

1. (30 points)

(i) Find the vectors \mathbf{T} , \mathbf{N} , \mathbf{B} for $\mathbf{r}(s) = \left(3 \cos\left(\frac{s}{3}\right), 3 \sin\left(\frac{s}{3}\right), 1\right)$ for $s \in [-\pi, \pi]$.

$$\vec{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} = \left(-\sin\frac{s}{3}, \cos\frac{s}{3}, 0\right) \quad \text{Note: } \left|\frac{d\vec{\mathbf{r}}}{ds}\right| = 1 \Rightarrow |\vec{\mathbf{T}}| = 1$$

$$\frac{d\vec{\mathbf{T}}}{ds} = -\frac{1}{3}\left(\cos\frac{s}{3}, \sin\frac{s}{3}, 0\right) = K \vec{\mathbf{N}}$$

$$\vec{\mathbf{N}} = \left(-\cos\frac{s}{3}, \sin\frac{s}{3}, 0\right) \quad |\vec{\mathbf{N}}| = 1$$

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\sin\frac{s}{3} & \cos\frac{s}{3} & 0 \\ -\cos\frac{s}{3} & -\sin\frac{s}{3} & 0 \end{vmatrix} = \vec{\mathbf{i}} 0 - \vec{\mathbf{j}} 0 + \vec{\mathbf{k}}$$

$$= (0, 0, 1)$$

Note: for planar curves $\vec{\mathbf{B}} = \pm \vec{\mathbf{k}}$

(ii) Determine the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve given in (i).

$$K = \left|\frac{d\vec{\mathbf{T}}}{ds}\right| = \frac{1}{3} \quad (\text{from } \frac{d\vec{\mathbf{T}}}{ds} = K \vec{\mathbf{N}})$$

$$\cancel{\text{if}} \quad \frac{d\vec{\mathbf{B}}}{ds} = -\tau \vec{\mathbf{N}} \quad \frac{d\vec{\mathbf{B}}}{ds} = \frac{d\vec{\mathbf{k}}}{ds} = \vec{\mathbf{a}} \quad \rightarrow \tau = 0$$

Note: for planar curves $\tau = 0$

(iii) Given $\omega(s) = \tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s)$ for any curve. Show one of

$$\omega \times \mathbf{T} = \mathbf{T}' \quad \text{or} \quad \omega \times \mathbf{N} = \mathbf{N}' \quad \text{or} \quad \omega \times \mathbf{B} = \mathbf{B}'$$

\mathbf{N}
 $\mathbf{B} \times \mathbf{T}$

(Note. Pick one and prove; do not try to do all. The prime ' indicates the derivative with respect to s. The results are true with any curve, not specific to the curve given in (i).)
(You may continue on the back of the sheet.)

$$\vec{\omega} \times \vec{\mathbf{T}} = \tau \vec{\mathbf{T}} \times \vec{\mathbf{T}} + \kappa \vec{\mathbf{B}} \times \vec{\mathbf{T}} = \vec{\mathbf{0}} + K \vec{\mathbf{N}} = \frac{d\vec{\mathbf{T}}}{ds} \quad \text{From formulas on cover page}$$

$$\vec{\omega} \times \vec{\mathbf{N}} = \tau \vec{\mathbf{T}} \times \vec{\mathbf{N}} + \kappa \vec{\mathbf{B}} \times \vec{\mathbf{N}} = \tau \vec{\mathbf{B}} - K \vec{\mathbf{T}} = \frac{d\vec{\mathbf{N}}}{ds}$$

$$\vec{\omega} \times \vec{\mathbf{B}} = \tau \vec{\mathbf{T}} \times \vec{\mathbf{B}} + \kappa \vec{\mathbf{B}} \times \vec{\mathbf{B}} = \tau (-\vec{\mathbf{N}}) + \vec{\mathbf{a}} = -\tau \vec{\mathbf{N}} = \frac{d\vec{\mathbf{B}}}{ds}$$

2. (20 points) Find the distance between the plane $x + y = 3$ and the line given as:

$$x = t$$

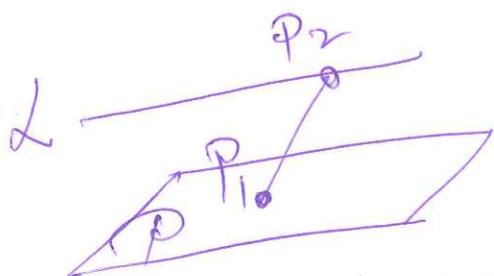
$$y = 1 - t$$

$$z = 2 + t$$

(Give a reason why you can calculate the distance between the given line and the plane.)

Reason 1: Substitute x, y, z into the Eq. of the plane: $t + 1 - t \stackrel{?}{=} 3 \rightarrow 1 \neq 3$

Conclusion: The line does not intersect the plane. Thus the line is parallel to the plane and does not lie in it.



A different reasoning: $\vec{n} = \text{normal vector of the plane} = (1, 1, 0)$
 $\vec{u} = \text{direction of the line} = \vec{u} = (1, -1, 1)$

$$\vec{n} \cdot \vec{u} = (1, 1, 0) \cdot (1, -1, 1) = 1 - 1 + 0 = 0 \quad \vec{n} \perp \vec{u}$$

Thus the line is parallel to the plane.

Take a point on the line: e.g. $P: (0, 1, 2)$

Substitute in the eq. of the plane: $0 + 1 \neq 3$
 Thus, the line is parallel to, but not in the plane.

So: We can talk about a distance.

$$\text{Take any } P_1 \in \mathcal{P}, P_2 \text{ ex: } \begin{array}{l} \text{eq. } P_1: (3, 0, 0) \\ P_2: (0, 1, 2) \end{array}$$

$$\vec{P_1 P_2} = (-3, 1, 2)$$

$$|d = \sqrt{2}|$$

$$d = \frac{|\vec{P_1 P_2} \cdot \vec{n}|}{|\vec{n}|} = \frac{|(-3, 1, 2) \cdot (1, 1, 0)|}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

3. (25 points) Given the function $f(x, y) = e^x y$

(i) Find an equation of the tangent plane at the point on the surface determined by $A : (1, 2)$ on the xy plane;

$$\text{Eq. of the tangent plane } z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(x_0, y_0) = e^x y \Big|_{\substack{x=1, \\ y=2}} = e \cdot 2$$

$$f_x = e^x y \quad f_x \Big|_{\substack{x=1, \\ y=2}} = e \cdot 2 \quad f_y = e^x \quad f_y \Big|_{\substack{y=2}} = e$$

$$z = 2e + 2e(x-1) + e(y-2) = e[2x+y+2-2-2] \Rightarrow z = e(2x+y-2)$$

(ii) Find a point $(x, y) \in \mathbb{R}^2$ so that the tangent plane of the surface $z = f(x, y)$ is parallel to the plane $ey - z = 203$.

We want to find x_0, y_0 such that
 $\vec{n}_T = t(0, 1, -1)$ $\left\{ \begin{array}{l} t: \text{a number to determine.} \\ (\text{From the given plane}) \end{array} \right.$
 Given plane

From the tangent plane: $\vec{n}_T = (-f_x^\circ, -f_y^\circ, 1)$
 (You can also take $\vec{n} = (f_x^\circ, f_y^\circ, -1)$)

$$\vec{n}_T = (e^{x_0} y_0, e^{x_0}, -1) = t(0, 1, -1)$$

To satisfy the equality: $-1 = -t \rightarrow \boxed{t=1}$

$$e^{x_0} y_0 = 0 \rightarrow y_0 = 0 \rightarrow e^{x_0} = t = 1 \rightarrow x_0 = 0$$

Ans: $P_0 : (0, 0, 0)$

4. (25 points) Given the points

$$A : (0, 0, 0) \quad B : (1, 2, 3) \quad C : (1, -2, 1) \quad D : (4, 1, -2)$$

(i) Find the angle between vectors \vec{AB} and \vec{AC} :

$$\vec{AB} = (1, 2, 3) - (0, 0, 0) = (1, 2, 3) \quad |\vec{AB}| = \sqrt{1+4+9} = \sqrt{14}$$

$$\vec{AC} = (1, -2, 1) - (0, 0, 0) = (1, -2, 1) \quad |\vec{AC}| = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{1-4+3}{\sqrt{14} \sqrt{6}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

(ii) Find the area of the triangle ABC :

$$\text{Area}(ABC) = \frac{|\vec{AB} \times \vec{AC}|}{2} \quad \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= (8, +2, -4)$$

ABC: Right-angle triangle
Area = $\frac{1}{2} |\vec{AB}| |\vec{AC}|$
 $= \frac{\sqrt{14} \sqrt{6}}{2} = 42$

$$|\vec{AB} \times \vec{AC}| = \sqrt{64+4+16} = \sqrt{84}$$

$$\text{Area } ABC = \frac{\sqrt{84}}{2} = \sqrt{\frac{84}{4}} = \sqrt{21}$$

(iii) Find the volume of the parallelopiped based on the vectors \vec{AB} , \vec{AC} and \vec{AD} .

$$\text{Volume} : (\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 4 & 1 & -2 \end{vmatrix} = 1 \cdot (4-1) - 2 \cdot (-2-4) + 3 \cdot (1+8) - 2 + 12 + 27 = 42$$

Volume = 42

Note: $\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = (4, 1, -2) \cdot (8, 2, -4) = 32 + 2 + 8 = 42$

Also Note: $\vec{AD} \cdot \vec{AB} = 0 \quad \vec{AB} \cdot \vec{AC} = 0 \quad = 42$

(Right-angle Prism)