MATH 211 Exam I Solutions

Question 1: (20 Points) Five horses are scheduled to run and the following probabilities of winning have been assigned:

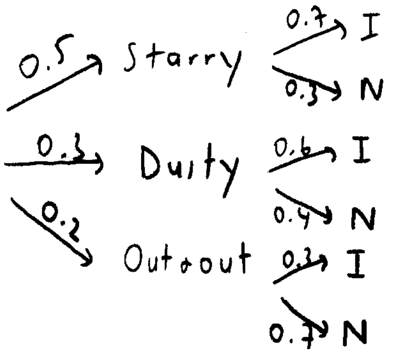
~~Scorpion~~	Starry	Dusty	~~Stake~~	Outandout
~~0.20~~	0.25	0.15	~~0.30~~	0.10

a) (7 points) Suppose Scorpion and Stake are removed from the race at the last minute. In this circumstance, what are the chances for the other horses to win?

$$\mathbb{P}\{\text{Starry wins}\} = \frac{0.25}{0.25+0.15+0.10} = \frac{0.25}{0.5} = \boxed{0.5}$$

$$\mathbb{P}\{\text{Dusty wins}\} = \frac{0.15}{0.50} = \boxed{0.3} \qquad \mathbb{P}\{\text{Outandout}\} = \frac{0.10}{0.50} = \boxed{0.2}$$

b) (8 points) If a horse wins this race, it may be invited to a more prestigious international race. For the remaining horses Starry, Dusty and Outandout, the probabilities of being invited are given by 0.70, 0.60, 0.30, respectively. If the winning horse is invited, what is the probability that it is Outandout? I : invited, N : not invited



$$\mathbb{P}\{\text{outandout} \mid I\} = \frac{\mathbb{P}\{I \mid O\}\,\mathbb{P}\{O\}}{P\{I\}}$$

$$= \frac{(0.3)(0.2)}{(0.3)(0.2) + (0.6)(0.3) + (0.7)(0.5)}$$

$$= \frac{6}{6+18+35} = \frac{6}{59} \cong 0.102 \cong \boxed{10\%}$$

c) (5 points) If the winning horse is invited, what is the probability that Starry lost the race?

$$\mathbb{P}\{\text{Duty or Out+out} \mid I\} = \mathbb{P}\{D \mid I\} + \mathbb{P}\{O \mid I\}$$

$$= \frac{(0.6)\cdot(0.3)}{0.59} + 0.102 \cong$$

$$\cong 0.305 + 0.102$$

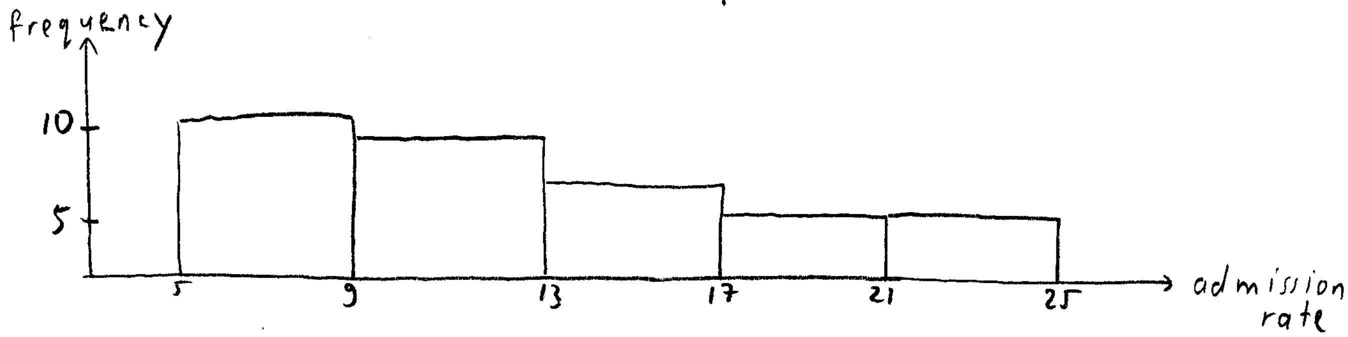
$$\cong \boxed{0.407}$$

Question 2: (20 Points) The following data show the admission rates of patients to the emergency room of health clinics, where the unit is "patients/day" and the clinics are similar in capacity.

5	5.8	6	6.4	6.5	7.1	7.7	7.9	8.1	8.6	9	9.2
10.4	11	11.5	11.7	12	12.1	12.8	13.1	13.3	15.2	15.8	15.9
16.5	16.8	17.2	18.7	19.4	19.5	19.7	22.1	23	24.6	24.6	24.9

a) (8 points) Draw a histogram of the data set. Label the axes.

[5, 9)	[9, 13)	[13, 17)	[17, 21)	[21, 25)
10	9	7	5	5



The joint probability distribution of the admission rate *X* and the type of clinic *Y* (1=private clinic, 2=publicly owned) is given in the following table, as predicted by health officials.

y \ X	1 (low, in[5,10))	2 (medium, in[10,20))	3 (high, in [20,25))	
1	0.10	0.28	0.06	0.44
2	0.22	0.22	0.12	0.56
	0.32	0.50	0.18	

b) (6 points) Is the admission rate independent from the type of clinic?

$$\mathbb{P}\{X=1 \text{ and } Y=1\} \overset{?}{=} \mathbb{P}\{X=1\}\,\mathbb{P}\{Y=1\}$$

$$0.10 \overset{?}{\neq} 0.32 \cdot 0.44 \quad \Rightarrow \quad \boxed{X \not\perp Y}$$

c) (6 points) In part b), what percent of clinics are predicted to have an admission rate within [10,20)? What is the percentage of clinics with admission rate within [10,20) in the data set?

$$\mathbb{P}\{X=2\} = \boxed{0.5}$$

in the data set : $\frac{19}{36} \cong \boxed{0.53}$

Question 3: (25 Points) Consider the joint probability density function of X and Y given by

$$f_{X,Y}(x,y) = ye^{-xy-y}, \quad x \geq 0,\ y \geq 0$$

a) (10 points) Find the marginal pdf of X.

$$f_X(x) = \int_0^{\infty} y e^{-y(x+1)}\, dy$$

$$u = y \qquad dv = e^{-y(x+1)}\,dy$$

$$du = dy \qquad v = \frac{e^{-y(x+1)}}{-(x+1)}$$

$$= \cancel{\frac{ye^{-y(x+1)}}{(x+1)}}\Bigg|_0^{\infty} + \frac{1}{x+1}\int_0^{\infty} e^{-y(x+1)}\,dy = -\frac{1}{(x+1)^2}\, e^{-y(x+1)}\Bigg|_0^{\infty}$$

$$= \boxed{\frac{1}{(x+1)^2}\ ,\ x \geq 0}$$

b) (8 points) Are X and Y independent? Show.

$$f_Y(y) = \int_0^{\infty} ye^{-y(x+1)}\,dx = \cancel{y}\,\frac{e^{-y(x+1)}}{-\cancel{y}}\Bigg|_0^{\infty} = e^{-y}\ ,\ y \geq 0$$

So, $f_{X,Y}(x,y) \neq f_X(x)\cdot f_Y(y) \implies \boxed{X \not\perp Y}$

c) (7 points) Find the probability density function of $W = 2X + 1$ if X has pdf $f_X(x) = \frac{1}{(x+1)^2}$, $x \geq 0$.

$$F_W(w) = \mathbb{P}\{2X + 1 \leq w\} = \mathbb{P}\{X \leq \tfrac{w-1}{2}\} = F_X\left(\tfrac{w-1}{2}\right)$$

$$F_W'(w) = \frac{1}{\left(\frac{w-1}{2}+1\right)^2}\cdot\frac{1}{2}$$

$$\|$$

$$f_W(w) = \frac{4}{(w+1)^2}\cdot\frac{1}{2} = \boxed{\frac{2}{(w+1)^2}\ ,\ w \geq 1}$$

Question 4: (15 Points) A radioactive material source emits 4 alpha particles per minute. Assume that the number of particles emitted has a Poisson distribution.

a) (5 points) What is the probability that exactly 3 particles will be counted in the next minute?

$X \sim Poi(4)$

$$\mathbb{P}\{X = 3\} = \frac{e^{-4} 4^3}{3!} \cong \boxed{0.1954}$$

b) (1 point) What is the expected $\underbrace{\text{number of particles counted in the next 30 minutes}}_{Y}$?

$$\mathbb{E}[Y] = 30 \cdot 4 = \boxed{120}$$

c) (2 points) What is the variance of the number of particles counted in the next 30 minutes?

$Y \sim Poi(120) \Rightarrow V(Y) = \boxed{120}$ since mean = variance for a Poisson r.v.

d) (7 points) Use central limit theorem to approximate the probability

$$P(X_1 + \cdots X_{30} \geq 150)$$

where X_i denotes the number of particles emitted at minute i, $i = 1, \ldots, 30$.

$$\mathbb{P}\left\{ \frac{X_1 + \ldots + X_{30} - 120}{\sqrt{120}} \geq \frac{149.5 - 120}{\sqrt{120}} \right\}$$

$$\cong \mathbb{P}\{Z \geq 2.69\}$$

$$\cong 1 - 0.9964 = \boxed{0.0036}$$

Question 5: (20 Points) In a casino game, a player chooses 10 distinct numbers from the set {1,2,3, … ,80} and writes them on a piece of paper. Only 20 of those numbers are winning numbers as announced by the "caller" (=an employee working in the casino) later.

a) (6 points) What is the probability that there are 6 winning numbers among those 10 chosen by the player?

$$\frac{\binom{20}{6}\binom{60}{4}}{\binom{80}{10}} = \boxed{0.0115}$$

Now, suppose the game is changed as follows: the player chooses 10 numbers *with replacement* from the set {1,2,3, … ,80}, randomly. That is, s/he can choose the same number(s) again and again. Only 20 of those numbers are winning numbers: say the numbers in A={1,…,20}, for simplicity.

(Ex. The choice of 27, 65, 44, 20, 9, 54, 31, 20, 65, 72 has three winning numbers, namely 20,9,20.)

b) (6 points) Among those 10 chosen by the player, what is the probability that at most 2 choices belong to set A?

$$p = \frac{20}{80} = 0.25$$

$$\mathbb{P}\{\text{desired}\} = \binom{10}{0}(0.25)^0(0.75)^{10} + \binom{10}{1}(0.25)^1(0.75)^9 + \binom{10}{2}(0.25)^2(0.75)^8 \cong \boxed{0.53}$$

c) (8 points) If the player chooses 40 numbers instead of 10, what is the approximate probability that at most 8 of them belong to set A?

$$X \sim \text{Bin}(40, 0.25)$$

$$\mathbb{P}\{X \leq 8\} \cong \mathbb{P}\left\{Z \leq \frac{8.5 - 40(0.25)}{\sqrt{40(0.25)(0.75)}}\right\}$$

$$= \mathbb{P}\left\{Z \leq \frac{8.5-10}{2.74}\right\}$$

$$\cong \mathbb{P}\{Z \leq -0.35\}$$

$$\cong \boxed{0.2912}$$