Math 211: Statistics for Sciences

November 8, 2012

**Fall 2012 Midterm #1**

**Closed book & notes; only a single-sided and handwritten A4 formula sheet and a calculator allowed; 100 minutes. No questions accepted!**

**Instructions**: There are six pages (one cover and five pages with questions) in this exam. Please inspect the exam and make sure you have all 6 pages. You may only use your calculator and your formula sheet. Do all your work on these pages. If you use the back of a page, make sure to indicate that. **You may not exchange any kind of material with another student.**

**Remember**: ***You must show your work to get proper credit.***

Academic Honesty Code: Koç University Academic Honesty Code stipulates that "copying from others or providing answers or information, written or oral, to others is cheating." By taking this exam, you are assuming full responsibility for observing the Academic Honesty Code.

**NAME and SURNAME:** KEY

**SIGNATURE:** ____________________

| | |
|---|---|
| 1 | /20 |
| 2 | /20 |
| 3 | /20 |
| 4 | /20 |
| 5 | /20 |
| Total: | /100 |

**Question 1: (20 Points)**

The sample space of a random experiment consists of the integers 0, 1, 2, 3, 4 (i.e., the outcomes of the trials are these integers). The probabilities assigned to each simple event (or outcome) are given below.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p(x)$ | 0.015 | 0.235 | 0.425 | 0.245 | 0.08 |

(a) Is this an acceptable probability assignment? Why?

Yes, because (1) $p(x) \geq 0$ for all $x$

(2) $\sum_{\text{all } x} p(x) = 0.015 + 0.235 + 0.425 + 0.245 + 0.08 = 1.00$

(b) Define the event $A$ as "The outcome of the trial is smaller than 3". What is $P(A)$?

$A =$ "$X < 3$", so $P(A) = P(X<3) = P(0) + P(1) + P(2)$

$= 0.015 + 0.235 + 0.425 = \underline{0.675}$

(c) Define the event $B$ as "The outcome of the trial is larger than 1". Are $A$ and $B$ independent?

$B =$ "$X > 1$", so $P(B) = P(X>1) = P(2) + P(3) + P(4)$

$= 0.425 + 0.245 + 0.08 = 0.75$

$A \cap B =$ "$1 < X < 3$", so $P(A \cap B) = P(2) = 0.425$

For independence, we check $P(A \cap B) \stackrel{?}{=} P(A)P(B)$

$0.425 \stackrel{?}{=} (0.675)(0.75)$

$0.425 \neq 0.50625$

so $A$ & $B$ are not independent

(d) Find also $E(X)$ and $Var(X)$.

$\mu_X = E(X) = \sum x\, p(x)$

$= 0(0.015) + 1(0.235) + 2(0.425) + 3(0.245) + 4(0.08)$

$\underline{= 2.14}$

and $E(X^2) = \sum x^2 p(x) = 0^2(0.015) + 1^2(0.235) + 2^2(0.425) + 3^2(0.245) + 4^2(0.08) = 5.42$

so $Var(X) = E(X^2) - \mu_X^2$

$= 5.42 - (2.14)^2 = \underline{0.8404}$

**Question 2: (20 Points)**

(a) Consider a class of 20 students. Assuming this class consists of randomly selected students, find the probability that at least two students share the same birthday? [Leave your answer in expression form; i.e., do not evaluate the expression to get the final answer.]

A = "at least two students share the same birthday"

then $A^c$ = "no two students share the same birthday".

$P(A^c) = \frac{{}_{365}P_{20}}{365^{20}}$ so $P(A) = 1 - \frac{{}_{365}P_{20}}{365^{20}}$

(b) An experiment consists of the following steps: Toss a fair coin; if the outcome is tails, roll a fair die and record the number appearing on the top face; if the outcome is heads, roll two fair dice and record the numbers appearing on the two top faces. Describe the sample space. How many elements does it have?

T → roll a die → $\{1, 2, \ldots, 6\}$

H → roll two dice $\{(1,1), (1,2), \ldots, (6,6)\}$

so sample space S is $S = \{(T,1), (T,2), \ldots, (T,6), (H,1,1), (H,1,2), \ldots, (H,6,6)\}$

so $n(S) = 6 + 36 = 42$

(c) Are the outcomes equally likely?

No, they are not, because $P(T,1) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

and $P(H,1,1) = \frac{1}{2} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{72}$ $(\Rightarrow P(T,1) \neq P(H,1,1))$

(d) Define the event $E$ as "the sum is 3". (That is, event $E$ consists of the following outcomes. If one die is rolled, the outcome is a 3; if two are rolled, the sum of the appearing two numbers is 3). What is the probability of $E$?

$E$ = "sum is 3" $\Rightarrow E = \{(T,3), (H,1,2), (H,2,1)\}$

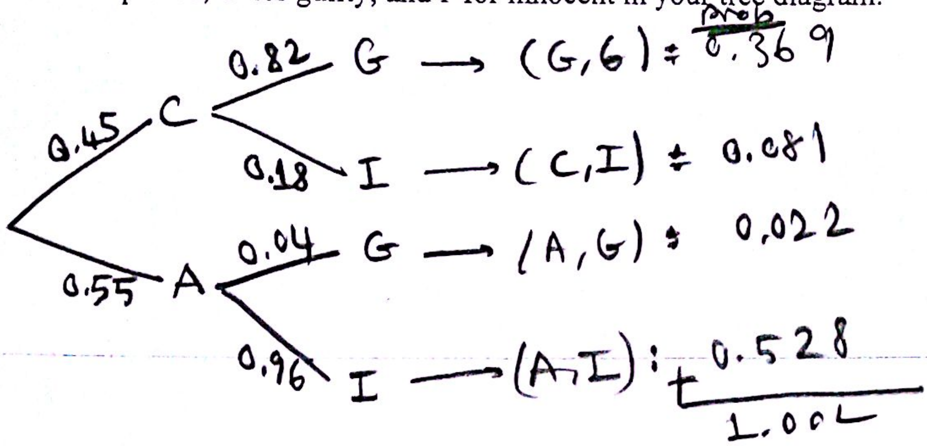
so $P(E) = P(T,3) + P(H,1,2) + P(H,2,1)$

$= \frac{1}{12} + \frac{1}{72} + \frac{1}{72} = \frac{8}{72} = \frac{1}{9}$

**Question 3: (20 Points)**

In Cleansland, 45% of the women who are tried in court (mahkemede yargılananlar) are convicted (mahkum ediliyorlar). 82% of convicted women are actually guilty, while 96% of acquitted (beraat eden) women ware actually innocent.

(a) Make a probability tree for women who are tried in court. Use the letters $C$ for convicted, $A$ for acquitted, $G$ for guilty, and $I$ for innocent in your tree diagram.




(b) What is the probability that a woman is acquitted?

$P(A) = 0.55$

(c) What is the probability that a woman who is convicted is innocent? (i.e., you are given that woman is convicted).

$$P(I|C) = \frac{P(C,I)}{P(C)} = \frac{0.081}{0.45} = 0.18$$

(d) What is the probability that a woman who is innocent is convicted?

$$P(C|I) = \frac{P(C,I)}{P(I)} = \frac{0.081}{0.081+0.528} = \frac{0.081}{0.609} \cong 0.133$$

(e) What is the probability that a woman is acquitted and is guilty?

$P(A,G) = 0.022$

**Question 4: (20 Points)**

Suppose $U$ is a uniform random variable over the unit interval $[0, 1]$.

(a) Compute the mean (i.e., expected value) and variance of $U$.

$U \sim \text{uniform}[0,1] \Rightarrow f_U(u) = \begin{cases} 1, & 0 \le u \le 1 \\ 0, & \text{o.w.} \end{cases}$

$\mu_U = E(U) = \int_0^1 u f_U(u)\, du = \int_0^1 u\, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

and $E(U^2) = \int_0^1 u^2 du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$

so $Var(U) = E(U^2) - \mu_U^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} \cong 0.083$

(b) The random variable $Y = (b - a)U + a$ is uniform over $[a, b]$. Find the variance of $Y$.

$Var(Y) = Var((b-a)U + a) = (b-a)^2 Var(U) + 0$

$= \frac{(b-a)^2}{12}$

(c) Let $Z$ be a random variable with the following pdf $f_Z(z) = 3\,z^2, 0 < z < 1$. Find the variance of $Z$.

$E(Z) = \mu_Z = \int_0^1 z(3z^2)\, dz = \frac{3z^4}{4}\Big|_0^1 = 3/4$

$E(Z^2) = \int_0^1 z^2(3z^2) dz = \frac{3z^5}{5}\Big|_0^1 = \frac{3}{5}$

so $Var(Z) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} = 0.0375$

(d) Which one of the variables, $U$ or $Z$, has larger spread? Why?

$Var(U) = \frac{1}{12} \cong 0.083$ and $Var(Z) = \frac{3}{80} = 0.0375$

so, since $Var(U) > Var(Z)$, U has a larger spread.

**Question 5: (20 Points)**

(a) For persons infected with a certain form of malaria (sıtma), the length of time spent in remission (gerileme) is described by the continuous pdf $f_Y(y) = \frac{1}{9}y^2$, $0 \le y \le 3$, where $Y$ is measured in years. What is the probability that a malaria patient's remission lasts longer than one year?

$$P(Y>1) = \int_1^3 \frac{1}{9}y^2\,dy = \frac{1}{9}\frac{y^3}{3}\Big|_1^3 = \frac{1}{27}(27-1) = \underline{\frac{26}{27}} \cong 0.96$$

(b) Find also the cdf of $Y$.

for $0 \le y \le 3$, $F_Y(y) = \int_0^y f_Y(t)\,dt = \int_0^y \frac{1}{9}t^2\,dt = \frac{t^3}{27}\Big|_{t=0}^{t=y} = \frac{y^3}{27}$

$$\text{so } F_Y(y) = \begin{cases} 0, & y<0 \\ \frac{y^3}{27}, & 0 \le y < 3 \\ 1, & y \ge 3 \end{cases}$$

(c) A fair die is rolled three times. Let $X$ denote the number of different faces showing. Find $E(X)$. (Hint: possible values for $X$ are 1, 2, 3.)

| $X$ | $P_X(x)$ |
|---|---|
| 1 | $\frac{6}{216} = \frac{1}{36}$ |
| 2 | $\frac{90}{216} = \frac{15}{36}$ |
| 3 | $\frac{120}{216} = \frac{20}{36}$ |

since $X=1$, $(1,1,1), (2,2,2), \ldots, (6,6,6)$

so $P(X=1) = \frac{6}{216}$

$X=2$, $\binom{3}{1} \times 6 \times 5 = 3 \times 6 \times 5 = 90$

so $P(X=2) = \frac{90}{216}$

and $X=3$, $6 \times 5 \times 4 = 120$

so $P(X=3) = \frac{120}{216}$

$$\text{so } E(X) = 1\left(\frac{1}{36}\right) + 2\left(\frac{15}{36}\right) + 3\left(\frac{20}{36}\right)$$

$$= \frac{1+30+60}{36} = \frac{91}{36} \cong \underline{2.53}$$