

1. (18 points) (a) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 1, \pi)$ for $3x + y \sin z + e^{xyz} = 1$

Take $\frac{\partial}{\partial x}$: $3 + y \cos z \frac{\partial z}{\partial x} + yz e^{xyz} + xy e^{xyz} \frac{\partial z}{\partial x} = 0$

At $(0, 1, \pi)$: $3 - \frac{\partial z}{\partial x} \Big|_{(0,1,\pi)} + \pi + 0 = 0 \rightarrow \boxed{\frac{\partial z}{\partial x} \Big|_{(0,1,\pi)} = 3 + \pi}$

Take $\frac{\partial}{\partial y}$: $0 + \sin z + y \cos z \frac{\partial z}{\partial y} + xz e^{xyz} + xy e^{xyz} \frac{\partial z}{\partial y} = 0$

At $(0, 1, \pi)$ $0 + 0 - \frac{\partial z}{\partial y} \Big|_{(0,1,\pi)} + 0 + 0 = 0 \rightarrow \boxed{\frac{\partial z}{\partial y} \Big|_{(0,1,\pi)} = 0}$

(b) Find the point(s) of the surface $z = xy + 5$ closest to the origin.

$$d^2 = x^2 + y^2 + z^2 = f(x, y, z) \quad g(x, y, z) = xy + 5 - z = 0$$

constraint

By Lagrange multiplier

$$\nabla f = \lambda \nabla g \rightarrow \begin{cases} 2x = \lambda y & (1) \\ 2y = \lambda x & (2) \\ 2z = -1 & (3) \end{cases} \quad xy + 5 - z = 0 \quad (4)$$

constraint

4 Eqs, 4 unknowns x, y, z, λ .

Solution

$$\text{Take } \frac{(1)}{(2)} \rightarrow \frac{x}{y} = \frac{y}{x} \rightarrow x^2 = y^2 \quad \begin{cases} y = x \\ y = -x \end{cases} \quad \begin{cases} x = c \\ x = 0 \end{cases} \quad \lambda \neq 2$$

$$\text{Use } y = x \text{ in (1)} \quad 2x = \lambda x \rightarrow (2-\lambda)x = 0 \quad \begin{cases} \lambda = 2 \\ x \neq 0 \end{cases}$$

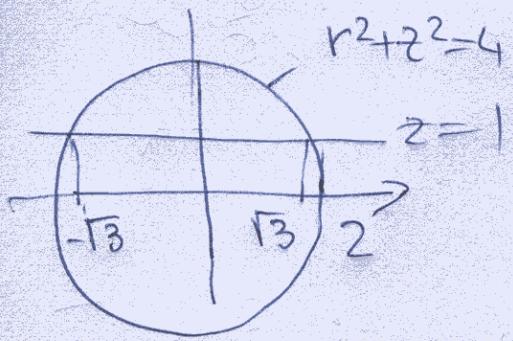
$$(1) \quad x = 0 = y \quad (1) \text{ & (2) satisfied from (4)} \quad 5 - z = 0 \rightarrow z = 5$$

$$(2) \quad x = y, x \neq 0 \quad \lambda = 2 \quad (1) \text{ & (2) satisfied from Eq 3: } z = -1$$

$$\text{Substitute in Eq (4)} \quad x^2 + 5 + 1 = 0 \quad \text{No solution}$$

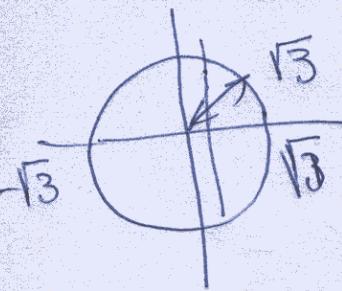
2. (18 points) Find the volumes of the following solids:

(a) The solid between the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$. (top part)

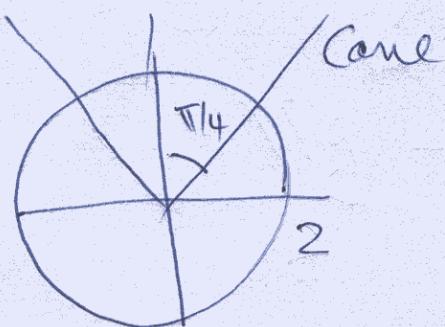


Side View

In the x,y plane



(b) The solid between the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$



Side view

Best choice: cylindrical coords

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \left[\int_{z=1}^{\sqrt{4-r^2}} dz \right] r dr d\theta$$

$$V = 2\pi \int_{r=0}^{\sqrt{3}} [\sqrt{4-r^2} - 1] r dr \quad \begin{aligned} &\text{let } 4-r^2=u \\ &-2rdr = du \end{aligned}$$

$$V = 2\pi \left(-\frac{1}{2}\right) \int_{u=1}^4 (u^{1/2} - 1) du = \pi \left(\frac{u^{3/2}}{3/2} - u\right) \Big|_{u=1}^4$$

$$\begin{aligned} &= \frac{2\pi}{3} (8 - 1) - \frac{3}{2}(3) \\ &= \frac{2\pi}{3} (7 - \frac{9}{2}) = \frac{5\pi}{3} \end{aligned}$$

Best choice:

spherical coords

$$V = \int_{\theta=0}^{2\pi} \int_{R=0}^2 \int_{\phi=0}^{\pi/4} \sin\phi R^2 dR d\phi d\theta$$

$$V = 2\pi \cdot \frac{R^3}{3} \Big|_0^2 \cdot (-\cos\phi) \Big|_0^{\pi/4} = \frac{16\pi}{3} \left(-\frac{\sqrt{2}}{2} + 1\right)$$

3. (17 points) Compute the following line integrals.

(a) $\int_C 2xy^3 dx + 3x^2y^2 dy$ where C is the curve $\mathbf{r}(t) = \langle \sin t, t \cos t \rangle$, $0 \leq t \leq \pi$.

A: $(0, 0)$

B: $(0, -\pi)$

(Open Curve.)

§ Direct integration: difficult & too long

§ Try to see if there is a potential.

$$P = 2xy^3 \quad Q = 3x^2y^2 \rightarrow \frac{\partial P}{\partial y} = 6xy^2 = \frac{\partial Q}{\partial x}$$

Conclusion: There is a potential f ,

$$\frac{\partial f}{\partial x} = 2xy^3 \rightarrow f = x^2y^3 + g(y); \quad \frac{\partial f}{\partial y} = 3x^2y + g' = 3x^2y^2 \rightarrow g = K \text{ const}$$

$$f = \left. x^2y^3 + g(y) \right|_{y=0}^{y=\pi} = \left. x^2y^3 + K \right|_{y=0}^{y=\pi} = 0 - 0 = 0$$

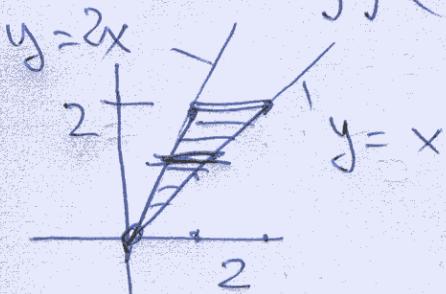
(b) $\int_C e^x dx + x^2y^2 dy$ where C is the triangle with vertices $(0, 0)$, $(1, 2)$ and $(2, 2)$. (Closed curve)

Too long to do by line integral.

Because of closed curve, use Green's Thm
(or Stokes in 2D)

$$\vec{F} = (e^x, x^2y^2)$$

$$I = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint (2xy^2 - 0) dx dy$$



$$= 2 \int_{y=0}^2 \left[\int_{x=y/2}^{x=y} x^2 y^2 dx \right] dy$$

$$I = 2 \int_{y=0}^2 (x^2 y^2 \Big|_{x=y/2}^{x=y}) dy = 2 \int_0^2 (y^4 - \frac{y^4}{4}) dy = 2 \int_0^2 \frac{3y^4}{4} dy = \frac{-2+3 \times 32}{4 \times 5} = \frac{48}{5}$$

4. (17 points) (a) Find a suitable $R(x, y, z)$ to make \mathbf{F} a conservative vector field.

$$\mathbf{F} = \langle 2xy + ze^{xz}, x^2 + 2yz^2, R(x, y, z) \rangle$$

\rightarrow potential if

$$\frac{\partial f}{\partial x} = 2xy + ze^{xz} \rightarrow f = x^2y + e^{xz} + g(y, z)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2yz^2 \rightarrow \frac{\partial g}{\partial y} = 2yz^2 \rightarrow g = y^2z^2 + h(z)$$

$$f = x^2y + e^{xz} + y^2z^2 + h(z)$$

$$\frac{\partial f}{\partial z} = xe^{xz} + 2y^2z + h'(z) = R(x, y, z)$$

- (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve $\mathbf{r}(t) = \langle t^2, \cos t, t \sin 2t \rangle$, $0 \leq t \leq \pi$.

Open curve: A: $(0, 1, 0)$, B: $(\pi^2, -1, 0)$

Because \vec{f} is conservative $I = f|_A^B$

$$I = \left. x^2y + e^{xz} + y^2z^2 + h(z) \right|_{0,1,0}^{\pi^2, -1, 0} = (-\pi^4 + 1 + 0 + h(0)) - (0 + 1 + 0 + h(0))$$

- (c) Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle on the xy -plane.

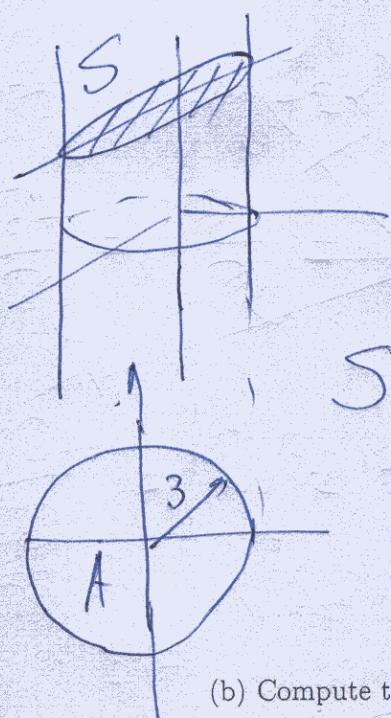
$$= -(\pi^4)^{\frac{1}{2}}$$

Because f is conservative

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

5. (15 points) Let S be the part of the plane $2x+5y+z=10$ inside the cylinder $x^2+y^2=9$.

(a) Find the area of S .



$$\text{Eq of the plane} \rightarrow z = f(x, y) = 10 - 2x - 5y$$

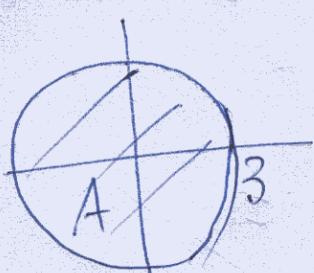
$$dS = \sqrt{1+f_x^2+f_y^2} dt = \sqrt{1+4+25} dA = \sqrt{30} dA.$$

$$S = \iint_A \sqrt{30} dA = \sqrt{30} \iint_A dA = \sqrt{30} \cdot \pi \cdot 3^2 = 9\sqrt{30}\pi$$

Circle
of Radius 3

(b) Compute the surface integral $\iint_S yz \, dS$

$$I = \iint_S yz \, dS = \iint_A y(10 - 2x - 5y) \sqrt{30} \, dA.$$



Use polar coords for an
easy integration

$$I = \sqrt{30} \int_0^{2\pi} r \sin \theta (10 - 2r \cos \theta - 5r \sin \theta) r dr d\theta$$

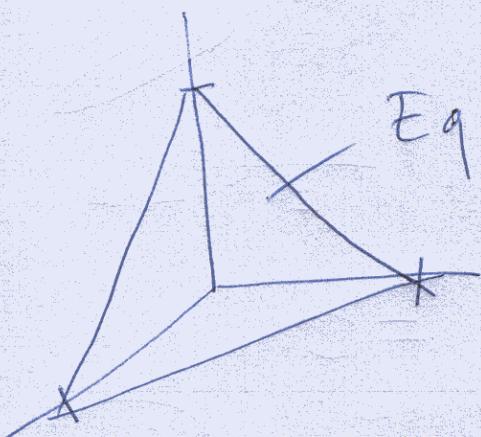
$$I = \sqrt{30} \int_{r=0}^3 \int_{\theta=0}^{\frac{\pi}{2}} (10r \sin \theta - 2r^2 \sin \theta \cos \theta - 5r^2 \sin^2 \theta) dr d\theta$$

$$= \sqrt{30} \int_{r=0}^3 (0 - 0 - 5r^3 \pi) dr = -\sqrt{30} 5\pi \left[\frac{r^4}{4} \right]_0^3 = -\frac{\sqrt{30} \times 5 \times \pi \times 81}{4}$$

6. (15 points) Calculate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{N} dS$ for

$$\mathbf{F} = \langle 3xy + z, y^2 - e^x, \sin xy \rangle$$

where S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.



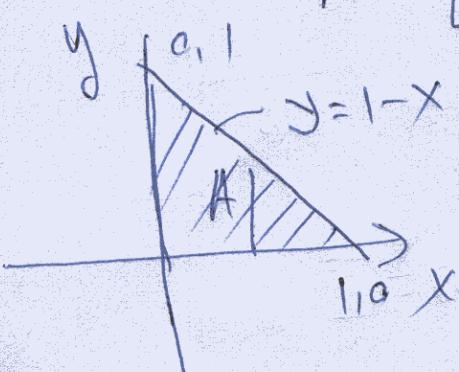
Eq of the surface : $x+y+z=1$

Direct surface integral
too long.

Use divergence Thm.

$$I = \iiint \nabla \cdot \mathbf{F} dV = \iiint (3y+2y+0) dV$$

$$I = 5 \iint_A \left[\int_{z=0}^{1-x-y} y \, dz \right] dx dy = 5 \iint_A y(1-x-y) dA$$



$$I = 5 \int_{x=0}^1 \left\{ \int_{y=0}^{1-x} \left[(1-x)y - \frac{y^2}{2} \right] dy \right\} dx$$

A in xy plane.

$$I = 5 \int_{x=0}^1 \left[\left(1-x \right) \frac{y^2}{2} - \frac{y^3}{6} \right]_{y=0}^{1-x} dx$$

$$I = 5 \int_0^1 \left[(1-x)^3 \left(\frac{3}{6} - \frac{1}{6} \right) \right] dx = \frac{5}{3} \left[-\frac{(1-x)^4}{4} \right]_0^1 = \frac{5}{3 \times 4} = \frac{5}{12}$$