>70) = P\left(Z > \frac{70-67}{2.5}\right) = P(Z>1.2)$

$= 0.5 - 0.3849 = 0.1151$

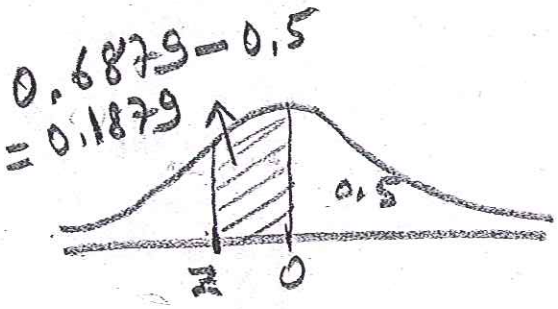
b) $P(X<x_0) = 0.20 \Rightarrow P\left(Z < \frac{x_0-67}{2.5}\right) = 0.20$



Normal table $\Rightarrow$ $z = -0.84$

$\Rightarrow x_0 = (2.5)(-0.84) + 67 = 64.9$

c) $P(Y>3) = P\left(Z > \underbrace{\frac{3-\mu}{1.3}}_{z}\right) = 0.6879$



$\Rightarrow z = -0.49$

$\frac{3-\mu}{1.3} = -0.49$

$\Rightarrow \mu = 3.637$ minutes

Question 3:

a) (4 points) A 95% confidence interval and a 98% confidence interval are calculated from the same sample. What is the same for both confidence intervals? In what way are they different? Explain by considering the formula for confidence intervals.

The formula is $\bar{x} \mp z_{\alpha/2} \frac{s}{\sqrt{n}}$. For 95% CI $z_{0.025} = 1.96$ and for 98% CI $z_{0.01} = 2.33$. These are different, but $\bar{x}$, s and n are the same. The centers of the intervals are the same. The 98% CI is wider than 95% CI.

b) (11 points) Suppose a random sample of size 78 taken from the population has an average of 5.79 and a standard deviation of 1.46. Construct the 95% and the 96% confidence intervals for the unknown population mean using this sample.

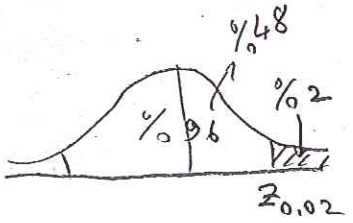
95% CI

$$\bar{x} \mp z_{0.025} \frac{s}{\sqrt{n}}$$

$$5.79 \mp (1.96) \frac{1.46}{\sqrt{78}} \Rightarrow (5.466, 6.114)$$

96% CI

$$\bar{x} \mp z_{0.02} \frac{s}{\sqrt{n}} \qquad z_{0.02} = 2.05 \quad \text{from z-table}$$



$$5.79 \mp (2.05) \frac{1.46}{\sqrt{78}}$$

$$(5.4511, 6.129)$$

Question 4.

The number of years that new graduates from the College of Administrative Sciences and Economics work in their first job has the following distribution.

Number of years, x	1	2	3	4
p(x)	0.3	0.4	0.2	0.1

a) A random sample of 50 new graduates is taken from this college. What is the approximate probability that the total number of years they work in their first job is at least 100 years?

b) The starting salaries of all new graduates from this college have a mean of 2100 TL per month and a standard deviation of 280 TL. What is the approximate probability that the mean salary in a sample of 50 new graduates is at least 2000 TL and at most 2150 TL?

c) If the salary [redacted] has a normal distribution, find the probability that the total salaries [redacted] of 3 new graduates from this college is at least 5000 TL.

a) $P(X_1+\cdots+X_{50} \geq 100) = ?$

$E(X_1+\cdots+X_{50}) = 50\mu$ where $\mu = 1(0.3)+2(0.4)+3(0.2)+4(0.1) = 2.1$

$Var(X_1+\cdots+X_{50}) = 50\sigma^2$

where $\sigma^2 = E(X^2)-\mu^2 = 1^2(0.3)+2^2(0.4)+3^2(0.2)+4^2(0.1) - (2.1)^2 = 0.89$

$\Rightarrow E(X_1+\cdots+X_{50}) = 105$

$\sqrt{Var(X_1+\cdots+X_{50})} = \sqrt{50}\sqrt{0.89} = 6.67$

$P(X_1+\cdots+X_{50} \geq 100) \cong P\left(Z \geq \frac{100-0.5-105}{6.67}\right)$

$= P(Z \geq -0.82) = 0.7939$

b) $P(2000 \leq \bar{Y} \leq 2150) \cong P\left(\frac{2000-2100}{280/\sqrt{50}} \leq Z \leq \frac{2150-2100}{280/\sqrt{50}}\right)$

$\boxed{Y: \text{salary}}$

$= P(-2.53 \leq Z \leq 1.26) = 0.4943 + 0.3962$

$= 0.8905$

c) $Y_1+Y_2+Y_3 \sim N(3(2100), \sqrt{3}(280))$

$\Rightarrow P(Y_1+Y_2+Y_3 \geq 5000) = P\left(Z \geq \frac{5000-6300}{484.97}\right)$

$= P(Z \geq -2.68)$

$= 0.5 + 0.4963 = 0.9963$

Question 5:

The joint probability distribution function of two random variables U and V is given below.

U \ V	0	3	5	$p(u)$
0	0.1	0.1	0.1	0.3
1	0.1	0.1	0.2	0.4
2	0.1	0.1	0.1	0.3
$p(u)$	0.3	0.3	0.4	

a) Find $E(2U+V)$, the expected value of $2U+V$

b) Find $Var(2U+V)$, the variance of $2U+V$

c) Find the probability that $U=1$ or $V=5$

d) If the marginal probabilities of U and V remain the same, how must their joint probability distribution be so that U and V are independent? (Construct a joint probability distribution table like the one above)

a) $E(2U+V) = 2E(U) + E(V) = 2(1) + 2.9 = 4.9$

since $E(U) = 0.4 + 2(0.3) = 1$, $E(V) = 3(0.3) + 5(0.4) = 2.9$

b) $Var(U) = E(U^2) - (E(U))^2$

$= 1(0.4) + 4(0.3) - 1^2 = 0.6$

$Var(V) = 9(0.3) + 25(0.4) - (2.9)^2 = 4.29$

$Cov(U,V) = (1)(3)(0.1) + (1)(5)(0.2) + (2)(3)(0.1) + (2)(5)(0.1) - (1)(2.9)$

$= 2.9 - 2.9 = 0$

$\Rightarrow Var(2U+V) = 4Var(U) + Var(V) + 4Cov(U,V)$

$= 4(0.6) + 4.29 + 0 = 6.69$

c) $P(U=1 \text{ or } V=5) = P(U=1) + P(V=5) - P(U=1, V=5)$

$= 0.4 + 0.4 - 0.2 = 0.6$

d) We must have $P(U=u, V=v) = P(U=u).P(V=v)$ for all u, v as given below:

	0	3	5	$p(x)$
0	0.09	0.09	0.12	0.3
1	0.12	0.12	0.16	0.4
2	0.09	0.09	0.12	0.3
$p(v)$	0.3	0.3	0.4	