

Fall 2013, EXAM 2
100 MinutesSOLUTIONS

Instructions: There are five questions in this exam. Please inspect the exam and make sure you have all the questions. You may only use your calculator and one side of a hand-written A4-size sheet of paper. Do all your work on the paper provided. **You may not exchange any kind of material with another student.**

Remember: *You must show your work to get proper credit.*

Academic Honesty Code: Koç University Academic Honesty Code stipulates that “copying from others or providing answers or information, written or oral, to others is cheating.” By taking this exam, you are assuming full responsibility for observing the Academic Honesty Code.

NAME: _____

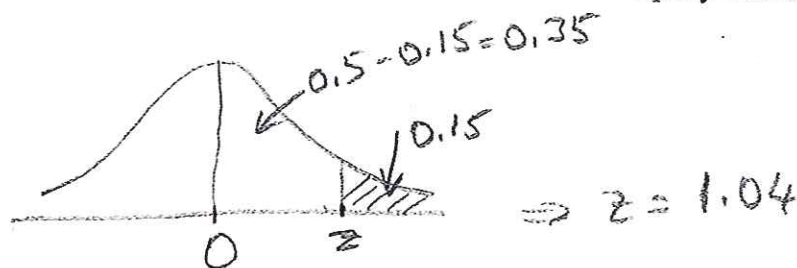
LECTURE: 1 (12:30) 2 (15:30) 3 (9:30) (Circle one of the numbers)

1	/20
2	/20
3	/30
4	/18
5	/22
Total:	/110

(10 points bonus)

Question 1 The scores on a job aptitude (= ability) test are normally distributed with mean 400 and standard deviation 100. The company will consider only those applicants scoring in the top 15%.

- a) Find the cutoff score (that is, the lowest score that the company will consider).



$$\frac{x - 400}{100} = 1.04 \Rightarrow x = 504$$

- b) Will a candidate with score 502 be considered? Why or why not?

No, because not better than 504.

- c) In a random sample of five applicants, what is the probability that their total score is at least 1500 and at most 2300?

$$\begin{aligned}
 & P(1500 \leq X_1 + \dots + X_5 \leq 2300) \quad n\mu = 5(400) = 2000 \\
 & \quad \sqrt{n}\sigma = \sqrt{5}(100) = 223.6 \\
 & = P\left(\frac{1500 - 2000}{223.6} \leq z \leq \frac{2300 - 2000}{223.6}\right) \\
 & = P(-2.24 \leq z \leq 1.34) \\
 & = 0.4875 + 0.4099 \\
 & = 0.8974
 \end{aligned}$$

- d) In a random sample of five applicants, what is the probability that at least 2 of them are in the top 15%?

$$\begin{aligned}
 & 1 - \left[\binom{5}{0} (0.15)^0 (0.85)^5 + \binom{5}{1} (0.15)^1 (0.85)^4 \right] \\
 & = 1 - 0.4437 - 0.3915 \approx 0.165
 \end{aligned}$$

Question 2 Military radar and missile detection systems are designed to warn a country of enemy attacks. A particular detection system has a 0.90 probability of detecting a missile attack.

- a) If 3 independent detection systems of this particular kind are installed, what is the probability that only one of the systems will detect the attack?
- b) If 20 detection systems are installed, what is the probability that at least fourteen, but fewer than seventeen systems will detect the attack? Use Binomial Table to answer this question.
- c) If 100 detection systems are installed, approximate the probability that at least 88, but at most 92 will detect the attack.

$$a) \binom{3}{1} (0.90)^1 (0.10)^2 = 0.027 \quad X \sim \text{Binom}(3, 0.90)$$

$$b) P(14 \leq X < 17) = P(X \leq 16) - P(X \leq 13)$$
$$= 0.133 - 0.002$$
$$X \sim \text{Bin}(20, 0.90)$$
$$= 0.131$$

$$c) \quad np = 100(0.90) = 90$$
$$npq = 90(0.10) = 9 \Rightarrow \sqrt{npq} = 3$$

$$P(88 \leq X \leq 92) \approx P\left(\frac{87.5 - 90}{3} \leq Z \leq \frac{92.5 - 90}{3}\right)$$
$$= P(-0.83 \leq Z \leq 0.83)$$
$$= 2(0.2967)$$
$$= 0.5934$$

Question 3. An investor in the stock exchange market assumes the following joint distribution for the rate of gain for the stocks of the companies X-Technologies and Y-Motors, denoted by X and Y , respectively, in a single period of time. Note that a negative rate of gain is actually rate of loss.

$Y \backslash X$	-0.20	0	0.20	$P(Y=y)$
-0.10	0.02	0.02	0.16	0.20
0	0.03	0.57	0.04	0.64
0.10	0.05	0.05	0.06	0.16
$P(X=x)$	0.10	0.64	0.26	

a) Are X and Y independent?

$$x = -0.20, y = -0.10 \Rightarrow 0.02 \stackrel{?}{=} (0.10)(0.20) \text{ Yes} \quad \text{Check } P(X=x, Y=y) \stackrel{?}{=} P(X=x)P(Y=y)$$

$$x = 0, y = -0.10 \Rightarrow 0.02 \stackrel{?}{=} (0.64)(0.20) = 0.128 \text{ No!}$$

So, X and Y cannot be independent.

b) Find the correlation coefficient $\rho(X, Y)$ and interpret. (Hint: Recall $\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$).

$$\mu_X = (-0.20)(0.1) + (0.20)(0.26) = -0.02 + 0.052 = 0.032$$

$$\mu_Y = -0.004 \quad (\text{similar calculation})$$

$$\sigma_Y^2 = (-0.10)^2(0.20) + (0.10)^2(0.16) - (-0.004)^2 = 0.003584$$

$$\Rightarrow \sigma_Y \approx 0.06$$

$$\sigma_X^2 = (-0.20)^2(0.10) + (0.20)^2(0.26) - (0.032)^2 = 0.013376$$

$$\Rightarrow \sigma_X = 0.116$$

$$\text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y$$

$$= (-0.10)(-0.20)(0.02) + (-0.10)(0.20)(0.16) + (0.10)(-0.20)(0.05) + (0.10)(0.20)(0.06) - (0.032)(-0.004) = -0.002472$$

$$\rho(X, Y) = \frac{-0.002472}{(0.116)(0.06)} \approx -0.355$$

c) What is the probability that X is -0.20 or Y is greater than 0?

$$P(X = -0.20) + P(Y > 0) - P(X = -0.20, Y > 0)$$

$$= 0.10 + 0.16 - 0.05 = 0.21$$

\Rightarrow Negative, weak correlation between X and Y

d) Suppose you invest 50 TL to X-Technologies and 100 TL to Y-Motors. Find

- your expected gain
- variance of your gain after one period.

$$\Rightarrow G = 50X + 100Y$$

$$\Rightarrow E(G) = 50E(X) + 100E(Y) = 50(0.032) + 100(-0.004) = 1.2 \text{ TL.}$$

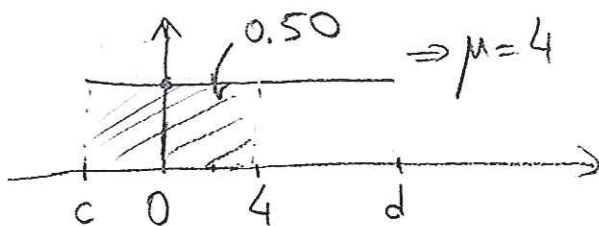
$$V(G) = 50^2 \sigma_X^2 + 100^2 \sigma_Y^2 + 2(50)(100) \text{Cov}(X, Y)$$

$$= 50^2(0.013376) + 100^2(0.003584) + 2(50)(100)(-0.002472)$$

$$= 44.56$$

Question 4. Let X be uniformly distributed with a standard deviation of 3.18

a) If $P(X < 4) = 0.50$, find $P(X < 2)$.



$$\sigma = 3.18 = \frac{d-c}{\sqrt{12}}$$

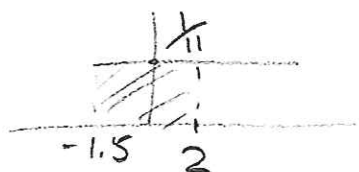
$$\Rightarrow d-c \approx 11$$

$$\Rightarrow 4-c = \frac{11}{2} = 5.5 \Rightarrow c = -1.5$$

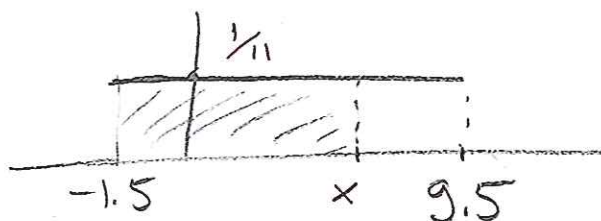
$$\Rightarrow d = 9.5$$

$$P(X < 2) = \frac{1}{d-c} \cdot (2-c)$$

$$= \frac{1}{11} \cdot (2 - (-1.5)) = \frac{3.5}{11} \approx 0.32$$



b) Find the 90th percentile of the distribution of X .



$$0.90 = \frac{(x - (-1.5))}{11}$$

$$\Rightarrow x + 1.5 = 9.9$$

$$x = 8.4$$

c) Let X_1, \dots, X_{40} be a random sample from the distribution of X . Find $P(150 < X_1 + \dots + X_{40} < 155)$.

$$\mu = 4$$

$$\Rightarrow P\left(\frac{150}{40} < \bar{X} < \frac{155}{40}\right) = P(3.75 < \bar{X} < 3.875)$$

$$\approx P\left(\frac{3.75-4}{3.18/\sqrt{40}} < Z < \frac{3.875-4}{3.18/\sqrt{40}}\right)$$

$$= P(-0.50 < Z < -0.25)$$

$$= 0.1915 - 0.0987$$

$$= 0.0928$$

