

1. (25 points) Let $f(x, y) = xy^2 + \frac{1}{6}x^3 - x + 1$.

(a) Find and classify all the critical points of f .

Critical points: $f_x = y^2 + \frac{1}{2}x^2 - 1 = 0$ $f_y = 2xy = 0$

$$x=0 \rightarrow y=\pm 1 ; \quad y=0 \rightarrow x=\pm \sqrt{2}$$

$$CP_1: (0, 1) \quad CP_2: (0, -1) \quad CP_3: (\sqrt{2}, 0) \quad CP_4: (-\sqrt{2}, 0)$$

$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} x & 2y \\ 2y & 2x \end{vmatrix} = 2x^2 - 4y^2$$

At CP_1 : $H = -4 < 0$; Saddlept; At CP_2 : $H = -4 < 0$ Saddle pt

At CP_3 : $H = 4 > 0$ Also $f_{xx}|_{\sqrt{2}} = \sqrt{2} > 0 \rightarrow$ Minimum

At CP_4 : $H = 4 > 0$ Also $f_{xx}|_{-\sqrt{2}} = -\sqrt{2} < 0 \rightarrow$ Maximum

(b) Find the extremum of f subject to the condition $\frac{x^2}{4} + y^2 = 1$ by a method other than substitution.

$$f = xy^2 + \frac{1}{6}x^3 - x + 1 \quad g = \frac{x^2}{4} + y^2 - 1 = 0$$

$$3 \text{ Eqs } f_x = \lambda g_x \rightarrow y^2 + \frac{1}{2}x^2 - 1 = \lambda \frac{x}{2} \quad (1)$$

$$f_y = \lambda g_y \rightarrow 2xy = \lambda 2y \quad (2)$$

$$g = 0 \rightarrow \frac{x^2}{4} + y^2 - 1 = 0 \quad (3)$$

Solving: Eliminate λ by ratio of (1) & (2)

$$\frac{y^2 + \frac{1}{2}x^2 - 1}{2xy} = \frac{\frac{x}{2}}{2y} \rightarrow 2y(y^2 + \frac{1}{2}x^2 - 1) = x^2y$$

Substitute $y^2 = 1 - \frac{x^2}{4}$ (or x^2)

$$y(2y^2 + x^2 - 2 - x^2) = 0 \rightarrow y(y^2 - 1) = 0$$

Extremum points: $\begin{cases} y=0 \rightarrow \text{From } g=0 \rightarrow x=\pm 2 \\ y=\pm 1 \rightarrow \text{From } g=0 \rightarrow x=0 \end{cases}$

2. (25 points) Evaluate the following double integrals:

$$(a) \int_{-1}^1 \int_{-2}^2 (xy^2 + y) dy dx$$

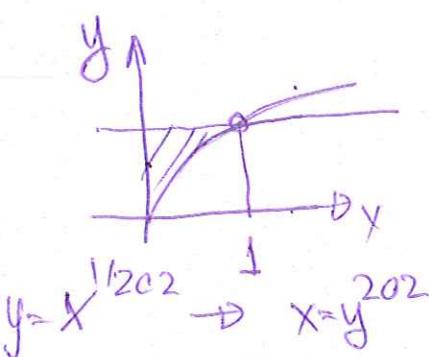
$$I = \int_{-1}^1 \left[\frac{xy^3}{3} + \frac{y^2}{2} \right]_{-2}^2 dx = \int_{-1}^1 \left[\frac{x(8) - x(-8)}{3} + \frac{4-4}{2} \right] dx \\ = \frac{16}{3} \int_{-1}^1 x dx = 0$$

You CAN ALSO FIND THIS RESULT BY SYMMETRY

$$(b) \int_0^1 \int_{x^{1/202}}^1 \sin(y^{203}) dy dx$$

Cannot do this integral in this order.

Changing the order: $y=1$



$$I = \int_{y=0}^1 \int_{x=0}^{y^{202}} \sin(y^{203}) dx dy = \int_{y=0}^1 \sin(y^{203}) y^{202} dy$$

$$\text{let } y^{203} = u \quad 203y^{202} dy = du \rightarrow I = \int_{u=0}^1 \frac{\sin u}{203} du$$

$$(c) \iint_D \sin(x^2 + y^2) \cos(x^2 + y^2) dA \quad \text{where } D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}.$$

Let's go to polar coord's: $x^2 + y^2 = r^2$

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^2 (\sin r^2)(\cos r^2) r dr d\theta$$

$$\theta=0 \quad r=0 \quad 4$$

$$I = 2\pi \int_{u=0}^4 \sin u \cos u \frac{du}{2} = \frac{\pi}{2} \int_{u=0}^4 \sin 2u du \\ = \frac{\pi}{2} \left(\frac{-\cos 2u}{2} \right) \Big|_0^4 = \frac{\pi}{4} (1 - \cos 8)$$

$$\text{let } r^2 = u \quad 2r dr = du$$

$$4$$

$$\frac{\pi}{2} \int_{u=0}^4 \sin 2u du$$

3. (25 points) Let $f(x, y, z) = x^y + y^z + z^x$.

(a) Compute ∇f .

Given formula: $\frac{d}{dx} a^x = a^x (\ln a)$
 Also $\frac{d}{dx} x^b = b x^{b-1}$

$$\frac{\partial f}{\partial x} = y x^{y-1} + 0 + z^x \ln z$$

$$\frac{\partial f}{\partial y} = x^y \ln x + z y^{z-1} + 0$$

$$\frac{\partial f}{\partial z} = 0 + y^z \ln z + x z^{x-1}$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

(b) Compute the directional derivative $D_{\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} f(1, 1, 1)$.

Note: $\ln x \Big|_{x=1} = 0$

$$\vec{u} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad |\vec{u}| = 1$$

$$\nabla f \Big|_{(1,1,1)} = (1, 1, 1)$$

$$D_u f \Big|_{(1,1,1)} = \vec{u} \cdot \nabla f \Big|_{(1,1,1)} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot (1, 1, 1) \\ = 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

4. (25 points) Suppose that x, y, z satisfies the equation $x^2y + y^2z + xz^2 = 203$.

(Note: You must determine the dependent and independent variables from the questions)

(a) Compute $\frac{\partial y}{\partial z}$.

Independent variables $z \neq x$

Method 1:

Implicit Deriv.
Formula

$$\frac{\partial y}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} = -\frac{y^2 + 2xz}{x^2 + 2yz}$$

$$y = f(x, z)$$

$$f(x, y, z) = x^2y + y^2z + xz^2 - 203$$

Method 2:

Directly differentiating \Rightarrow

$F = 203$ w.r.t z
keeping x fixed $y = f(x, z)$

(b) Compute $\frac{\partial x}{\partial z}$.

$$x^2y + 2yz \frac{\partial y}{\partial z} + y^2 + 2xz = 0$$

$$(x^2 + 2yz) \frac{\partial y}{\partial z} + (y^2 + 2xz) = 0$$

$\frac{\partial y}{\partial z}$ = same as above.

Independent variables $y, f(z)$

Method 1: Implicit Deriv formula. $x = f(y, z)$

$$F(x, y, z) = x^2y + y^2z + xz^2 - 203$$

$$\frac{\partial x}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial x}} = -\frac{y^2 + 2xz}{2xy + z^2}$$

Method 2: Directly differentiating $F(x, y, z) = 203$
w.r.t z ,
keeping y fixed

$$\frac{\partial}{\partial z} (x^2y + y^2z + xz^2) = 2x \frac{\partial x}{\partial z} y + y^2 + 2xz + xz^2 = 0$$

$$\frac{\partial x}{\partial z} (2xy + z^2) + (y^2 + 2xz) = 0$$

$\frac{\partial x}{\partial z}$ = same as above