



LINEAR PROGRAMMING MODELS

Common terminology for linear programming:

- linear programming models involve
 - . resources denoted by i , there are m resources
 - . activities denoted by j , there are n activities
 - . performance measure denoted by z

An LP Model:

$$\begin{aligned} \max \quad & z = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i = 1, \dots, m \end{aligned}$$

z : value of overall performance measure

x_j : level of activity j ($j=1, \dots, n$)

c_j : performance measure coefficient for activity j

b_j : amount of resource i available ($i=1, \dots, m$)

a_{ij} : amount of resource i consumed by each unit of activity j

Decision Variables: x_j

Parameters: c_j, a_{ij}, b_j

Standard Form of the LP Model

A Linear programming problem can be expressed in the following standard form:

$$\begin{aligned} \max \quad & z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1 \\ & a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2 \\ & \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\ & \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\ & a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m \\ & \quad x_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \geq 0 \\ & \quad \quad \quad x_2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \geq 0 \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \dots \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_n \geq 0 \end{aligned}$$

Objective functions: overall performance measure

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Constraints:

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \leq b_i \quad \forall i=1, \dots, m \quad (\text{Functional constraints})$$

$$x_j \geq 0 \quad \forall j=1, \dots, n \quad (\text{Nonnegativity constraints})$$



Variations in LP Model

An LP model can have the following variations:

1. *Objective Function:* **minimization** or **maximization** problem.

2. *Direction of constraints*

$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i \quad \forall i=1, \dots, m$	less than or equal to
$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \geq b_i \quad \forall i=1, \dots, m$	greater than or equal to
$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad \forall i=1, \dots, m$	equality

3. *Non-negativity constraints*

$$-\infty \leq x_j \leq \infty$$

Terminology for solutions of the LP Model

Solution: any specification of values for the decision variables, x_j , is called a solution

- *Infeasible Solution:* a solution for which at least one constraint is violated.
- *Feasible Solution:* a solution for which all of the constraints are satisfied

Corner - Point Feasible (CPF) solution: a solution that lies at the corner of the feasible region.

Optimal Solution: a feasible solution that has the most favorable value of the objective function.

- maximization → largest z
- minimization → smallest z

Multiple Optimal Solutions: infinite number of solutions with the most favorable value of the objective function

The best CPF = optimal solution



Assumptions of Linear Programming

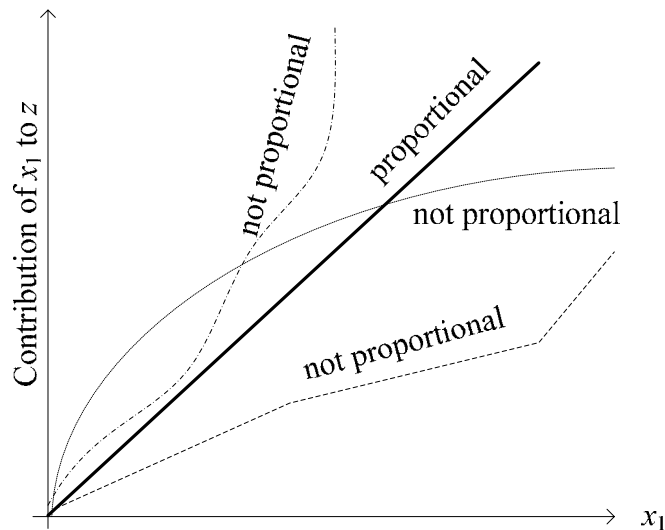
1. Proportionality:

- contribution of each activity to the objective function, z , is proportional to its level.

$$c_j x_j$$

- contribution of each activity to each functional constraint is proportional to its level.

$$a_i x_j$$



2. Additivity:

- every function is the sum of the individual contribution of the respective activities.

$$z = \sum_{j=1}^n c_j x_j$$

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i = 1, \dots, m$$

3. Divisibility:

- decision variables are allowed to have any real values that satisfy the functional and non-negativity constraints.

4. Certainty:

- the parameter values are assumed to be known constants.

Examples of LP

- Radiation Therapy Design
- Regional Planning
- Controlling Air Pollution
- Reclaiming Solid Water
- Personnel Scheduling
- Distribution Network
- Product Mix
- Planning

read pp.44-67

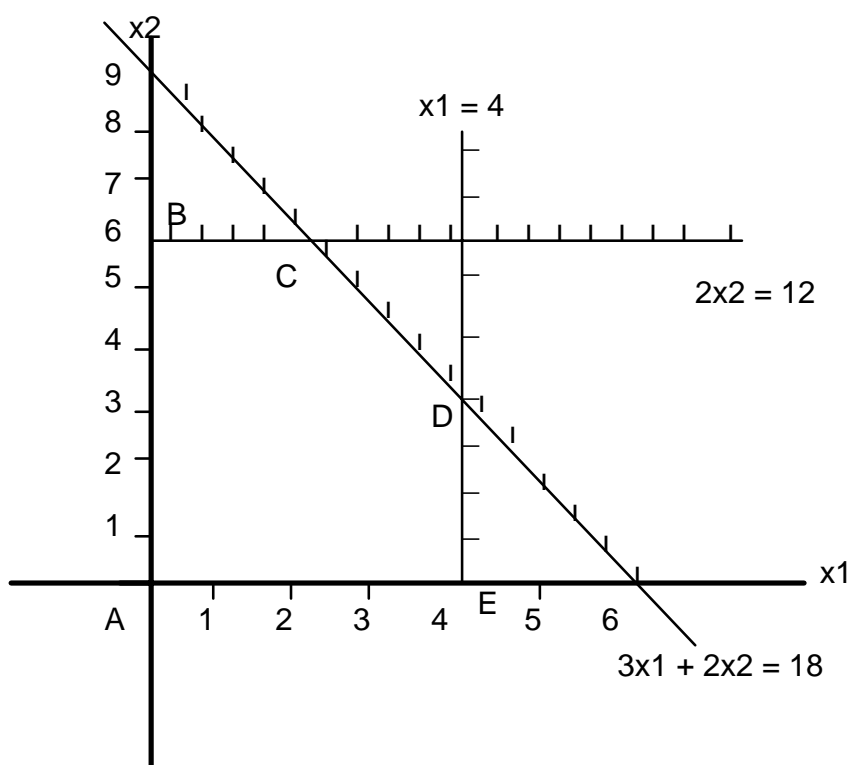
**SOLVING LP PROBLEMS**

Consider the Wyndor Glass Co. problem:

$$\max z = 3x_1 + 5x_2$$

s.t.

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



CPF's (Corner-point Feasible Solutions): intersection points of the system of equations that define the feasible region.

Definition: Adjacent CPF solutions

For any linear programming problem with n decision variables, two CPF solutions are adjacent to each other if they share $n-1$ constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line is referred to as an edge on the feasible region.

<u>Point</u>	<u>Coordinates (x_1, x_2)</u>
A	(0,0)
B	(0,6)
C	(2,6)
D	(4,3)
E	(4,0)



<u>Point</u>	<u>Coordinates (x_1, x_2)</u>	<u>Its Adjacent CPF solutions</u>
A	(0,0)	B (0,6) E (4,0)
B	(0,6)	A (0,0) C (2,6)
C	(2,6)	B (0,6) D (4,3)
D	(4,3)	C (2,6) E (4,0)
E	(4,0)	A (0,0) D (4,3)

Optimality Test: If a CPF solution has no adjacent CPF solutions that are better, then it must be an optimal solution.

Solution Algorithm:

1. Initialize: choose an initial CPF solution.
2. Optimality Test: evaluate the performance measure at the current solution. If its value is larger than all of its adjacent CPF solutions; the current solution is optimal. Otherwise go to step 3.
3. Iteration: Move to a better adjacent CPF solution. Go to step 2.

Example: Wyndor Glass Co.

- Initialization: select point A (0,0) as the initial CPF solution.
- Optimality Test: $z = 0$. There may be better solutions.
- Iteration: move to B. Why?
 $z = 3x_1 + 5x_2$
Moving along x_2 would make z larger.
- Optimality Test: $z = 30$. It is better than A but is it the best?
- Iteration: Move to C. Why?
There is no point else but C to move.
- Optimality Test: $z = 36$. It is better than B, but is it the best?
- Iteration: move to D. Why?
- Optimality Test: $z = 27$. Optimal Solution corresponds to CPF solution C.

Some observations on the solution algorithm:

- Simplex method focuses solely on CPF solutions.
- Simplex method is an iteration algorithm.
- Whenever possible, the initialization of the simplex method chooses the origin as the initial CPF solution.
- Given a CPF solution, it is much quicker to gather information about its adjacent CPF solutions than its non-adjacent CPF solutions.
- After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. The most promising edge is selected and this edge is chosen for the next iteration.
- A positive rate of improvement in Z implies that the adjacent CPF solution is better; a negative rate of improvement in Z implies that the adjacent CPF solution is worse. The optimality test consists of simply of checking whether any of the edges give a positive rate improvement in Z . If none do, then the current CPF solution is optimal.