
Math 204 - Differential Equations

Final Exam January 7, 2012

Duration: 150 minutes

Instructions: Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers and show your work** to receive full credit. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: _____

Section (Check One):

Section 1: E. Ceyhan (Mon-Wed 12:30)

Section 2: E. Ceyhan (Mon-Wed 15:30)

Section 3: B. Coşkunüzer (Tue-Thu 12:30)

Section 4: B. Coşkunüzer (Tue-Thu 15:30)

PROBLEM	1	2	3	4	5	6	7	TOTAL
POINTS	15	12	14	14	16	15	14	100
SCORE								

1. (15 points) Solve the following differential equations.

$$(a) \frac{dy}{dx} = \frac{x^4 - 2y}{x}$$

$$y' = x^3 - 2\frac{y}{x} \Rightarrow y' + \frac{2}{x}y = x^3, \mu(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

$$\Rightarrow x^2y' + 2xy = x^5$$

$$\Rightarrow (x^2 \cdot y)' = x^5$$

$$\Rightarrow x^2 \cdot y = \frac{1}{6}x^6 + c, c \in \mathbb{R}$$

$$\Rightarrow \boxed{y(x) = \frac{1}{6}x^4 + c \cdot \frac{1}{x^2}}, c \in \mathbb{R}$$

$$(b) (2y + 3x)dx = -x dy$$

$$\underbrace{(2y + 3x)}_M dx + \underbrace{x \cdot dy}_N = 0$$

$$\frac{My - Nx}{N} = \frac{2-1}{x} = \frac{1}{x} \Rightarrow \frac{d\mu}{dx} = \frac{1}{x} \cdot \mu$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{dx}{x}$$

$$\Rightarrow \mu = x$$

$$\Rightarrow (2xy + 3x^2)dx + x^2 dy = 0$$

(exact) $\Rightarrow \Psi(x, y) = c$

$$\Psi_x = 2xy + 3x^2 \Rightarrow \Psi = x^2y + x^3 + g(y)$$

$$\Rightarrow \Psi_y = x^2 + g'(y) = x^2 \Rightarrow g'(y) = 0$$

$$\Rightarrow \Psi(x, y) = x^2y + x^3$$

$$\Rightarrow \text{solution: } \boxed{x^2y + x^3 = c}, c \in \mathbb{R}$$

2. (12 points) Find the general solution of the given differential equation.

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$$

We will use the Variation of Parameters method.

Solutions of homogenous equation; $y'' + 4y' + 4y = 0$:

$$y_1(t) = e^{-2t}$$

$$y_2(t) = te^{-2t}$$

$$Y = u_1 \cdot y_1 + u_2 \cdot y_2, \text{ where}$$

$$u_1 = - \int \frac{y_2 \cdot g}{W} \quad \text{and} \quad u_2 = \int \frac{y_1 \cdot g}{W}$$

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{2t} - 2te^{2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}$$

$$u_1 = - \int \frac{te^{-2t} \cdot t^{-2} \cdot e^{-2t}}{e^{-4t}} = - \int \frac{1}{t} = -\ln t + C$$

$$u_2 = \int \frac{e^{-2t} \cdot t^{-2} \cdot e^{-2t}}{e^{-4t}} = \int \frac{1}{t^2} = -\frac{1}{t} + C$$

$$\Rightarrow Y_p = -\ln t \cdot e^{-2t} + \left(-\frac{1}{t}\right) \cdot t \cdot e^{-2t}$$

$$\Rightarrow \boxed{y(t) = c_1 \cdot e^{-2t} + c_2 \cdot t \cdot e^{-2t} - \ln t \cdot e^{-2t}}$$

3. (14 points) Determine the general solution of the given differential equations.

(a) $y''' - y' = 2 \cos t$

$$r^3 - r = 0 \Rightarrow r(r^2 - 1) = 0, \quad r_1 = 0, \quad r_2 = 1, \quad r_3 = -1$$

$$y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$$

$$y_p = a \cos t + b \sin t$$

$$y_p' = -a \sin t + b \cos t$$

$$y_p'' = -a \cos t - b \sin t$$

$$y_p''' = a \sin t - b \cos t$$

$$y_p(t) = -\sin t$$

$$y_p''' - y_p' = 2 \cos t$$

$$(a \sin t - b \cos t) - (-a \sin t + b \cos t) = 2 \cos t$$

$$(2a) \sin t - (2b) \cos t = 2 \cos t$$

$$a = 0 \quad b = -1$$

$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} - \sin t$$

(b) $y^{(4)} - 4y'' = t^2 + e^t$

$$r^4 - 4r^2 = 0 \Rightarrow r^2(r^2 - 4) = 0, \quad r_1 = r_2 = 0, \quad r_3 = 2, \quad r_4 = -2$$

$$y_c(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t}$$

$$y^{(4)} - 4y'' = t^2 \Rightarrow y_{p_1} = (At^2 + Bt + C)t^2$$

$$y_{p_1}' = 4At^3 + 3Bt^2 + 2Ct$$

$$y_{p_1}'' = 12At^2 + 6Bt + 2C$$

$$y_{p_1}''' = 24At + 6B$$

$$y_{p_1}^{(4)} = 24A$$

$$y_{p_1}^{(4)} - 4y_{p_1}'' = t^2$$

$$24A - 48At^2 - 24Bt - 8C = t^2$$

$$A = -1/48 \quad B = 0$$

$$C = -1/16$$

$$y_{p_1}(t) = -\frac{1}{48}t^4 - \frac{1}{16}t^2$$

$$y^{(4)} - 4y'' = e^t \Rightarrow y_{p_2} = D e^t = y_{p_2}' = y_{p_2}'' = y_{p_2}''' = y_{p_2}^{(4)}$$

$$\Rightarrow y_{p_2}^{(4)} - 4y_{p_2}'' = A e^t, \quad A e^t - 4A e^t = e^t \Rightarrow A = 1/3$$

$$y_{p_2}(t) = \frac{1}{3}e^t \quad \text{So, the general solution is:}$$

$$y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} - \frac{1}{48}t^4 - \frac{1}{16}t^2 - \frac{1}{3}e^t$$

4. (14 points) Find the power series solution of the following differential equation about $x_0 = 0$. Determine also the radius of convergence of the series solution you found.

$$y'' + xy' + 2y = 0$$

$$\left. \begin{array}{l} y = \sum_{n=0}^{\infty} a_n \cdot (x-0)^n = \sum_{n=0}^{\infty} a_n \cdot x^n \\ y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} \\ y'' = \sum_{n=2}^{\infty} n(n-1) \cdot a_n \cdot x^{n-2} \end{array} \right\} \text{Substituting in the equation,}$$

$$\sum_{n=2}^{\infty} n(n-1) \cdot a_n \cdot x^{n-2} + x \cdot \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} + 2 \cdot \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) \cdot a_{n+2} \cdot x^n + \sum_{n=1}^{\infty} n \cdot a_n \cdot x^n + \sum_{n=0}^{\infty} 2a_n \cdot x^n = 0$$

↓
can be changed
to $n=0$

$$\Rightarrow \sum_{n=0}^{\infty} ((n+2)(n+1) a_{n+2} + n a_n + 2a_n) x^n = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + (n+2) a_n = 0 \quad \Rightarrow \quad a_{n+2} = \frac{-a_n}{n+1}$$

$$a_2 = -a_0$$

$$a_4 = -\frac{a_2}{3} = \frac{a_0}{3}$$

$$a_6 = -\frac{a_4}{5} = -\frac{a_0}{5 \cdot 3}$$

⋮

$$a_{2n} = \frac{(-1)^n \cdot a_0}{(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1}$$

$$a_3 = -\frac{a_1}{2}$$

$$a_5 = -\frac{a_3}{4} = \frac{a_1}{4 \cdot 2}$$

$$a_7 = -\frac{a_5}{6} = -\frac{a_1}{6 \cdot 4 \cdot 2}$$

⋮

$$a_{2n+1} = \frac{(-1)^n \cdot a_1}{2n \cdot (2n-2) \dots 6 \cdot 4 \cdot 2}$$

$$y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n = a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n-1) \dots 5 \cdot 3 \cdot 1} + a_1 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2n \dots 4 \cdot 2}$$

RADIUS OF CONVERGENCE:

$$y'' + x \cdot y' + 2y = 0$$

↓ ↓ ↓
1 x 2 ⇒ analytic $\Rightarrow r = \infty$

5. (16 points)

(a) Express the solution of the given initial value problem in terms of a convolution integral.

$$y'' + 2y' + y = g(t), \quad y(0) = 2, \quad y'(0) = -3$$

Laplace transform of the D.E. together with the initial conditions:

$$s^2 Y(s) - 2s + 3 + 2s Y(s) - 4 + Y(s) = G(s)$$

$$\Rightarrow (s+1)^2 \cdot Y(s) = 2s + 1 + G(s) \Rightarrow Y(s) = \frac{2s+1}{(s+1)^2} + \frac{G(s)}{(s+1)^2}$$

By partial fractions

$$\frac{2s+1}{(s+1)^2} = \frac{2}{s+1} - \frac{1}{(s+1)^2}, \quad \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} = 2e^{-t} \text{ and}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t \cdot e^{-t}. \text{ By the convolution thm, we have:}$$

$$y(t) = e^{-t} \cdot (2-t) + \int_0^t (t-\tau) \cdot e^{-(t-\tau)} \cdot g(\tau) d\tau$$

(b) Use Laplace transform to solve the following initial value problem.

$$y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1$$

Laplace transform of the D.E. is

$$s^2 Y(s) - s \cdot y(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = e^{-2s}/s, \text{ with}$$

$$\text{the initial conditions: } s^2 Y(s) - 1 + 3s Y(s) + 2Y(s) = e^{-2s}/s.$$

$$(s^2 + 3s + 2) Y(s) = 1 + \frac{e^{-2s}}{s} \Rightarrow Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s \cdot (s^2 + 3s + 2)}$$

Using partial fractions,

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2} \text{ and } \frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2} \left(\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1} \right)$$

Taking the inverse transform term-by-term,

$$y(t) = e^{-t} - e^{-2t} + \left(\frac{1}{2} - e^{-t-2} + \frac{1}{2} e^{-2(t+2)} \right) u_2(t)$$

6. (15 points) Find the general solution of the following systems of equations (if the general solution is complex-valued, express it in terms of real-valued functions).

$$(a) \mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

Setting $\mathbf{x} = \xi e^{rt}$, we get $\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = r^2 - 1 = 0, \Rightarrow r_1 = 1, r_2 = -1 \text{ are eigenvalues.}$$

For $r=1$, system reduces to $\xi_1 = \xi_2$, so $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $r=-1$, system reduces to $3\xi_1 = \xi_2$, so $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Since eigenvalues are real & distinct, general solution is

$$X(t) = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

$$(b) \mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$$

Setting $\mathbf{x} = \xi e^{rt}$, we get $\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i \text{ are eigenvalues.}$$

For $r = -1-i$, system reduces to $(2+i)\xi_1 - \xi_2 = 0$, so $\xi^{(1)} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$ and

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

Hence one of the complex-valued solutions is

$$x^{(1)} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(-1-i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} (\cos t - i \sin t)$$

$$= e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \cdot e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

so the general (real-valued) solution is

$$X(t) = c_1 \cdot e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \cdot e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

7. (14 points) Find the general solution of the following system of equations.

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$$

Setting $\mathbf{x} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot e^{rt}$ $\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = r^2 = 0$
 $\Rightarrow r_{1,2} = 0$ (double root)

With $r=0$, the system reduces to

$$4\xi_1 - 2\xi_2 = 0 \Rightarrow 2\xi_1 = \xi_2 \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So, one solution is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (a constant vector)

In order to find a second linearly independent solution, we need to solve

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow 4\eta_1 - 2\eta_2 = 1$$

$$\begin{aligned} \text{Letting } \eta_1 = k, \quad \eta_2 = 2k - 1/2 &\Rightarrow \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} k \\ 2k - 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + k \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \end{aligned}$$

$k \in \mathbb{R}$.

Hence, the general solution becomes

$$\boxed{\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right)}, \quad c_1, c_2 \in \mathbb{R}$$