SPECIAL CASES OF LP PROBLEM

The Assignment Problem

Fig. 1. Schematic representation of the Assignment Problem

\[ x_{ij} = \begin{cases} 
1 & \text{if job } i \text{ is assigned to machine } j \\
0 & \text{else} 
\end{cases} \]

Find minimum cost assignment, given \( c_{ij} \)

\[ \text{min } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \]

s.t.

\[ \sum_{j=1}^{n} x_{ij} = 1 \quad i=1, \ldots, n \]

\[ \sum_{i=1}^{n} x_{ij} = 1 \quad j=1, \ldots, n \]

\[ x_{ij} = 0, 1 \]

Note: assignment problem can be solved as an LP \( 0 \leq x_{ij} \leq 1 \) (unimodular matrix \( A \), identify vector \( b \), every basic feasible solution is integer)

Solution Of the Assignment Problem:

It is possible to exploit the special structure of the assignment problem and develop custom solution algorithms. The Hungarian method is the most widely used algorithm.

**Hungarian Method (Kuhn Algorithm):**

Step1: For the original cost matrix, identify each row’s minimum, and subtract it from all the entries of the row.

Step2: For the matrix resulting from step 1, identify each column’s minimum, and subtract it from all the entries of the column.

Step3: Identify the optimal assignment as the one associated with the zero elements of the matrix obtained in step 2. Optimal assignment is found, stop.

Step4: If no feasible assignment (with all zero entries) can be secured from steps 1, 2, and 3.
i. Draw the minimum number of horizontal and vertical lines in the reduce matrix that will cover all of the zero entries.

ii. Select the smallest uncovered element, and subtract it from every uncovered element; then add it to every element at the intersection of two lines.

iii. If no feasible assignment can be found among the resulting zero entries, repeat step 4. Otherwise go to step 3 to determine the optimal assignment.

Example:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 4 & 6 & 3 \\
2 & 9 & 7 & 10 & 9 \\
3 & 4 & 5 & 11 & 7 \\
4 & 8 & 7 & 8 & 5 \\
\end{array}
\]

Step 1:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 & \text{min} \\
1 & 0 & 3 & 5 & 2 & 1 \\
2 & 2 & 0 & 3 & 2 & 7 \\
3 & 0 & 1 & 7 & 3 & 4 \\
4 & 3 & 2 & 3 & 0 & 5 \\
\end{array}
\]

Step 2:
Step 3: No feasible assignment exists

Step 4:
   i. draw minimum number of horizontal and vertical lines to cover all of the zeros.

   ii. the smallest uncovered element is equal to 1, add this to every element at the intersection of two lines and subtract it from every uncovered element.

   iii. A feasible and optimal solution exists. (1,1), (2,3), (3,2), (4,4)
Optimal solution has an objective function value of 21.

Extensions of the Assignment Problem

Facility Location Problem:

Parameters:

- $f_i$: fixed cost for selecting plant $i$
- $c_{ij}$: variable cost for assigning plant $i$ to market $j$
- $d_j$: demand in market $j$
- $U_i$: maximum capacity of plant $i$

Decision Variables:

- $y_i$: 1 if plant $i$ is selected, 0 else
- $x_{ij}$: amount plant $i$ produces for market $j$

Optimization Model:

$$\min \ z = \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$
The Transportation Problem

The transportation problem deals with the situation in which a commodity is shipped from sources (e.g. plants) to destinations. The objective is to determine the amounts shipped from each source to each destination that minimizes the total shipping cost while satisfying both the supply limits and demand requirements.

\[
\begin{align*}
\text{s.t.} & \quad \sum_{j=1}^{m} x_{ij} = d_j & j = 1, \ldots, m \\
& \quad \sum_{j=1}^{m} x_{ij} - U_i \leq 0 & i = 1, \ldots, n \\
& \quad y_i = 0, 1 & i = 1, \ldots, n \\
& \quad x_{ij} \geq 0 & i = 1, \ldots, n, j = 1, \ldots, m
\end{align*}
\]

\[c_{ij}: \text{unit transportation cost from source } i \text{ to destination } j\]
\[x_{ij}: \text{amount shipped from source } i \text{ to destination } j\]

\[
\begin{align*}
\min \quad z &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{s.t} & \quad \sum_{j=1}^{n} x_{ij} = a_i & i = 1, \ldots, m \\
& \quad \sum_{j=1}^{n} x_{ij} = b_j & j = 1, \ldots, m \\
& \quad x_{ij} \geq 0 & i = 1, \ldots, m, j = 1, \ldots, n
\end{align*}
\]
**Customized Implementation of the Simplex Method for Transportation Problems**

**Step 1. Determination of the Starting Solution:**
There are three common methods to find an initial basic feasible solution to the transportation problems:

1. Northwest-corner method
2. Least-cost method
3. Vogel approximation method (improved version of least-cost method)

**Northwest-corner Method:**
The northwest-corner rule requires that we start in the upper left-hand corner and allocate units.

1. Exhaust the supply of each row before moving down to the next row.
2. Exhaust the requirement of each column before moving to the next column on the right.
3. Check to ensure that all supplies and demands are met.

**Lowest-Cost Method:**
1. Identify the cell with the lowest cost. Break any ties arbitrarily.
2. Allocate as many units as possible to that cell without exceeding the supply or demand; then cross out that row or column (or both) that is exhausted by this assignment.
3. Find the cell with the lowest cost from the remaining (not crossed out) cells.
4. Repeat steps 2 and 3 until all units have been allocated.

**Step 2. Optimality Check:**
The optimality check requires the calculation of reduced cost elements for each nonbasic variable. The following procedure is as follows:

1. Select any nonbasic variable (i.e., a cell that was not assigned flow).
2. Beginning at this cell, trace a closed path back to this cell via cells that are assigned flow (only horizontal and vertical moves are allowed, you may also step over cells that may or may not have assigned flow).
3. Beginning with a (+) sign at the selected cell, place alternate (+) and (-) signs on each corner cell of the closed path just traced.
4. Calculate the reduced cost by adding the unit-cost figures for each cell containing (+) sign and by subtracting the unit-cost figures for each cell containing (-) sign.
5. Repeat steps 1 to 4 for each nonbasic variable.
6. If all reduced costs are nonnegative, current solution is optimal; otherwise go to Step 3.

**Step 3. Iteration:**
The iterations of the transportation simplex method consists of three steps:

1. Determine the entering basic variable among the nonbasic variables that has the most negative reduced cost calculated in Step 2 (break ties arbitrarily).
2. Determine the leaving basic variable by selecting the cell in the closed path that has a (-) sign with the minimum flow assigned.
3. obtain new basic feasible solution by adding the value determined in the previous step to the cells with (+) sign in the closed path and subtracting the same value from the cells with (-) sign. Go to Step 2.

**Example:**

Transportation Cost:

<table>
<thead>
<tr>
<th>From/To</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$5</td>
<td>$4</td>
<td>$3</td>
<td>100</td>
</tr>
<tr>
<td>B</td>
<td>$8</td>
<td>$4</td>
<td>$3</td>
<td>300</td>
</tr>
<tr>
<td>C</td>
<td>$9</td>
<td>$7</td>
<td>$5</td>
<td>300</td>
</tr>
<tr>
<td>Demand</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

The transportation problem given above can be represented in the following transportation table:

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3</td>
<td></td>
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</tr>
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<td></td>
<td>300</td>
</tr>
<tr>
<td>C</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td></td>
<td>300</td>
</tr>
<tr>
<td>Demand</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Step 1: Initialization**

*The Lowest-cost Method:*

<table>
<thead>
<tr>
<th>From</th>
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<td></td>
<td>300</td>
</tr>
<tr>
<td>Demand</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Northwest-corner Method:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{From} & \text{To} & 1 & 2 & 3 & \text{Supply} \\
\hline
A & 100 & & & & 100 \\
B & & & & & 300 \\
C & & & & & 300 \\
\hline
\end{array}
\]

We will continue with the initial solution obtained by the Northwest-corner method (this choice is arbitrary).

**Step 2: Optimality Check**
Find the reduced cost elements for all nonbasic variables.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Nonbasic Cells:} & \text{reduced cost for} & \text{reduced cost for} & \text{reduced cost for} & \text{reduced cost for} \\
\text{A2} & 4 - 5 + 8 - 4 = 3 & \text{A3} & 3 - 5 + 8 - 4 + 7 - 5 = 4 & \text{B3} & = 3 - 4 + 7 - 5 = 1 & \text{C1} & 9 - 7 + 4 - 8 = -2 \\
\text{reduced cost is negative, room for improvement in the objective.} & & & & & & & \\
\hline
\end{array}
\]

**Step 3: Iteration 1**
1. Entering Basic Cell: since on cell C1 has a negative reduced cost, it is the only candidate as entering basic cell
2. Leaving Basic Cell: select the smallest number of units found in cells containing (-) signs.
   Cell C2 is the leaving basic cell with a value of 100.
3. New BFS:
**Step 2: Optimality Check**
reduced cost for $A_2 = 4 - 5 + 8 - 4 = 3$
reduced cost for $A_3 = 3 - 5 + 9 - 5 = 2$
reduced cost for $B_3 = 3 - 5 + 9 - 8 = -1$  reduced cost is negative, room for improvement
reduced cost for $C_2 = 7 - 9 + 8 - 4 = 2$

**Step 3: Iteration 2**
1. Entering Basic Cell: since on cell B3 has a negative reduced cost, it is the only candidate as entering basic cell
2. Leaving Basic Cell: select the smallest number of units found in cells containing (-) signs.
   Cell B1 is the leaving basic cell with a value of 100.
3. New BFS:

   **Special Issues**
   **Unbalanced Demand and Supply:**
   When the total demand is not equal to the total supply, the can convert this unbalanced problem into a balanced problem by introducing dummy sources or dummy destinations. If total demand is larger than the total supply, we can add a dummy source that will balance the
problems. Similarly, if the total supply is larger than the total demand, we can add a dummy destination to balance the problem. The cost of dummy sources or destinations is zero.

Degeneracy:
In order to apply the method described in this chapter, the number of shipping routes must be equal to the number of rows plus the number of columns minus 1.

# of basic variables = m + n - 1

A solution that does not satisfy this rule is called a degenerate solution. In order to eliminate problems associated with degenerate solutions, we must create artificially occupied cells by placing a zero flow.

Example:
Given the following data for a transportation problem, find the optimal solution.

<table>
<thead>
<tr>
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<th>Supply</th>
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<td>$2</td>
<td>$6</td>
<td>100</td>
</tr>
<tr>
<td>B</td>
<td>$10</td>
<td>$9</td>
<td>$9</td>
<td>120</td>
</tr>
<tr>
<td>C</td>
<td>$7</td>
<td>$10</td>
<td>$7</td>
<td>80</td>
</tr>
<tr>
<td>Demand</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

The transportation table is constructed and initialized using the Northwest-corner method as follows:

<table>
<thead>
<tr>
<th>From</th>
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<td>100</td>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

# basic variables = 4 < 3+3-1 Degenerate Solution

assign a 0 unit shipment to a cell with the following criteria:
- it should be possible to trace a closed-path for each unused cell.
- avoid the cells that have a (-) sign in the closed path.
<table>
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HW: solve this problem