

MATH 204 - DIFFERENTIAL EQUATIONS: MIDTERM SOLUTIONS.

(1) Consider the following initial value problem (IVP).

$$y' + 3y = te^{-3t}, \quad y(1) = 0$$

(a) Find the solution to the above IVP.

SOLUTION: In this equation, $p(t) = 3$ and $g(t) = te^{-3t}$ for the form: $y' + p(t)y = g(t)$.

So the integrating factor is $\mu(t) = \exp\left(\int 3dt\right) = e^{3t}$.

Multiplying both sides by e^{3t} , the equation becomes $(e^{3t}y)' = t$.

Integrating both sides, we get $e^{3t}y = \frac{t^2}{2} + C$. So $y(t) = \frac{t^2}{2}e^{-3t} + Ce^{-3t}$.

With I.C.: $y(1) = 0$, $0 = \frac{1}{2}e^{-3} + C \cdot e^{-3} \Rightarrow C = -\frac{1}{2}$.

So the solution is

$$\boxed{y(t) = \left(\frac{t^2-1}{2}\right)e^{-3t}}$$

(b) Is this solution unique? If so, explain why and state where the solution exists.

If not, explain why not.

SOLUTION: In this differential equation, $p(t)$ and $g(t)$ are continuous for all \mathbb{R} . So, there exists a unique solution $y = \phi(t)$ over all \mathbb{R} . That is, the solution in (a) is the unique solution.

(c) Describe the behavior of the solution you found in part (a) for large t .

SOLUTION: As $t \rightarrow \infty$, the solution $y(t) \rightarrow 0$, since e^{-3t} dominates the solution and is going to zero.

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(2) Consider the differential equation $y' + ky = \frac{x - e^{-x}}{y + e^y}$ where k is a constant.

(a) For which value of k is the above differential equation separable?

SOLUTION: The differential equation is separable only for $k=0$.

(b) For the value you found for k in part (a), find the solution to the differential equation.

SOLUTION: $y' = \frac{x - e^{-x}}{y + e^y} \Rightarrow (y + e^y)dy = (x - e^{-x})dx$

Integrating both sides, we get $\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C_1$.

that is,
$$\boxed{y^2 + 2e^y = x^2 + 2e^{-x} + C.}$$

(c) Specify the solution that satisfies the initial condition $y(0)=1$.

SOLUTION: With $y(0)=1$, we have $1+2e=0+2+C$. So $C=2e-1$.

So the specific solution is

$$\boxed{y^2 + 2e^y = x^2 + 2e^{-x} + 2e - 1.}$$

(3a) Without solving the problem, give the longest interval in which the solution of the IVP is certain to exist:

$$t(t-5)y' + y = 0, \quad y(2) = 1.$$

SOLUTION: Rewrite the differential equation as $y' + \frac{1}{t(t-5)}y = 0$, so $p(t) = \frac{1}{t(t-5)}$. $p(t)$ is continuous everywhere except $t=0$ and $t=5$. Initial condition is at $t=2$, so a theorem in class guarantees the existence of a unique solution on $(0, 5)$.

(3b) Determine whether the following differential equation is exact. If it is exact, find the

solution.

$$(2x+3) + (2y-2)y' = 0.$$

SOLUTION: Here $M(x,y) = 2x+3$, and $N(x,y) = 2y-2$.

So $M_y = 0$ and $N_x = 0$, hence, $M_y = N_x$. So the DE is exact.

- Integrate M with respect to x while holding y constant, we get $\varphi(x,y) = x^2 + 3x + h(y)$
- Differentiating φ with respect to y , we get $\varphi_y = h'(y) = N = 2y - 2$. So we must have $h'(y) = 2y - 2$, so $h(y) = y^2 - 2y$.

• So $\boxed{\varphi(x,y) = x^2 + 3x + y^2 - 2y}$ and the solution is defined implicitly as $\varphi(x,y) = c$.

(3c) Determine whether the following differential equation is exact. If it is exact, find the solution.

$$\frac{dy}{dx} = -\frac{ax-by}{bx-cy}$$

where a and c are arbitrary constants and b is a nonzero constant.

SOLUTION: $y \cdot (bx-cy) = - (ax-by)$, so $(ax-by) + (bx-cy)y' = 0$.

So $M(x,y) = ax-by$ and $N(x,y) = bx-cy$, then $M_y = -b$ and $N_x = b$. Since $b \neq 0$, $M_y \neq N_x$, hence the DE is not exact.

(u) Find the general solution for the following differential equations.

(a) $y'' + 5y' + 4y = 0$

(b) $4y'' - 4y' + y = 0$

(c) $y'' - 4y' + 13y = 0$

SOLUTIONS

(a) The characteristic equation of DE is $r^2 + 5r + 4 = 0$. Solving this equation, we have $r_1 = -4$ and $r_2 = -1$. So $y_1(t) = e^{-4t}$ and $y_2(t) = e^{-t}$.

$$\Rightarrow \boxed{y(t) = c_1 e^{-4t} + c_2 e^{-t}}$$

(b) The characteristic equation of DE is $4r^2 - 4r + 1 = 0$. Solving this equation, we have $r_1 = r_2 = \frac{1}{2}$. So $y_1(t) = e^{\frac{1}{2}t}$ and $y_2(t) = t e^{\frac{1}{2}t}$.

$$\Rightarrow \boxed{y(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}}$$

(c) The characteristic equation of DE is $r^2 - 4r + 13 = 0$. Solving this equation, we have $r_1 = 2 + 3i$ and $r_2 = 2 - 3i$. So $y_1 = e^{2t} \cos 3t$ and $y_2 = e^{2t} \sin 3t$.

$$\Rightarrow \boxed{y(t) = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t.}$$

(5a) Find the general solution for the following differential equation.

$$y'' - 5y' + 6y = 2e^{5t} + t.$$

SOLUTION: Let us first find the solution to homogenous equation $y'' - 5y' + 6y = 0$.

The characteristic equation of this DE is $r^2 - 5r + 6 = 0$, so $r_1 = 2$ and $r_2 = 3$. So

$$y_1(t) = e^{2t} \text{ and } y_2(t) = e^{3t}.$$

$$\bullet \quad y'' - 5y' + 6y = 2e^{5t} \Rightarrow Y = Ae^{5t} \Rightarrow 25Ae^{5t} - 5.5Ae^{5t} + 6Ae^{5t} = 2e^{5t}$$

$$\Rightarrow A = \frac{1}{3}.$$

$$\bullet \quad y'' - 5y' + 6y = t \Rightarrow Y = At + B \Rightarrow 0 - 5A + 6At + 6B = t$$

$$\Rightarrow A = \frac{1}{6} \text{ and } B = \frac{5}{36}.$$

$$\Rightarrow y_p = \frac{e^{5t}}{3} + \frac{t}{6} + \frac{5}{36}$$

$$\Rightarrow \boxed{y = c_1 e^{2t} + c_2 e^{3t} + \frac{e^{5t}}{3} + \frac{t}{6} + \frac{5}{36}.}$$

(5b) Find the solution for the following IVP.

$$y'' - 3y' - 4y = 2e^{4t} \quad y(0) = 1 \text{ and } y'(0) = 5.$$

SOLUTION: Let us first solve the homogenous equation $y'' - 3y' - 4y = 0$. The char-eq. is $r^2 - 3r - 4 = 0$. So $r_1 = 4$ and $r_2 = -1$. So $y_1 = e^{4t}$ and $y_2 = e^{-t}$.

$$\Rightarrow y'' - 3y' - 4y = 2e^{4t} \Rightarrow Y = Ate^{4t} \Rightarrow Y' = (A + 4At)e^{4t}, \quad \text{...}$$

$$\Rightarrow Y'' = (4A + 4A + 16At)e^{4t}$$

$$\Rightarrow (16At + 8A)e^{4t} - 3(4At + A)e^{4t} - 4Ate^{4t} = 2e^{4t}.$$

$$\Rightarrow (8A - 3A)e^{4t} = 2e^{4t} \Rightarrow A = \frac{2}{5} \Rightarrow Y_p = \frac{2}{5}te^{4t}.$$

$$\text{The general solution is } y = c_1 e^{4t} + c_2 e^{-t} + \frac{2}{5}te^{4t}.$$

$$\text{Since } y(0) = 1, \quad 1 = c_1 + c_2 + 0, \text{ that is } c_1 + c_2 = 1. \quad (*)$$

Since $y'(t) = 4c_1 e^{4t} - c_2 e^{-t} + \frac{2}{5} e^{4t} + \frac{8}{5} t e^{4t}$, and $y'(0) = 5$, we have

$$y'(0) = 5 = 4c_1 - c_2 + \frac{2}{5}, \text{ so } 4c_1 - c_2 = \frac{23}{5}. \quad (**)$$

$$\text{By } (*) \text{ and } (**), \quad c_1 = \frac{28}{25} \text{ and } c_2 = \frac{-3}{25}.$$

So the solution of DE is

$$y = \frac{28}{25} e^{4t} + \frac{(-3)}{25} e^{-t} + \frac{2}{5} t e^{4t}.$$

(6) Given that $y_1(t) = t^2$ is a solution for the differential equation

$$t^2 y'' - 4t y' + 6y = 0$$

for $t > 0$. Find the general solution.

SOLUTION: We shall use the method of "reduction of order".

$$\begin{aligned} \bullet \quad y_2(t) = v_1 t^2. \Rightarrow y_2' = v_1' t^2 + 2v_1. \Rightarrow y_2'' = v_1'' t^2 + 2t v_1' + 2v_1. \\ = (t^2 v_1'' + 4t v_1' + 2v_1) \end{aligned}$$

Put y_2 in the DE to get:

$$t^2(t^2 v'' + 4t v' + 2v) - 4t \cdot (t^2 v' + 2t v) + 6t^2 v = 0.$$

$$\Rightarrow t^4 v'' + \underbrace{(4t^3 - 4t^3)}_0 v' + \underbrace{(2t^2 - 8t^2 + 6t^2)}_0 v = 0$$

$$\Rightarrow t^4 v'' = 0 \quad , \quad v = c_1 t + c_2 \quad \text{say} \quad v = t \Rightarrow y_2(t) = t^3.$$

$$y = c_1 t^2 + c_2 t^3$$