# Fall 2017 Math 211 MT 1 Solutions

**Question 1: (25 Points)** Two cards are drawn in succession without replacement from a deck of cards. A face card is Jack, Queen, or King, and the others are number cards (the Ace is counted as the number 1).

a) How many elements does the sample space have?

$52 \cdot 51$

b) How many elements does the event "a face card on second draw" have?

$12 \cdot 11 + 40 \cdot 12$

c) What is the probability of getting a face card on first draw (A) or getting a number card on second draw (B)?

$P(A \cap B) = \frac{12}{52} \cdot \frac{40}{51} = 0.18$ $P(A) = \frac{12}{52} = 0.23$ $P(B) = \frac{40 \cdot 39 + 12 \cdot 40}{52 \cdot 51} = 0.77$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.82$

d) Let $X$ denote the sum of the numbers on the cards. You are told that both cards are number cards which are less than or equal to 2. Under this condition, find the probability distribution of $X$.

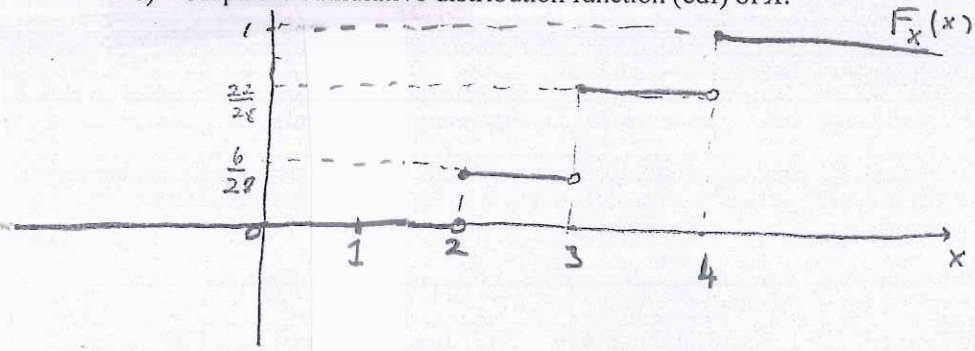
$X = 2, 3, 4$

$P(X=2) = \frac{\binom{4}{2}}{\binom{8}{2}} = \frac{6}{28}$

$P(X=3) = \frac{\binom{4}{1}\binom{4}{1}}{\binom{8}{2}} = \frac{16}{28}$ $P(X=4) = \frac{\binom{4}{2}}{\binom{8}{2}} = \frac{6}{28}$

$$P_X(x) = \begin{cases} \frac{6}{28} & x = 2 \\ \frac{16}{28} & x = 3 \\ \frac{6}{28} & x = 4 \end{cases}$$

e) Graph the cumulative distribution function (cdf) of $X$.



**Question 2: (20 Points)** A jewelery shop window (=vitrin) contains three kinds of gems (=değerli taş), namely diamond, ruby and sapphire. There are 20 jewels with diamond, 5 jewels with ruby, and 15 with sapphire.

a) If 7 jewels are selected from the window at random in order to show to a customer, what is the probability that 4 of them are with diamond?

$$\frac{\binom{20}{4}\binom{20}{3}}{\binom{40}{7}}$$

b) A random customer visiting the shop asks for ruby with probability 0.2. Suppose 10 customers visit the shop on a given day. What is the probability that at least 3 of them ask for ruby?

Let $X$ be number of customers asking for ruby.

$P(X=0) = \left(\frac{8}{10}\right)^{10} = 0.10737$ $\quad P(X=2) = \binom{10}{2}\left(\frac{2}{10}\right)^2\left(\frac{8}{10}\right)^8 = 0.302$

$P(X=1) = \binom{10}{1}\cdot\left(\frac{2}{10}\right)\cdot\left(\frac{8}{10}\right)^9 = 0.2684$ $\quad P(X\geq 3) = 1 - \sum_{i=0}^{2} P(X=i)$

$\cong 0.32$

c) How many customers are expected to visit the shop for encountering (=seeing) the first one who asks for ruby?

Hint: $\sum_{k=1}^{\infty} kr^{k-1} = \frac{1}{(1-r)^2}$ if $|r| < 1$.

Let $X$ number of customers visiting the shop until (including) the first one asking for ruby.

$$E[X] = \sum_{k=1}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} k \cdot (0.8)^{k-1} \cdot (0.2)$$

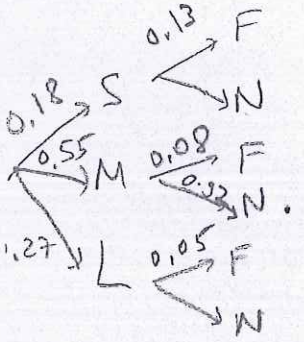
$$= 0.2 \cdot \frac{1}{(1-0.8)^2} = 5$$

$P(F|S) = 0.13 \qquad P(F|M) = 0.08 \qquad P(F|B) = 0.05$

**Question 3: (20 Points)** An investigation of the vehicles on the road showed that 18% are small, 55% are medium size and, 27% are bigger cars. If a small car is involved in an accident, the fatality (=death) probability is 0.13. The fatality probability is 0.08 if the car is medium size, and it is 0.05 for a bigger car.

a) If an accident is reported to be fatal, what is the probability that a small car was involved?

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F|S) \cdot P(S)}{P(F|S) \cdot P(S) + P(F|M) \cdot P(M) + P(F|B) \cdot P(B)}$$




$$= \frac{13.18}{13.18 + 8.55 + 5.27} = \frac{234}{803} \cong 0.29$$

b) Are the events A="small car is involved in an accident" and B="an accident is fatal" independent from each other?

$P(A \cap B) = P(B|A) \cdot P(A) = (0.13) \cdot (0.18) = 0.0234$

$P(A) = 0.18$

$P(B) = 0.0809$

$P(A) \cdot P(B) \neq P(A \cap B)$ so they are not independent.

c) Find the probability that a medium size car is involved in the accident or the accident is not fatal.

$$P(M \cup F^c) = P(M) + P(F^c) - P(M \cap F^c)$$

$$= 0.55 + (1 - 0.0809) - (0.55) \cdot (0.92)$$

$$\simeq 0.96$$

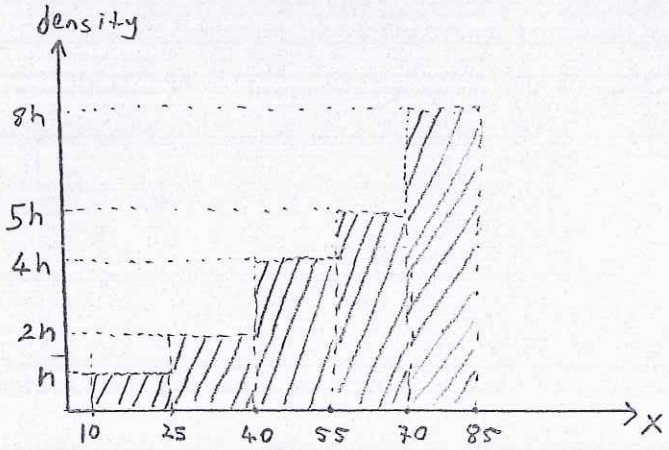
**Question 4: (25 Points)** The reaction time (in seconds) in a chemical reaction is recorded by the repetition of the experiment 25 times under the same conditions and the following data set is obtained.

11.5
27.2
38.8
47.5
49.6
53.4
54.1
59.3
66.3
68.7
69.3
69.9
71.2
77.5
78.2
79.7
80.1
81.5
82.6
84.3

a) Form a frequency table by grouping the data in class intervals of size 15.

| class | 10-25 | 25-40 | 40-55 | 55-70 | 70-85 |
|---|---|---|---|---|---|
| frequency | 1 | 2 | 4 | 5 | 8 |

b) Draw a density scale histogram.




$$h = \frac{1}{20.15} = 0.0033$$

c) The following is a possible pdf (probability density function) that one could fit as a model for this data set:

$$f(x) = a(-10 + x) \qquad 10 \le x \le 85$$

where $a > 0$ is a constant. Find $a$.

$$1 = \int_{10}^{85} f(x)\,dx = a \cdot \int_{10}^{85} (x-10)\,dx = a \cdot \left( \frac{x^2}{2} - 10x \Big|_{10}^{85} \right)$$

$$= a \cdot \left[ \left( \frac{85^2}{2} - 850 \right) - \left( \frac{10^2}{2} - 100 \right) \right]$$

$$= a \cdot (2812.5)$$

$$\Rightarrow a = \frac{1}{2812.5} = 0.000356$$

**Question 5: (20 Points)** Let $\lambda > 0$ be a fixed parameter. Consider a random variable with pdf

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|} \qquad x \in \mathbb{R}.$$

a) Find the cdf of $X$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

if $x < 0$, $F_X(x) = \int_{-\infty}^{x} \frac{\lambda}{2} e^{+\lambda t}\,dt = \frac{\lambda}{2} \frac{e^{\lambda t}}{\lambda}\Big|_{-\infty}^{x} = \frac{e^{\lambda x}}{2}$

if $x \geq 0$, $F_X(x) = \int_{-\infty}^{0} \frac{\lambda}{2} e^{\lambda t}\,dt + \int_{0}^{x} \frac{\lambda}{2} e^{-\lambda t}\,dt$

$$= \frac{1}{2} + \frac{\lambda}{2} \frac{e^{-\lambda t}}{-\lambda}\Big|_{0}^{x} = 1 - \frac{e^{-\lambda x}}{2}$$

b) What is the median of $X$?

Since $F_X(0) = \frac{1}{2}$, $m = 0$

from part a)

c) Show that $E(aX + b) = aE(X) + b$ for a continuous random variable $X$, when $a, b \in \mathbb{R}$.

$$\int_{-\infty}^{\infty} (ax+b) f(x)\,dx = a \int x f(x)\,dx + b \underbrace{\int f(x)\,dx}_{1}$$

$$= a E(X) + b$$