ONLINE SUPPLEMENT TO
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

INCENTIVE AND PRODUCTION DECISIONS FOR REMANUFACTURING OPERATIONS

Volume, Issue, (year)

Onur Kaya
Koc University, Turkey
Dept. of Industrial Engineering
okaya@ku.edu.tr

Elsevier, publication year.
A Appendix

Proof of Proposition 1

First, we prove the following lemma.

**Lemma 1.** In the optimal solution, either $Q(t) = q_r$ or $t = 0$.

**Proof.** Assume that $t > 0$ and $Q(t) > q_r$ in the optimal solution. Then, the manufacturer can increase his profit by decreasing $t$ which decreases $Q(t)$ until $Q(t) = q_r$ or $t = 0$.

Now, assume that $t > 0$ and $Q(t) = q_r$ in the optimal solution. Then, the manufacturer’s problem becomes:

$$
\max_t \pi = \int_0^{Q(t)} p_r f_r(x) dx + p_r Q(t) \bar{F}_r(Q(t)) - (t + c_r) Q(t)
$$

Since $Q(t)$ is a deterministic function of $t$, we can write the objective function in terms of $Q(t)$. Looking at the first and second order derivatives w.r.t. $Q(t)$, we observe that the objective function is concave w.r.t $Q(t)$ and thus equating the first order derivative to 0 will give us the optimal values of $t$ and $q_r = Q(t)$ if the resulting $t \geq 0$.

$$
\frac{\partial \pi}{\partial Q(t)} = p_r Q(t) f_r(Q(t)) + p_r - p_r F_r(Q(t)) - p_r Q(t) f_r(Q(t)) - c_r - t - Q(t) \frac{\partial t}{\partial Q(t)} = 0
$$

(A.1)

$$
\frac{\partial^2 \pi}{\partial Q(t)^2} = -p_r f_r(Q(t)) - \frac{\partial t}{\partial Q(t)} - \frac{\partial t}{\partial Q(t)} - Q(t) \frac{\partial^2 t}{\partial Q(t)^2} < 0
$$

If the value of $t$ that satisfies the equation A.1 is less than 0, then $t = 0$ is the optimal value of the incentive to offer since $Q(t)$ is an increasing function of $t$ and the objective function is a concave function of $Q(t)$. In this case, the manufacturer will want to produce $q_r = F_r^{-1}\left(\frac{p_r - c_r}{p_r}\right)$, however for $t = 0$, he can produce at most $Q(0)$. Thus, $q^*_r = \min\{F_r^{-1}\left(\frac{p_r - c_r}{p_r}\right), Q(0)\}$ will be the optimal production quantity.
Proof of Proposition 2

First, observe that, in the optimal solution, either \( Q(t) = q_r \) or \( t = 0 \). Otherwise, if \( t > 0 \) and \( Q(t) > q_r \) in the optimal solution, then the collection agency can increase his profit by decreasing \( t \) which decreases \( Q(t) \) until \( Q(t) = q_r \) or \( t = 0 \).

Then, assuming \( Q(t) = q_r = F_r^{-1}\left(\frac{p_r - c_r - w}{p_r}\right) \), we can write \( t = Q^{-1}[F_r^{-1}(\frac{p_r - c_r - w}{p_r})] \) and \( w = p_r - c_r - p_r F_r(q_r) \). Then, the collection agency’s problem is to maximize:

\[
\pi_t = (w - Q^{-1}[F_r^{-1}(\frac{p_r - c_r - w}{p_r})]) F_r^{-1}(\frac{p_r - c_r - w}{p_r}) = (p_r - c_r - p_r F_r(q_r) - Q^{-1}(q_r))q_r
\]

Since \( \pi_t \) is concave w.r.t \( q_r \), equating \( \frac{\partial \pi_t}{\partial q_r} \) to 0:

\[
\frac{\partial \pi_t}{\partial q_r} = p_r - c_r - p_r F_r(q_r) - Q^{-1}(q_r) = 0
\]

We can find the value of \( q_r \) from this equation and the value of \( t \) from the relation \( q_r = Q(t) \) and these values will be the optimal ones if \( t \geq 0 \) in this solution.

However, if \( t < 0 \), then \( t = 0 \) will be optimal. In that case, the collection agency tries to maximize \( \pi_t = w q_r = (p_r - c_r - p_r F_r(q_r))q_r \). Looking at the first order derivative, the optimal \( q_r \) will be the solution to the equation \( p_r - c_r - p_r F_r(q_r) - p_r q_r f_r(q_r) = 0 \) if it satisfies our assumption that \( q_r \leq Q(0) \). Otherwise \( q_r = Q(0) \) when \( t = 0 \) will be the optimal solution. In all cases, the collection agency will set the wholesale price such that \( w^* = p_r - c_r - p_r F_r(q_r) \).

Proof of Proposition 3

Observe that, similar to the previous cases, in the optimal solution, either \( Q(t) = q_r \) or \( t = 0 \). First, we assume that \( t > 0 \) and \( Q(t) = q_r \) in the optimal solution. Then, the manufacturer’s problem becomes:

\[
\max_t \pi = \int_0^{q_r+q_m} px f(x) dx + p(q_r + q_m) F(q_r + q_m) - (t + c_r)q_r - c_m q_m
\]
We know that the objective function is convex w.r.t \( q_r \) and \( q_m \). Looking at the first order derivatives, we get:

\[
\frac{\partial \pi}{\partial q_r} = p(q_r + q_m)f(q_r + q_m) + p - pF(q_r + q_m) - p(q_r + q_m)f(q_r + q_m) - c_r - t - q_r \frac{\partial t}{\partial q_r}
\]

(A.2)

\[
\frac{\partial \pi}{\partial q_m} = p(q_r + q_m)f(q_r + q_m) + p - pF(q_r + q_m) - p(q_r + q_m)f(q_r + q_m) - c_m
\]

(A.3)

Solving them simultaneously, we get: \( c_m = c_r + g^{-1}(q_r) + q_r \frac{\partial g^{-1}(q_r)}{\partial q_r} \) which has a solution denoted by \( q_r'' \). Then, we can find \( t'' \) from the relation \( Q(t'') = q_r'' \).

If \( t'' > 0 \) and \( F^{-1}(\frac{p-c_m}{p}) - q_r'' \geq 0 \), from equation (A.3), \( q_m^* = F^{-1}(\frac{p-c_m}{p}) - q_r'' \) and \( q_r^* = q_r'' \). However, if \( F^{-1}(\frac{p-c_m}{p}) - q_r'' < 0 \), then \( q_m^* = 0 \) and using equation (A.2), \( q_r^* = q_r' \) if \( t' = Q^{-1}(q_r') \geq 0 \).

However, if the above conditions are not satisfied and the resulting incentive values in the above cases ar less than 0, then \( t^* = 0 \) in the optimal solution. In that case, the manufacturer produces all products using the used materials since \( c_r < c_m \) and \( q_r^* = F^{-1}(\frac{p-c_m}{p}) \) and \( q_m^* = 0 \) if \( F^{-1}(\frac{p-c_m}{p}) \leq Q(0) \). Otherwise, \( q_r^* = Q(0) \) and \( q_m^* = max\{F^{-1}(\frac{p-c_m}{p}) - Q(0), 0\} \) due to equation (A.3).

Proof of Proposition 4

Similar to previous cases, either \( Q(t) = q_r \) or \( t = 0 \) should be satisfied in the optimal solution.

First, we solve the problem assuming \( w < c_m - c_r \). Under this assumption, the problem is to maximize: \( \pi_l = (p - c_r - pF(q_r) - Q^{-1}(q_r))q_r \). Equating the first order derivative to 0, we’ll get the following equation which has the solution \( q_r' \):

\[
\frac{\partial \pi_l}{\partial q_r} = p - c_r - pF(q_r) - Q^{-1}(q_r) - pq_rf(q_r) - q_r \frac{\partial Q^{-1}(q_r)}{\partial q_r} = 0
\]
Then, \( w' = p - c_r - pF(q'_r) \) and \( t' = Q^{-1}(q'_r) \). If \( w' < c_m - c_r \) and \( t' \geq 0 \), then this solution is optimal.

Otherwise, assume that, \( w = c_m - c_r \). Then the problem is to maximize:

\[
\pi_t = \max_t (c_m - c_r - t) \min_t (F^{-1}(\frac{p-c_m}{p}), Q(t))
\]

Then For this problem, if \( Q(t'') \leq F^{-1}(\frac{p-c_m}{p}) \) and \( t'' \geq 0 \) where \( t'' \) is the value that satisfies the equation \((c_m - c_r - t) \frac{\partial Q(t)}{\partial t} - Q(t) = 0\), then \( t^* = t'' \) and \( w^* = c_m - c_r \) will be the optimal solution. However, if \( Q(t'') > F^{-1}(\frac{p-c_m}{p}) \), then \( t'' = Q^{-1}(F^{-1}(\frac{p-c_m}{p})) \) will be the optimal incentive value if \( t'' \geq 0 \).

If none of the above conditions are satisfied, then \( t^* = 0 \) will be the optimal incentive value. In this case, the optimal value of the wholesale price will be \( w'' = p - c_r - pF(q''_r) \) where \( q''_r \) is the value that satisfies the following relation: \( p - c_r - pF(q_r) - pq_r f(q_r) = 0 \) if \( w'' \leq c_m - c_r \) and \( q''_r \leq Q(0) \). Otherwise, we can increase \( w \) until \( w = c_m - c_r \) or \( w = p - c_r - pF(Q(0)) \) such that \( q_r = Q(0) \).

\[\Box\]

**Proof of Proposition 5**

In the optimal solution, either \( Q(t) = q_r \) or \( t = 0 \) is satisfied. First, we assume that \( t > 0 \) and \( Q(t) = q_r \) in the optimal solution. Then, the manufacturer’s problem is:

\[
\max_{q_r, q_m} \pi = p_r E[R_r] - p_r E[R_r - q_r]^+ - c_r q_r - t q_r + p_m E[R_m] - p_m E[R_m - q_m]^+ - c_m q_m \tag{A.4}
\]

As demonstrated in Parlar and Goyal (1984), this objective function is jointly concave w.r.t \( q_r \) and \( q_m \). Thus, solving the following two equations simultaneously will give the optimal solution if the resulting values satisfy our assumptions.

\[
\frac{\partial \pi}{\partial q_r} = p_r - p_r H_r(q_r) - c_r - t - q_r \frac{\partial t}{\partial q_r} - p_m \alpha m P(D_r > q_r) + p_m \alpha m P(R_m > q_m, D_r > q_r) = 0
\]

\[
p_r H_r(q_r) + p_m \alpha m P(D_r > q_r) - p_m \alpha m P(R_m > q_m, D_r > q_r) = p_r - c_r - t - q_r \frac{\partial t}{\partial q_r}
\]
\[ p_r P(R_r < q_r) + p_m \alpha^{rm} P(R_m < q_m, D_r > q_r) = p_r - c_r - t - q_r \frac{\partial t}{\partial q_r} \]

\[ p_m P(R_m < q_m) + p_r \alpha^{mr} P(R_r < q_r, D_m > q_m) = p_m - c_m \]

If \( t = Q^{-1}(q_r) > 0 \) then this solution is optimal. Otherwise, \( t = 0 \) will be optimal and we need to solve the following two equations to find \( q_r \) and \( q_m \).

\[ p_r P(R_r < q_r) + p_m \alpha^{rm} P(R_m < q_m, D_r > q_r) = p_r - c_r \]

\[ p_m P(R_m < q_m) + p_r \alpha^{mr} P(R_r < q_r, D_m > q_m) = p_m - c_m \]

If the resulting \( q_r \leq Q(0) \), then this solution is optimal. Otherwise, \( t = 0, q_r = Q(0) \) will be optimal and \( q_m \) will be the solution to the equation

\[ p_m P(R_m < q_m) + p_r \alpha^{mr} P(R_r < q_r, D_m > q_m) = p_m - c_m \]

\[ \text{Proof of Proposition 6} \]

The proof follows from the same steps of the determination of optimal incentive values and production quantities in the above sections and from comparing the results of the centralized and decentralized model solutions. We outline the proof for Model 1 below and the proofs for Models 2 and 3 will be similar.

In Model 1, the optimal incentives to offer and the optimal production quantities for centralized and decentralized cases are given in section 3.1 of the main text. First consider the case where \( t^c > 0 \). In this case, the optimal value of the incentive to offer is given by the equation \( F_r(Q(t)) = \frac{p_r-c_r-t-Q(t)\alpha q_r}{p_r} \) and the optimal production quantity is \( q_r^C = Q(t_C) \). In the decentralized model, for a given \( w \), the optimal production quantity is equal to
\( q_r^D = F_r^{-1}(\frac{p_r - c_r - w}{p_r}) \). Thus \( w = t^c + q_r^c \frac{\partial c_r}{\partial q_r} \) will result in the same production quantity as in the centralized model. Then, the collection agency’s problem is:

\[
\max_{t \geq 0, w} \pi_l = w \min\{q_r^c, Q(t)\} - tQ(t)
\]

which has the optimal solution \( t^* = t^c \).

Next, we consider the case \( t^c = 0 \). In this case, \( q_r^c = \min\{F_r^{-1}(\frac{p_r - c_r}{p_r}), Q(0)\} \) for the centralized model. If \( w = 0 \) is offered in the contract, the collection agency offers \( t^* = t^c = 0 \) as the incentive since he has no benefit in collecting any used products. Also, the optimal production quantity under this contract will be equal to \( q_r^c \).

**Proof of Proposition 7**

For a given \( Q \), the manufacturer’s problem is:

\[
\begin{align*}
\text{Max} & \quad \int_0^{q_r} p_r x f_r(x) dx + p_r q_r \tilde{F}_r(q_r) - c_r q_r \\
\text{s.t.} & \quad q_r \leq Q
\end{align*}
\]

(A.5)

Without the constraint, the optimal production quantity is \( q_r' = F_r^{-1}(\frac{p_r - c_r}{p_r}) \). So, if \( Q \geq q_r' \), \( q_r' \) is the optimal solution. However, if \( Q < q_r' \), since the profit function is a convex function of \( q_r \), the optimal production quantity will be equal to \( Q \). Thus, the optimal production quantity, \( q_r^* \) satisfies \( q_r^* = \min\{F_r^{-1}(\frac{p_r - c_r}{p_r}), Q\} \).

References