
KOÇ UNIVERSITY
MATH 203 - CALCULUS 2
Midterm I Monday, March 30, 2015
Duration of Exam: 75 minutes

Each question case (i) or (ii) or (iii) is 8 points; 4 points are a bonus.

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always explain your answers and show your work to receive full credit. Use the back of these pages if necessary. Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.

Name: _____

Surname: _____

Signature: _____

Section (Check One):

- | | |
|-------------------------------------|------|
| Section 1: Attila Aşkar T-Th (8:30) | ____ |
| Section 2: Attila Aşkar T-Th(11:30) | ____ |
| Section 3: Ayşe Soysal M-W (8:30) | ____ |
| Section 4: Ayşe Soysal M-W(10:00) | ____ |

PROBLEM	POINTS	SCORE
1	24	
2	24	
3	24	
4	16	
5	16	
TOTAL	104	

1. (24 points) Given the points $A : (1, 0, -1)$, $B : (0, 1, 1)$ and the vectors $\mathbf{u} = \mathbf{i} - \mathbf{k} \equiv (1, 0, -1)$, $\mathbf{v} = \mathbf{i} + \mathbf{j} \equiv (1, 1, 0)$:

(i) Find the vector \overrightarrow{AB} ;

$$\boxed{\overrightarrow{AB} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}}$$

(ii) Find the area of the triangle formed by the vectors \mathbf{u} and \mathbf{v} ;

$$\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\text{Area } \Delta = \frac{1}{2} |\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}}| = \boxed{\frac{1}{2} \sqrt{3}}$$

(iii) Find the angle between the vectors \mathbf{u} and \mathbf{v} .

$$\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{v}} = |\overrightarrow{\mathbf{u}}| |\overrightarrow{\mathbf{v}}| \cos \alpha$$

$$= \sqrt{2} \sqrt{2} \cos \alpha$$

$$\cos \alpha = \frac{1}{2}$$

$$\boxed{\alpha = \frac{\pi}{3}}$$

2. (24 points) Given the planes $\mathcal{P}_1: x - y + z = 5$ and $\mathcal{P}_2: x + ay + z = b$ where a and b are constants:

(i) Determine a and b for the two planes to be perpendicular (orthogonal) to each other;

The normals of the planes are

$$\vec{N}_1 = \vec{i} - \vec{j} + \vec{k}, \quad \vec{N}_2 = \vec{i} + a\vec{j} + \vec{k}$$

$$\vec{N}_1 \cdot \vec{N}_2 = 1 - a + 1 = 0$$

$$\therefore \boxed{a = 2, \text{ all } b.}$$

(ii) Determine a and b for the two planes to be (distinct) parallel to each other;

$$\vec{N}_1 \parallel \vec{N}_2 \Rightarrow \vec{N}_1 = r\vec{N}_2 \text{ for some } r \in \mathbb{R}$$

$$r(\vec{i} + a\vec{j} + \vec{k}) = \vec{i} - \vec{j} + \vec{k} \Rightarrow r=1 \text{ and}$$

$$\boxed{a = -1} ; \quad \boxed{b = \text{any real no } \neq 5}$$

(iii) Find the equation of the line of intersection of the two planes with $a = 2, b = 0$.

$$\begin{array}{r} x - y + z = 5 \\ -x + 2y + z = 0 \\ \hline -3y = 5 \end{array} \Rightarrow y = -\frac{5}{3}$$

Parametric eq's of the line of intersection:

$$\boxed{L: \left\{ \begin{array}{l} x = \frac{10}{3} - \frac{1}{3}t \\ y = -\frac{5}{3} \\ z = \frac{1}{3}t \end{array} \right.}$$

(Other parametrizations are possible)

3. (24 points) Given the surface defined by $z = x^2 - \frac{1}{3}x^3 + \frac{1}{3}y^3 + xy$, the point $A : (1, 1, 2)$ and the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} \equiv (1, -2)$ in the plane:

(i) Find the equation of the tangent plane to the surface at the point A :

$$\frac{\partial z}{\partial x} = 2x - x^2 + y \Rightarrow \left(\frac{\partial z}{\partial x}\right)_A = 2$$

$$\frac{\partial z}{\partial y} = y^2 + x \Rightarrow \left(\frac{\partial z}{\partial y}\right)_A = 2$$

$$2(x-1) + 2(y-1) = z-2$$

(ii) The directional derivative on the surface at the point A in the direction of the vector

$$(\vec{\nabla} z)_A = 2\mathbf{i} + 2\mathbf{j}$$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{5}} (\mathbf{i} - 2\mathbf{j})$$

$$\left(\frac{\partial z}{\partial s}\right)_A = (\vec{\nabla} z)_A \cdot \frac{\vec{u}}{|\vec{u}|} = \frac{2}{\sqrt{5}} (1-2) = \boxed{-\frac{2}{\sqrt{5}}}$$

(iii) Find the direction where the directional derivative to the surface at the point A is a maximum and the value of this maximum.

This is the direction of the gradient at A ,
 \therefore it is the direction of $\boxed{\mathbf{i} + \mathbf{j}}$

The maximum value of the directional derivative is

$$|(\vec{\nabla} z)_A| = \sqrt{4+4} = \boxed{2\sqrt{2}}$$

4. (16 points) Given the function $f(x, y) = x^3 + y^3 - 3xy + 18$:

(i) Find the location of the critical points;

$$\begin{aligned} f_x &= 3x^2 - 3y = 0 \Rightarrow y = x^2 \\ f_y &= 3y^2 - 3x = 0 \Rightarrow x^4 - x = 0 \\ &\Rightarrow x(x^3 - 1) = 0 \\ &\therefore x = 0 \text{ or } 1 \end{aligned}$$

Critical points $P_1(0,0), P_2(1,1)$

(ii) Determine the nature of the critical points.

Second derivative test:

$$f_{xx} = 6x, \quad f_{xy} = f_{yx} = -3, \quad f_{yy} = 6y$$

$$\Delta(P_1) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0 \Rightarrow P_1(0,0) \text{ is a saddle point}$$

$$\Delta(P_2) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0, \quad f_{xx}(P_2) > 0$$

$\therefore P_2(1,1)$ is a local minimum point.

5. (16 points) Given the function $e^{yz} - x^2z \ln y = 1$.

(i) Calculate $\frac{\partial z}{\partial y}$; $z = f(x, y)$ implicitly

$$ze^{yz} + ye^{yz} \frac{\partial z}{\partial y} - \frac{x^2 z}{y} - x^2 \ln y \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} (ye^{yz} - x^2 \ln y) = \frac{x^2 z}{y} - ze^{yz}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{x^2 z - ye^{yz}}{y^2 e^{yz} - x^2 y \ln y}}$$

(ii) Calculate the value of $\frac{\partial z}{\partial y}$ at the point A: (0, 1, 1).

$$\left(\frac{\partial z}{\partial y} \right)_A = \frac{0 - e}{e - 0} = \boxed{-1}$$