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Math 203 - Multivariable Calculus

Midterm 2      May 4, 2012

Duration: 90 minutes

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Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always explain your answers and show your work to receive full credit. If necessary, you can use the back of these pages, but make sure you have indicated doing so. Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.

Name, Surname: K E Y

Signature: \_\_\_\_\_

Section (Check One):

Section 1: A. Aşkar (Mon-Wed 9:30)

Section 2: A. Aşkar (Mon-Wed 12:30)

Section 3: B. Coşkunüzer (Tue-Thu 9:30)

Section 4: B. Coşkunüzer (Tue-Thu 12:30)

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PROBLEM	1	2	3	4	5	6	TOTAL
POINTS	18	12	20	15	20	15	100
SCORE							

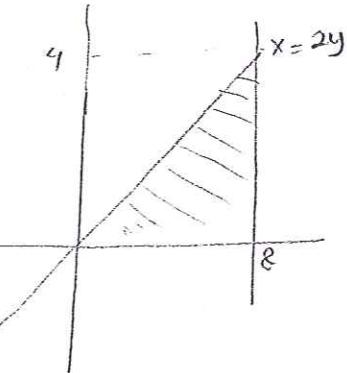
1. (18 points) (a) Find the volume of the solid lying under the paraboloid  $z = x^2 + y^2$  and above the rectangle  $R = [-2, 2] \times [-3, 3]$

$$\begin{aligned}
 V &= \int_{-3}^3 \int_{-2}^2 (x^2 + y^2) dx dy = \int_{-3}^3 \left[ \frac{x^3}{3} + xy^2 \right]_{-2}^2 dy = \int_{-3}^3 \left( \frac{8}{3} + 2y^2 - \left( -\frac{8}{3} - 2y^2 \right) \right) dy \\
 &= \int_{-3}^3 \frac{16}{3} + 4y^2 dy = \left[ \frac{16}{3}y + \frac{4}{3}y^3 \right]_{-3}^3 = \left( \frac{16}{3}, 3 + \frac{4}{3}, 3, 9 \right) \cdot 2 \\
 &= \underbrace{(16+36)}_{52} \cdot 2 = 104 //
 \end{aligned}$$

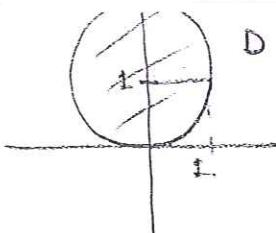
(b) Find volume of the solid bounded by the coordinate planes and the plane  $3x + 2y + z = 6$ .

$$\begin{aligned}
 z &= 6 - 3x - 2y \\
 \text{Diagram: } &\text{A 3D plot showing a triangular region in the } xy\text{-plane bounded by } x=0, y=0, \text{ and } 3x+2y=6. \text{ The vertical axis } z \text{ ranges from 0 to 6. The plane } z = 6 - 3x - 2y \text{ intersects the } z\text{-axis at } 6 \text{ and the } xy\text{-plane at } (2, 3). \\
 y &= \frac{6-3x}{2} \\
 V &= \int_0^2 \int_0^{3-\frac{3}{2}x} (6 - 3x - 2y) dy dx = \int_0^2 \left[ 6y - 3xy - y^2 \right]_0^{3-\frac{3}{2}x} dx \\
 &= \int_0^2 \left( 6 \left( 3 - \frac{3}{2}x \right) - 3x \left( 3 - \frac{3}{2}x \right) - \left( 3 - \frac{3}{2}x \right)^2 \right) dx \\
 &= \int_0^2 \left( 18 - 9x - 9x + \frac{9}{2}x^2 - 9 + 9x - \frac{9}{4}x^2 \right) dx = \int_0^2 \left( \frac{9}{4}x^2 - 9x + 9 \right) dx = \left[ \frac{3}{4}x^3 - \frac{9}{2}x^2 + 9x \right]_0^2 \\
 &= \frac{3}{4} \cdot 8^2 - \frac{9}{2} \cdot 4^2 + 9 \cdot 2 = 6 //
 \end{aligned}$$

2. (12 points) Evaluate the following integral. (Hint: Change the order of integration).



$$\begin{aligned}
 & \int_0^4 \int_{2y}^8 \sin x^2 \, dx \, dy \\
 &= \int_0^8 \int_0^{x/2} \sin x^2 \, dy \, dx = \int_0^8 y \sin x^2 \Big|_0^{x/2} \, dx \\
 &= \left( \frac{x}{2} \sin x^2 \Big|_0^8 \right) = -\frac{1}{4} \cos x^2 \Big|_0^8 = -\frac{1}{4} (\cos 64 - 1)
 \end{aligned}$$



$$z = 12 - 6x - 4y, \quad x^2 + (y-1)^2 = 1$$

3. (20 points) Use polar coordinates to find the volume of the solid under the plane  $6x + 4y + z = 12$  and above the disk with boundary circle  $x^2 + y^2 = 2y$ .

$$(\text{Half-angle formulas: } \sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \cos^2 x = \frac{1+\cos 2x}{2})$$

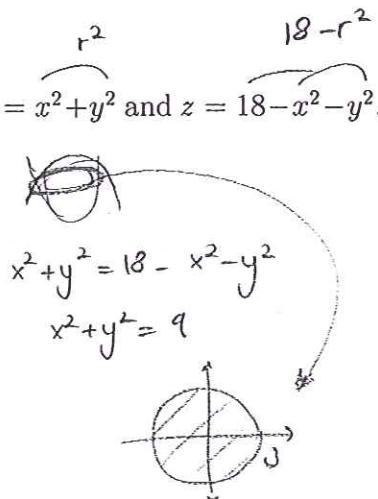
$$r^2 = 2r \sin \theta \Rightarrow r = 2 \sin \theta$$

$$\begin{aligned}
 V &= \iint_D (12 - 6x - 4y) dA = \int_0^\pi \int_0^{2\sin\theta} (12 - 6r\cos\theta - 4r\sin\theta) r dr d\theta \\
 &\stackrel{\text{Polar.}}{=} \int_0^\pi \int_0^{2\sin\theta} (12r - 6r^2\cos\theta - 4r^2\sin\theta) dr d\theta = \int_0^\pi \left[ 6r^2 - 2r^3\cos\theta - \frac{4}{3}r^3\sin\theta \right]_{0}^{2\sin\theta} d\theta \\
 &= \int_0^\pi \left[ 24\sin^2\theta - 16\sin^3\theta\cos\theta - \frac{4}{3}\cdot 8\sin^4\theta \right] d\theta = \int_0^\pi 24\left(\frac{1-\cos 2\theta}{2}\right) d\theta - 16 \int_0^\pi u^3 du \\
 &\quad - \frac{32}{3} \int_0^\pi \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \left( \int_0^\pi 12 - 12\cos 2\theta d\theta \right) - \frac{32}{3} \int_0^\pi \frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{(\cos 2\theta)^2}{4} d\theta \\
 &= 12\theta - 6\sin 2\theta \Big|_0^\pi - \frac{32}{3} \left[ \frac{\theta}{4} - \frac{\sin 2\theta}{4} \Big|_0^\pi + \int_0^\pi \frac{1+\cos 4\theta}{8} d\theta \right] \\
 &= 12\pi - \frac{32}{3} \left[ \frac{\pi}{4} + \frac{\pi}{8} + \frac{\sin 4\theta}{16} \Big|_0^\pi \right] = 12\pi - \frac{32}{3} \cdot \frac{3\pi}{8} = 12\pi - 4\pi = 8\pi
 \end{aligned}$$

4. (15 points) Let  $S$  be the solid enclosed by the paraboloids  $z = x^2 + y^2$  and  $z = 18 - x^2 - y^2$ .

(a) Write a triple integral to interpret the volume of  $S$ .

$$V = \int_0^{2\pi} \int_0^3 \int_{r^2}^{18-r^2} dz \, r \, dr \, d\theta$$



(b) Compute the volume of  $S$ .

$$V = \int_0^{2\pi} \int_0^3 (18 - 2r^2) r \, dr \, d\theta = \int_0^{2\pi} \int_0^3 18r - 2r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[ 9r^2 - \frac{2}{4} r^4 \right]_0^3 \, d\theta = \int_0^{2\pi} \left( 9.9 - \frac{1}{2}.81 \right) \, d\theta = \frac{81}{2} \int_0^{2\pi} \, d\theta = \underline{\underline{81\pi}}$$

5. (20 points) Evaluate the following line integral where  $C$  is the straight line segment from  $(-1, 0)$  to  $(5, 1)$ .

$$\int_C \underbrace{2x \sin y \, dx}_{P} + \underbrace{(x^2 \cos y - 3y^2) \, dy}_{Q}$$

$F = P \, dx + Q \, dy$  defined on  $\mathbb{R}^2$ , simply connected.

$$\frac{\partial P}{\partial y} = 2x \cos y = \frac{\partial Q}{\partial x} \Rightarrow F \text{ conservative} \Rightarrow F \text{ path independent}$$

↓

$$\exists f \text{ s.t } F = \nabla f \text{ Find potential.}$$

$$F = \langle 2x \sin y, x^2 \cos y - 3y^2 \rangle = \langle f_x, f_y \rangle.$$

$$f_x = 2x \sin y \Rightarrow f = x^2 \sin y + g(y)$$

$$f_y = x^2 \cos y + g'(y) = x^2 \cos y - 3y^2 \Rightarrow g(y) = -y^3 + c$$

$$\Rightarrow f(x, y) = x^2 \sin y - y^3 + c \quad \text{for some constant } c.$$

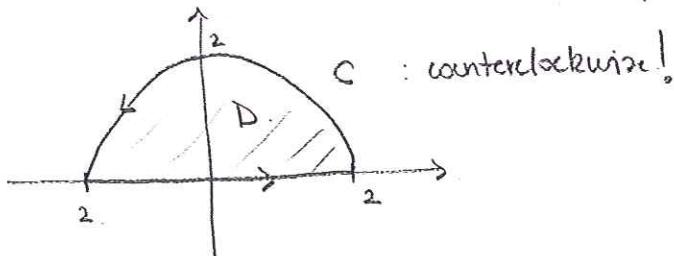
$$\text{By Fund. Thm of Line Integrals: } \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

$$\Rightarrow \int_C F \cdot dr = \int_C \nabla f \cdot dr = f(5, 1) - f(-1, 0) = 25 \sin(1) - 1 + c - 0 - c$$

$$= 25 \sin(1) - 1.$$

6. (15 points) Evaluate the following line integral where  $C$  consists of the line segment from  $(-2, 0)$  to  $(2, 0)$  and the top half of the circle  $x^2 + y^2 = 4$ .

$$\oint_C xy \, dx + 2x^2 \, dy$$



Use Green's Thm:  $\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$$= \iint_D (4x - x) \, dA = \int_0^\pi \int_0^2 3r \cos \theta \, r \, dr \, d\theta = \int_0^\pi r^3 \cos \theta \Big|_0^2 \, d\theta$$

polar

$$= \int_0^\pi 8 \cos \theta \, d\theta = 8 \sin \theta \Big|_0^\pi = 0 //$$