

## Math 204 - Differential Equations

Midterm 2      December 9, 2011

**Duration: 90 minutes**

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**Instructions:** Calculators are not allowed. No books, no notes, no questions, and no talking allowed. You must always **explain your answers** and **show your work** to receive full credit. If necessary, you can use the back of these pages, but make sure you have indicated doing so. **Print (i.e., use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name, Surname: KEY

Signature: Key

**Section (Check One):**

Section 1: E. Ceyhan (Mon-Wed 12:30)

Section 2: E. Ceyhan (Mon-Wed 15:30)

Section 3: B. Coşkunüzer (Tue-Thu 12:30)

Section 4: B. Coşkunüzer (Tue-Thu 15:30)

PROBLEM	1	2	3	4	5	6	TOTAL
POINTS	10	20	15	15	18	22	100
SCORE							

1. (10 points) Find the particular solution of the following differential equation

$$t^2 y'' - 2y = 3t^2 - 1 \quad t > 0$$

where  $y_1(t) = t^2$  and  $y_2(t) = t^{-1}$  are the solutions for the corresponding homogeneous equation.

Write the D.E. as  $y'' - \frac{2}{t^2}y = \frac{3t^2-1}{t^2}$

so, here  $p(t) = 0$ ,  $q(t) = -2/t^2$  and  $g(t) = (3t^2-1)/t^2$

Wronskian of the given solutions is;

$$W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3$$

By V.O.P. method, the particular soln. is

$$Y(t) = u_1(t) y_1(t) + u_2(t) \cdot y_2(t),$$

where  $u_1(t) = - \int \frac{t^{-1} \cdot (3t^2-1)}{t^2(-3)} dt = \frac{1}{3} \int \left( \frac{3}{t} - \frac{1}{t^3} \right) dt$   
 $= \ln t + t^{-2}/6,$

and

$$u_2(t) = \int \frac{t^2(3t^2-1)}{t^2(-3)} dt = -\frac{1}{3} \int (3t^2-1) dt = \frac{t-t^3}{3}$$

So  $Y(t) = t^2 \cdot \ln t + \frac{1}{6} + \frac{1}{3} - \frac{t^2}{3} = t^2 \left( \ln t - \frac{1}{3} \right) + \frac{1}{2}$

$$Y(t) = t^2 \cdot \ln t + \frac{1}{2}$$

since  $t^2$  is included in  $y_2(t)$ .

2. Find the general solutions of the following differential equations.

(a) (10 points)  $y''' - 4y' = 2t + 5 \sin(t)$

Char. eq'n is  $r^3 - 4r = r(r-2)(r+2) = 0 \Rightarrow r_1=0, r_2=2, r_3=-2$

So,  $y_c(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}$ , let  $\underbrace{g_1(t)}_{g_1(t)} = \underbrace{2t}_{g_2(t)} + 5 \sin t$

and  $Y(t) = Y_1(t) + Y_2(t)$ , for  $g_1(t) = 2t$ ,

assume  $Y_1(t) = At^2 + Bt + C$  (since  $At+B$  fails)

$$Y_1'(t) = 2At + B$$

$$Y_1''(t) = 2A, Y_1'''(t) = 0 \Rightarrow -4(2At + B) = -8At - 4B = 2t \Rightarrow B = 0$$

$$A = -1/4$$

$$B = 0$$

$$Y_1(t) = -\frac{t^2}{4}$$

For  $g_2(t) = 5 \sin t$ , assume  $Y_2(t) = D \cos t + E \sin t$

$$Y_2'(t) = -D \sin t + E \cos t, Y_2''(t) = -D \cos t - E \sin t$$

$$Y_2'''(t) = D \sin t - E \cos t$$

$$\Rightarrow 5D \sin t - 5E \cos t = 5 \sin t \Rightarrow D = 1, E = 0$$

$$\Rightarrow Y_2(t) = \cos t$$

So general sol'n is:

$$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t} - \frac{t^2}{4} + \cos t.$$

(b) (10 points)  $y^{(4)} - y = 3t + e^t$

Char. eq'n is  $r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0 \Rightarrow r_1=1, r_2=-1, r_{3,4}=\pm i$

So  $y_c(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$ , let  $\underbrace{g_1(t)}_{g_1(t)} = 3t + e^t$

Assume  $Y_1(t) = At + B \Rightarrow Y_1' = A, Y_1'' = Y_1''' = Y_1^{(4)} = 0$

$$\Rightarrow -At - B = 3t \Rightarrow A = -3, B = 0 \Rightarrow Y_1(t) = -3t$$

Assume  $Y_2(t) = (C e^t) \cdot t \Rightarrow Y_2'(t) = C(1+t)e^t$

$$Y_2''(t) = C(2+t)e^t$$

$$Y_2'''(t) = C(3+t)e^t$$

$$Y_2^{(4)}(t) = C(4+t)e^t$$

$$\Rightarrow C(4+t)e^t - C(1+t)e^t = 4Ce^t = e^t \Rightarrow C = 1/4 \Rightarrow Y_2(t) = \frac{1}{4}t e^t$$

So,  $Y(t) = -3t + \frac{1}{4}t e^t$ . General sol'n becomes:

$$y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t - 3t + \frac{1}{4}t e^t$$

3. (a) (5 points) Determine whether the following three functions are linearly independent or not (explain your answer):

$$f_1(t) = 5, \quad f_2(t) = \sin^2(t), \quad f_3(t) = \cos 2t$$

$$f_3(t) = \cos 2t = 2\cos^2 t - 1 = 1 - 2\sin^2 t, \text{ so } f_3(t) = \frac{1}{5} f_1(t) - 2f_2(t),$$

so these functions are not linearly independent.

OR

Show  $W = \begin{vmatrix} 5 & \sin^2 t & \cos 2t \\ 0 & 2\sin t \cos t & -2\sin 2t \\ 0 & 2\cos^2 t - 2\sin t & -4\cos 2t \end{vmatrix} = 0.$

(b) (10 points) Given that  $x$ ,  $x^2$ , and  $1/x$  are the solutions of the homogeneous equation corresponding to

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4 \quad x > 0$$

determine the particular solution.

(Wronskian of the given solutions are  $W = \begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} = 6/x$ .)

Write the eq'n as  $y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = 2x, \quad x > 0$

Compute  $W_1(x) = \begin{vmatrix} 0 & x^2 & x^{-1} \\ 0 & 2x & -x^{-2} \\ 1 & 2 & 2x^{-3} \end{vmatrix} = -3$

$$W_2(x) = \begin{vmatrix} x & 0 & x^{-1} \\ 1 & 0 & -x^{-2} \\ 0 & 1 & -x^{-3} \end{vmatrix} = 2/x \quad \text{and} \quad W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = x^2$$

Hence  $u_1'(x) = \frac{2x(-3)}{6/x} = -x^2 \quad u_2'(x) = \frac{2x(2/x)}{6/x} = 2x/3$

$$u_3'(x) = \frac{2x(x^2)}{6/x} = x^4/3$$

So,  $u_1(x) = \int -x^2 dx = -x^3/3, \quad u_2(x) = \int \frac{2x}{3} dx = \frac{x^2}{3}, \quad u_3(x) = \int \frac{x^4}{3} dx = \frac{x^5}{15}$

$$\Rightarrow Y(t) = x \cdot \left(\frac{-x^3}{3}\right) + x^2 \cdot \left(\frac{x^2}{3}\right) + \frac{1}{x} \cdot \left(\frac{x^5}{15}\right) = \frac{x^4}{15}$$

4. (15 points) Find the power series solution of the following differential equation about  $x_0 = 0$ . Determine also the radius of convergence of the series solution you found.

$$y'' - xy' - y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substituting,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \cdot \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n - a_n] x^n = 0 \Rightarrow (n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{a_n}{n+2} \Rightarrow a_2 = \frac{a_0}{2} \quad a_3 = \frac{a_1}{3}$$

$$a_4 = \frac{a_2}{4} = \frac{a_0}{4 \cdot 2}$$

$$a_5 = \frac{a_3}{5} = \frac{a_1}{5 \cdot 3}$$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2}$$

$$a_7 = \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 3}$$

$$\Rightarrow a_{2n} = \frac{a_0}{2n(2n-2) \dots 4 \cdot 2}$$

$$a_{2n-1} = \frac{a_1}{(2n+1)(2n-1) \dots 5 \cdot 3}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$= a_0 \cdot \sum_{n=0}^{\infty} \frac{1}{2n \dots 6 \cdot 4 \cdot 2} x^{2n} + a_1 \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1) \dots 7 \cdot 5 \cdot 3} x^{2n+1}$$

Radius of Convergence:  $y'' - xy' - y = 0 \Rightarrow p = -\frac{x}{1}, q = -1 \quad (y'' + p(x)y' + q(x)y = 0)$

By the theorem in class,  $p, q$  are analytic everywhere,  $R_p = \infty$ ,  $R_q = \infty$ ,  
then  $R_{\text{solution}} \geq \frac{R_p}{R_q} > \infty$ , so the radius of convergence of solution is  $\infty$ .

5. (a) (8 points) Determine a lower bound for the radius of convergence of the series solution about  $x_0 = -1$  for the following differential equation.

$$(x^2 - 4x + 5)y'' - xy' + 5y = 0$$

$$y'' - \frac{x}{x^2 - 4x + 5} y' + \frac{5}{x^2 - 4x + 5} y = 0, \quad \rho_{\text{soln}} > \text{distance}(x_0, \text{roots of the polynomial})$$

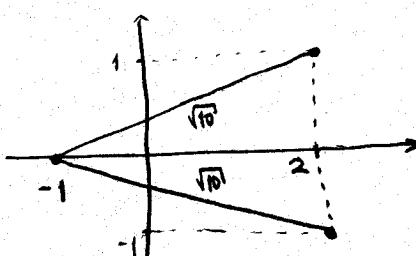
$$x^2 - 4x + 5 = 0$$

$$x^2 - 4x + 4 + 1 = 0$$

$$(x-2)^2 + 1 = 0$$

$$r_1 = 2+i$$

$$r_2 = 2-i$$



$$\downarrow \\ (x^2 - 4x + 5)$$

$$\rho_{\text{soln}} \geq \sqrt{10}$$

- (b) (10 points) Find the solution of the following initial value problem.

$$x^2 y'' + 7xy' + 9y = 0, \quad y(1) = 3, \quad y'(1) = 5$$

Given equation is an Euler equation:

$$y = x^r \Rightarrow (r^2 + br + g) x^r = 0$$

$$(r+3)^2 = 0$$

$$r_1 = r_2 = -3 \Rightarrow$$

$$y_1 = x^{-3} \quad y_2 = \ln x \cdot x^{-3}$$

$$y(x) = c_1 x^{-3} + c_2 \ln x \cdot x^{-3}$$

$$y(1) = c_1 \cdot 1 + c_2 \cdot 0 = 3 \Rightarrow c_1 = 3$$

$$y'(x) = -3c_1 x^{-4} - 3c_2 \ln x \cdot x^{-4} + c_2 x^{-4}$$

$$y'(1) = -3c_1 + c_2 \cdot 0 + c_2 = 5$$

$$-3 \cdot 3 + c_2 = 5 \Rightarrow c_2 = 14$$

$$y(x) = 3 \cdot x^{-3} + 14 \cdot \ln x \cdot x^{-3}$$

### LAPLACE TRANSFORM TABLE:

$$\mathcal{L}\{1\} = \frac{1}{s} \quad s > 0 \quad | \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad s > a \quad | \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad s > 0 \quad | \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad s > 0 \quad | \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2} \quad s > a \quad | \quad \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad s > a$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$$

6. (a) (7 points) Find the inverse Laplace transform of  $F(s)$ , i.e.,  $f(t)$  for which  $\mathcal{L}\{f(t)\} = F(s)$ .

$$F(s) = \frac{4s+3}{(s+2)(s^2+2s+5)}$$

$$\frac{4s+3}{(s+2)(s^2+2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5} \Rightarrow A(s^2+2s+5) + (Bs+C)(s+2) = 4s+3$$

$$(A+B)s^2 + (2A+2B+C)s + (5A+2C) = 4s+3$$

$$\left. \begin{array}{l} A+B=0 \\ 2A+2B+C=4 \\ 5A+2C=3 \end{array} \right\} C=4 \left. \begin{array}{l} A=-1 \\ B=1 \end{array} \right\} \Rightarrow F(s) = \frac{-1}{s+2} + \frac{s+4}{(s+1)^2+4} = \frac{-1}{s+2} + \frac{(s+1)}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}$$

$$\Rightarrow y(t) = -e^{-2t} + e^{-t} \cdot \cos 2t + \frac{3}{2} e^{-t} \cdot \sin 2t$$

- (b) (15 points) Use Laplace transform to solve the following initial value problem.

$$y'' - 2y' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0$$

Taking the Laplace Transform,  $[s^2 F(s) - s \cdot y(0) - y'(0)] - 2[s \cdot F(s) - y(0)] + 2 \cdot F(s) = \frac{5}{s^2+1}$

$$(s^2 - 2s + 2) F(s) - s + 2 = \frac{5}{s^2+1} \Rightarrow F(s) = \frac{s-2}{s^2-2s+2} + \frac{s}{(s^2+1)(s^2-2s+2)}$$

$$\frac{s}{(s^2+1)(s^2-2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2} \Rightarrow (As+B)(s^2-2s+2) + (Cs+D)(s^2+1) = s$$

$$(A+C)s^3 + (-2A+B+D)s^2 + (2A-2B+C)s + (2B+D) = s$$

$$\left. \begin{array}{l} A+C=0 \\ -2A+B+D=0 \\ 2A-2B+C=1 \\ 2B+D=0 \end{array} \right\} \left. \begin{array}{l} A=1/5 \\ B=-2/5 \\ C=-1/5 \\ D=4/5 \end{array} \right\} F(s) = \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1} + \frac{1}{5} \cdot \frac{s-2}{s^2+1} + \frac{1}{5} \cdot \frac{-s+4}{(s-1)^2+1}$$

$$\Rightarrow y(t) = e^t \cdot \cos t - e^t \cdot \sin t + \frac{1}{5} \cos t - \frac{2}{5} \sin t - \frac{1}{5} e^t \cdot \cos t + \frac{3}{5} e^t \cdot \sin t$$

$$y(t) = \frac{1}{5} \cos t - \frac{2}{5} \sin t - \frac{2}{5} e^t \cdot \sin t + \frac{4}{5} e^t \cdot \cos t$$