KOÇ UNIVERSITY

MATH 203 - CALCULUS 2

Midterm II   Monday, May 13, 2015 starting at 19:00

Duration of Exam: 75 minutes

Each question case (i) or (ii) or (iii) is 7 points; 5 points are a bonus.

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers and show your work** to receive **full credit.** Use the back of these pages if necessary. **Print** (use **CAPITAL LETTERS**) **and sign your name, and indicate your section below.**

Name: ______________________

Surname: ______________________

Signature: ______________________

Section (Check One):

Section 1: Attila Aşkar T-Th (8:30) ___
Section 2: Attila Aşkar T-Th(11:30) ___
Section 3: Ayşe Soysal M-W (8:30) ___
Section 4: Ayşe Soysal M-W(10:00) ___

PROBLEM	POINTS	SCORE
1	14	
2	28	
3	21	
4	21	
5	21	
TOTAL	105	

1. (14 points)

(i) Given two numbers whose total is 100. Find the ~~largest value~~ least of the sum of the squares of these two numbers by Lagrange multipliers.

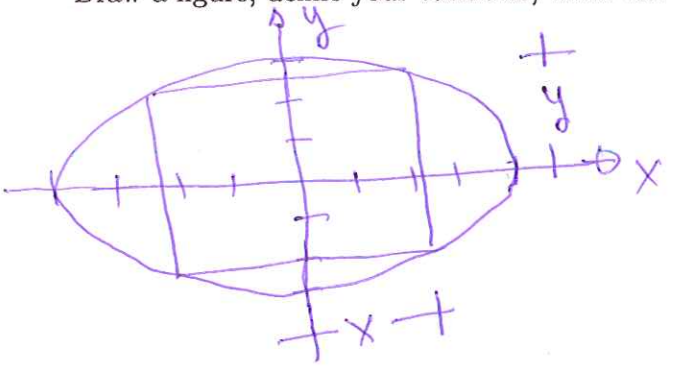
Let the numbers be x & y.

$f(x,y) = x^2 + y^2$; constraint: $x + y = 100$

$z = x^2 + y^2 - \lambda(x+y-100) \Rightarrow z_x = 2x - \lambda = 0 \quad z_y = 2y - \lambda$

$\Rightarrow x = y \rightarrow x + y = 2x = 100 \Rightarrow x = y = 50$

(ii) Find by the Lagrange multiplers method the rectangle of maximum area, which has its sides parallel to the coordinate axes and which is inscribed into the ellipse $x^2/16 + y^2/9 = 1$. Draw a figure, define your variables, write the constraint and function to be maximized.



Constraint: $\frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$

function to optimize:

$f(x,y) = xy$

By Lagrange multipliers:

$\boxed{xy_{max} = 2\sqrt{2} \cdot \frac{3}{\sqrt{2}} = 6}$

$z = xy - \lambda\left(\frac{x^2}{16} + \frac{y^2}{9} - 1\right)$

$z_x = y - \lambda \frac{x}{8} = 0 \qquad z_y = x - \lambda \frac{2y}{9} = 0$

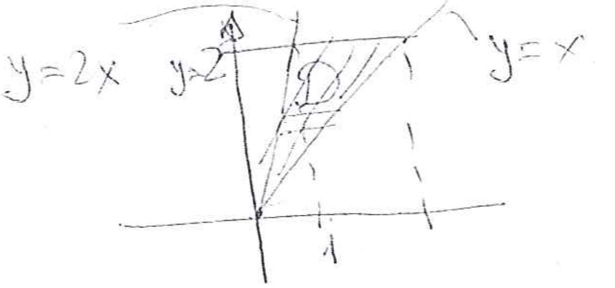
$y = \lambda \frac{x}{8} \qquad x = \lambda \frac{2y}{9}$ Take ratio to eliminate λ: $\frac{y}{x} = \frac{x \cdot 9}{8 \cdot 2y}$

$\frac{y^2}{9} = \frac{x^2}{16}$ Eq.(1) Eq(2): $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\Rightarrow 2\frac{x^2}{16} = 1 \quad x^2 = 8 \quad x = 2\sqrt{2} \Rightarrow y^2 = \frac{9}{2} \Rightarrow y = \frac{3}{\sqrt{2}}$

2. (28 points) The domain $\mathcal{D}$ is defined by the boundaries $\{y = x, \quad y = 2x, \quad y = 2\}$

(i) Draw the domain of integration in the xy plane;



$y = x \Rightarrow x = y$

$y = 2x \Rightarrow x = \frac{1}{2} y$

(ii) Write (Do not calculate) the double integral $\int\int xy \, dA$ over $\mathcal{D}$ with the integration over y first;

$$I = \int_{x=0}^{1} \left[\int_{y=x}^{y+2x} x y \, dy \right] dx + \int_{x=1}^{2} \left[\int_{y=x}^{2} x y \, dy \right] dx$$

(iii) Write (Do not calculate) the double integral $\int\int xy \, dA$ over $\mathcal{D}$ with the integration over x first;

$$I = \int_{y=0}^{2} \left[\int_{x=\frac{1}{2}y}^{x=y} x y \, dx \right] dy$$

(iv) Calculate the value of the integral defined in one of (ii) or (iii) of your choice.

By (iii) Easier: one integral

$$I = \int_{y=0}^{2} \left\{ \left[\frac{x^2}{2} y \right]_{x=\frac{1}{2}y}^{x=y} \right\} dy = \frac{1}{2} \int_{y=0}^{2} \left[y^3 - \frac{y^3}{4} \right] dy$$

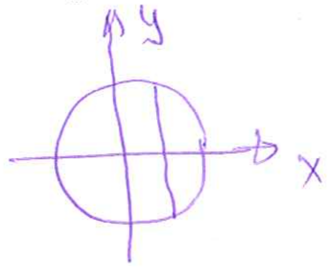
$$= \frac{1}{2} \cdot \frac{3}{4} \int_{0}^{2} y^3 dy = \frac{3}{8} \left(\frac{y^4}{4} \right) \Big|_0^2 = \frac{3}{32} \, 16$$

$$\boxed{I = \frac{3}{2}}$$

3. (21 points) Let $\mathcal{D}$ be the region inside the circle of radius a that is centered at the origin. Let $f(x,\ y) \ = \ h - \sqrt{x^2 + y^2}$, h is a constant.

(i) Write (Do not calculate) the double integral $\int\int f(x,\ y)\ dA$ over $\mathcal{D}$ with the integration over Cartesian coordinates x and y;

$$I = \int_{x=-a}^{a} \left\{ \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-y^2}} \left[h - \sqrt{x^2+y^2}\right] dy \right\} dx$$



(ii) Write (Do not calculate) the double integral $\int\int f(x,\ y)\ dA$ over $\mathcal{D}$ with the integration over polar coordinates r and θ ;

$$I = \int_{\theta=0}^{2\pi} \left[\int_{r=0}^{r=a} (h-r)\, r\, dr\right] d\theta \equiv \int_{\theta=0}^{\theta=2\pi} 1\, d\theta \cdot \int_{r=0}^{r=a} (h-r)\, r\, dr$$

(iii) Calculate the value of the integral defined in one of (i) or (ii) of your choice.

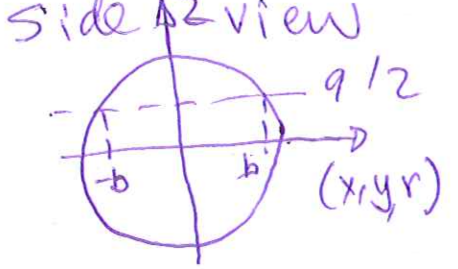
From (ii)

$$I = 2\pi \left[h\frac{r^2}{2} - \frac{r^3}{3}\right]_{r=0}^{r=a} = 2\pi \frac{a^2}{6}(3h - 2a)$$

4. (21 points) Let $\mathcal{D}$ be the solid region bounded above by the sphere of radius a and centered at the origin and below by the horizontal plane $z = a/2$.

(i) Write (Do not calculate) the triple integral for the volume of $\mathcal{D}$ in the Cartesian coordinates x, y and z;

Intersection $x^2+y^2+\left(\frac{a}{2}\right)^2=a^2 \Rightarrow x^2+y^2=\frac{3a^2}{4}$

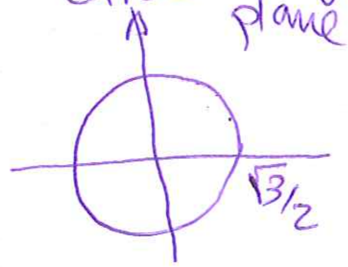


$$V=\int_{x=-\frac{\sqrt{3}}{2}a}^{x=\frac{\sqrt{3}}{2}a}\left\{\int_{y=-\sqrt{\frac{3}{4}a^2-x^2}}^{y=\sqrt{\frac{3}{4}a^2-x^2}}\left[\int_{z=a/2}^{z=\sqrt{a^2-x^2-y^2}} dz\right]dy\right\}dx$$

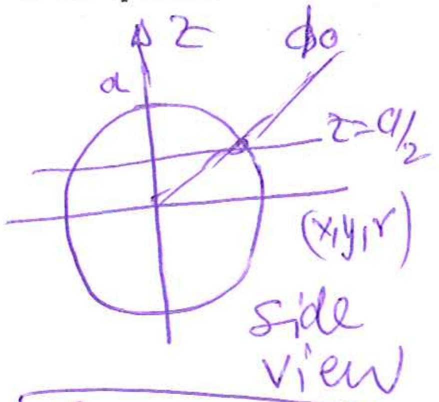
(ii) Write (Do not calculate) the triple integral for the volume of $\mathcal{D}$ in the cylindrical coordinates r, θ and z;

$$V=\int_{\theta=0}^{2\pi}\left\{\int_{r=0}^{\frac{\sqrt{3}}{2}a}\left[\int_{z=a/2}^{z=\sqrt{a^2-r^2}} dz\right]r\,dr\right\}d\theta$$

Circle in xy plane



(iii) Write (Do not calculate) the triple integral for the volume of $\mathcal{D}$ in the spherical coordinates R, ϕ and θ;



Equation of the horizontal plane in spherical coords:

$z=R\cos\phi \rightarrow \frac{a}{2}=R\cos\phi$

$R=\frac{a}{2\cos\phi}$

Angle ϕ_0: $R=a=\frac{a}{2\cos\phi_0} \rightarrow \cos\phi_0=\frac{1}{2}$

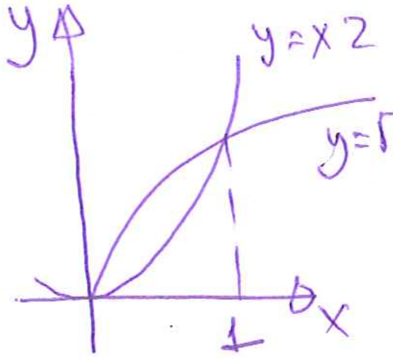
$\boxed{\text{Can use } R \text{ or } \rho}$

$\phi_0=\frac{\pi}{3}=60^\circ$

$$V=\int_{\theta=0}^{2\pi}\left\{\int_{\phi=0}^{\pi/3}\sin\phi\left[\int_{R=0}^{\frac{a}{2\cos\phi}}R^2\,dR\right]d\phi\right\}d\theta$$

5. (21 points) The domain $\mathcal{D}$ is defined by the parabolas $y = x^2$ and $y = \sqrt{x}$ as its boundaries. $f(x, y) = 1 - 4xy$ is a given function.

(i) Write (Do not calculate) the double integral $\int\int f(x, y)\, dA$ over $\mathcal{D}$ with an integration over the Cartesian coordinates x and y and draw the domain of integration in the xy plane;



Intersection: $x^2 = \sqrt{x} \Rightarrow x = 0, 1$

$$I = \int_{x=0}^{1} \left[\int_{y=x^2}^{y=\sqrt{x}} (1-4xy)\, dy \right] dx$$

(ii) Write (Do not calculate) the double integral $\int\int f(x, y)\, dA$ over $\mathcal{D}$ with the integration over the generalized coordinates u and v that are suggested by the boundary curves and draw the domain of integration in the uv plane; (FROM the BOOK: p. 832)

$u = \frac{x^2}{y}$ $\quad v = \frac{y^2}{x}$ $\quad J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{y^2}{x^2} \\ -\frac{x^2}{y^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$

$J = \frac{1}{J'} = \frac{1}{3}$

$I = \frac{1}{3} \int\int f(x,y)\, du\, dv$ (is enough for a full point) Boundaries are hard to define

(iii) Calculate the value of the integral defined in one of (i) or (ii).

Following (i)

$$I = \int_{x=0}^{1} \left[y - \frac{4xy^2}{2} \right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \left[(\sqrt{x} - 2x^2) - (x^2 - 2x^5) \right] dx$$

$$= \frac{x^{3/2}}{(3/2)} - \frac{3x^3}{3} + \frac{2x^6}{6} \Big|_{x=0}^{1} = \frac{2}{3} - 1 + \frac{1}{3} = 0$$