

# MATH 203 - Midterm 1 KEY

SPRING 2012

1. (15 points) Given the points:  $A = (-1, 2, 1)$ ,  $B = (2, 0, 1)$ ,  $C = (1, -1, 3)$ , and  $D = (-1, d, 2)$ . Determine,

- (a) The length and the unit direction vector corresponding to  $\vec{AB}$ .

$$\vec{AB} = \langle 2 - (-1), 0 - 2, 1 - 1 \rangle = \langle 3, -2, 0 \rangle$$

$$|\vec{AB}| = \sqrt{9+4} = \sqrt{13}$$

$$\Rightarrow \frac{\vec{AB}}{|\vec{AB}|} = \left\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}, 0 \right\rangle$$

- (b) The area of the triangle  $ABC$ .

$$|\Delta ABC| = \frac{1}{2} |\vec{AB} \times \vec{AC}| \Rightarrow \vec{AC} = \langle 2, -3, 2 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 3 & -2 & 0 \\ 2 & -1 & 2 \end{vmatrix} = \langle -4, -6, -5 \rangle \Rightarrow |\vec{AB} \times \vec{AC}| = \sqrt{16 + 36 + 25} = \sqrt{77}$$

$$\text{Area} = \frac{\sqrt{77}}{2}$$

- (c) The value(s) of  $d$  in order for  $\vec{AB}$  and  $\vec{CD}$  to be perpendicular;

$$\vec{AB} = \langle 3, -2, 0 \rangle \quad \vec{AB} \cdot \vec{CD} = 0 \Rightarrow 6 - 2(d+1) + 0 = 0 \Rightarrow d = 2$$

$$\vec{CD} = \langle -1, d+1, 1 \rangle$$

- (d) The value of  $d$  for the points  $A, B, C$  and  $D$  to be in the same plane.

Plane equation for the plane  $P$  containing  $A, B, C$ :

$$\vec{N} = \vec{AB} \times \vec{AC} = \langle -4, -6, -5 \rangle, \text{ point } A = (-1, 2, 1)$$

$$\Rightarrow P: -4(x+1) - 6(y-2) - 5(z-1) = 0 \Rightarrow 4x + 4y + 5z - 13 = 0$$

$$4x + 4y + 5z = 13$$

$$D = (-1, d, 2) \Rightarrow 4(-1) + 4d + 5(2) = 13 \Rightarrow 6d = 13 - 6$$

$$d = \frac{7}{6}$$

2. Given the planes  $P : x - 2y + 4z = 2$  and  $P' : x - y - 2z = 2$  and the point  $A : (2, -3, 4)$ .

(a) (8 points) Find the shortest distance from the point  $A$  to the plane  $P$ , and the closest point in plane  $P$  to the point  $A$ .

$$\text{distance} : d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1 \cdot 2 + (-2) \cdot (-3) + 4 \cdot 4 + (-2)|}{\sqrt{1^2 + (-2)^2 + 4^2}} = \frac{22}{\sqrt{21}}$$

closest point: We use the method of Lagrange multipliers to the function:

$$d(x, y, z) = (x-2)^2 + (y+3)^2 + (z-4)^2 \text{ subject to the constraint } g(x, y, z) = x - 2y + 4z - 2 = 0.$$

$$\nabla d = \lambda \nabla g.$$

$$\begin{aligned} & \langle 2x-4, 2y+6, 2z-8 \rangle = (\lambda, -2\lambda, 4\lambda) \\ \Rightarrow & \begin{cases} 2x-4 = \lambda \\ 2y+6 = -2\lambda \\ 2z-8 = 4\lambda \end{cases} \Rightarrow \begin{cases} x = \frac{\lambda+4}{2} \\ y = -3-\lambda \\ z = 2\lambda+4 \end{cases} \Rightarrow \begin{cases} x-2y+4z=2 \\ \frac{\lambda+4}{2} - 2(-3-\lambda) + 4(2\lambda+4) = 2 \\ \lambda = -\frac{44}{21} \end{cases} \Rightarrow \left( x_0, y_0, z_0 \right) = \left( \frac{20}{21}, \frac{-19}{21}, \frac{-4}{21} \right) \end{aligned}$$

(b) (7 points) Find the parametric equation of the line which is the intersection of  $P$  and  $P'$ .

We will find a point on the intersection of  $P$  and  $P'$  and the direction vector on the line.

Intersection point:

Since such a point must satisfy  $x - 2y + 4z = 2$  and  $x - y - 2z = 2$  taking  $x=0$  we get a point  $r_0 = (0, -\frac{3}{2}, -\frac{1}{4})$

direction vector:

Let  $\vec{n}_1, \vec{n}_2$  be the normal vectors of  $P$  and  $P'$  respectively. Then a direction vector is given by

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & -2 & 4 \\ 1 & -1 & -2 \end{vmatrix} = i \begin{vmatrix} -2 & 4 \\ -1 & -2 \end{vmatrix} - j \begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} + k \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = (8, 6, 1)$$

So the parametric equation of the line is

$$\begin{cases} x = 0 + 8t \\ y = -\frac{3}{2} + 6t \\ z = -\frac{1}{4} + t \end{cases}$$

3. (20 points) Evaluate;

(a)  $\frac{\partial z}{\partial x}$  for  $z = e^{xy} + x^2 + y$  at  $x = 2, y = 3$ .

$$\frac{\partial z}{\partial x} = y \cdot e^{xy} + 2x \Rightarrow \frac{\partial z}{\partial x} \Big|_{(2,3)} = 3 \cdot e^6 + 4.$$

(b)  $\frac{\partial z}{\partial x}$  for  $2 - xyz^2 + z = 0$  at  $x = 0, y = 1$ .

$$F(x, y, z) = 2 - xyz^2 + z \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \text{ Now that } F_x = -yz^2 \text{ and}$$

$$F_z = -2xyz + 1, \quad \frac{\partial z}{\partial x} = -\frac{-yz^2}{-2xyz+1} = \frac{y^2}{-2xyz+1}. \text{ At } x=0, y=1 \text{ we}$$

observe that  $z = -2$ . Thus  $\frac{\partial z}{\partial x} = \frac{1 \cdot (-2)^2}{1} = 4$  at  $x=0, y=1$ .

(c)  $\frac{dz}{dt}$  for  $z = (x^2 + y^2)(x + 1)$  when  $x = \cos t, y = e^t$  at  $t = 0$ .

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}. \quad \frac{\partial z}{\partial x} = 3x^2 + 2x + y^2, \quad \frac{dx}{dt} = -\sin t,$$

$$\frac{\partial z}{\partial y} = 2yx + 2y, \quad \frac{dy}{dt} = e^t \Rightarrow \frac{dz}{dt} = -(3x^2 + 2x + y^2)\sin t + 2y(x+1)e^t. \text{ Put}$$

ting  $t=0$  (hence  $x=y=1$ ) in place,  $\frac{dt}{dt} \Big|_{t=0} = 1$ .

(d)  $\frac{dz}{dx}$  for  $z = xe^{-xy}$  when  $y = e^{-x}$  at  $x = 0$ .

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}. \quad \text{Now that } \frac{\partial z}{\partial x} = e^{-xy} - xy \cdot e^{-xy}, \quad \frac{dx}{dx} = 1$$

$$\frac{\partial z}{\partial y} = -x^2 \cdot e^{-xy} \quad \text{and} \quad \frac{dy}{dx} = -e^{-x}, \quad \text{we get} \quad \frac{dz}{dx} = (1-xy) e^{-xy} + x^2 \cdot e^{-x} \cdot e^{-xy}$$

Putting  $x=0$  and  $y=1$  in place,  $\frac{dz}{dx} \Big|_{x=0} = 1$ .

4. (a) (12 points) Find an equation of the plane which is parallel to plane  $12x - 8y + 2z = 0$  and tangent to the graph of  $f(x, y) = x^2 + 2y^2$ .

Equation of the tangent plane to the graph of  $f$  at a point  $(x_0, y_0, z_0)$  is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)$$

Hence, the normal vector is given by

$$\mathbf{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle = \langle 2x_0, 4y_0, -1 \rangle$$

If two planes are parallel, then their normals are parallel:

$$\langle 2x_0, 4y_0, -1 \rangle = \lambda \langle 12, -8, 2 \rangle \Rightarrow \lambda = \frac{1}{2} \Rightarrow \begin{aligned} x_0 &= -3 \\ y_0 &= 1 \\ z_0 &= x_0^2 + 2y_0^2 = 11 \end{aligned}$$

The eqn. of the plane with  $\mathbf{n} = \langle -6, 4, -1 \rangle$  passing through  $(-3, 1, 11)$  is

$$-6(x+3) + 4(y-1) - (z-11) = 0$$

- (b) (8 points) Find the direction in which the function  $f(x, y, z) = xe^y + z^2$  increases most rapidly at the point  $(1, \ln 2, 0.5)$ . Then find the value of the derivative of  $f$  in that direction.

The function  $f$  increases most rapidly in the direction of  $\nabla f(1, \ln 2, 0.5)$

at the point  $(1, \ln 2, 0.5)$ .

$$\text{So } \vec{u} = \frac{\nabla f}{\|\nabla f\|}. \quad \nabla f = \langle e^y, xe^y, 2z \rangle$$

$$\nabla f(1, \ln 2, 0.5) = \langle 2, 2, 1 \rangle$$

$$\|\nabla f\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3.$$

$$\vec{u} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

The value of the derivative of  $f$  in the direction of  $\vec{u}$ :

$$D_u f = \nabla f \cdot \vec{u} = \langle 2, 2, 1 \rangle \cdot \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{1}{3}$$

$$= 3$$

5. (15 points) Given the function  $z = f(x, y) = x^3 + y^3 - 3xy$ .

(a) Find the critical points of the function.

$$\text{critical pt: } f_x = f_y = 0$$

$$f_x = 3x^2 - 3y = 0 \Rightarrow y = x^2 \quad > \quad x^4 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x-1)(x^2+x+1) = 0 \Rightarrow x=0 \quad x=1$$

$$f_y = 3y^2 - 3x = 0 \Rightarrow x = y^2$$

$$\begin{aligned} y = x^2 &\Rightarrow x=0 \Rightarrow y=0 \\ &\Rightarrow x=1 \Rightarrow y=1 \end{aligned}$$

$(0,0)$  &  $(1,1)$

critical pts

(b) Determine the nature of the critical points (local max, local min, or saddle).

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

$$\begin{aligned} f_{xx} &= 6x \\ f_{yy} &= 6y \end{aligned} \Rightarrow D = 6x \cdot 6y - (-1)^2 = 36xy - 9$$

$$f_{xy} = -3$$

$$\text{at } \underline{(0,0)} \quad D = 36 \cdot 0 - 9 = -9 < 0 \Rightarrow \underline{\text{saddle pt}}$$

$$\text{at } \underline{(1,1)} \quad D = 36 \cdot 1 \cdot 1 - 9 = 27 > 0 \quad \& \quad f_{xx} = 6x = 6 > 0 \Rightarrow \underline{\text{local min.}}$$

6. (15 points) Find the maximum and minimum values of the function  $f(x, y) = x^2 + 2y$  on the ellipse  $x^2 + 4y^2 = 1$ .

We will solve by Lagrange multipliers

$$f(x, y) = x^2 + 2y \quad g(x, y) = x^2 + 4y^2 - 1 = 0 \quad : \text{constraint}$$

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 2 \rangle = \lambda \langle 2x, 8y \rangle$$

$$2x = \lambda 2x \Rightarrow (\lambda - 1)2x = 0 \Rightarrow \lambda = 1 \text{ or } x = 0.$$

$$2 = \lambda 8y$$

$$\text{if } \underline{x=0} \text{ then } 0^2 + 4y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm \frac{1}{2}$$

$$(0, \frac{1}{2}), (0, -\frac{1}{2})$$

$$\text{if } \underline{\lambda=1} \text{ then } y = \frac{1}{4} \Rightarrow x^2 + 4 \cdot \frac{1}{16} = 1 \Rightarrow x^2 = \frac{12}{16} \Rightarrow x = \pm \frac{\sqrt{3}}{2}$$

$$(\frac{\sqrt{3}}{2}, \frac{1}{4}), (-\frac{\sqrt{3}}{2}, \frac{1}{4})$$

4 points:  $(0, \frac{1}{2}), (0, -\frac{1}{2}), (\frac{\sqrt{3}}{2}, \frac{1}{4}), (-\frac{\sqrt{3}}{2}, \frac{1}{4})$

$$f(0, \frac{1}{2}) = 0^2 + 2 \cdot \frac{1}{2} = 1$$

$$f(0, -\frac{1}{2}) = 0^2 + 2(-\frac{1}{2}) = -1 \rightarrow \text{min.}$$

$$f(\frac{\sqrt{3}}{2}, \frac{1}{4}) = \frac{3}{4} + \frac{2}{4} = \frac{5}{4} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{max}$$

$$f(-\frac{\sqrt{3}}{2}, \frac{1}{4}) = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$$