ON \( p \)-ADIC PERIODS FOR MIXED TATE MOTIVES OVER A NUMBER FIELD

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Abstract. For a number field, we have a Tannaka category of mixed Tate motives at our disposal. We construct \( p \)-adic points of the associated Tannaka group by using \( p \)-adic Hodge theory. Extensions of two Tate objects yield functions on the Tannaka group, and we show that evaluation at our \( p \)-adic points is essentially given by the inverse of the Bloch-Kato exponential map.

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Introduction

For a number field \( E \), one has an abelian category of mixed Tate motives \( MT(E) \) [DG05]. A mixed Tate motive comes equipped with a weight filtration \( W \), and the associated graded pieces are sums of Tate objects. There is a natural fibre functor \( \omega \) defined by

\[
\omega(M) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathbb{Q}(n), \text{gr}_{-2n}^W(M));
\]

we denote by \( G_{\omega} \) the corresponding Tannaka group.

If \( \mathcal{O} \) denotes the ring of integers of \( E \) and \( x \in \text{Spec}(\mathcal{O}) \) is a closed point, then Deligne and Goncharov construct a Tannaka subcategory \( MT(\mathcal{O}_x) \) of \( MT(E) \) consisting of motives which are unramified at \( x \) [DG05, 1.6]. We will denote its group of tensor automorphisms by \( G_x \).

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To a mixed Tate motive $M$ we can attach its $p$-adic realization $M_p$ which is a representation of the Galois group of $E$ with coefficients in $\mathbb{Q}_p$. If the point $x$ lies over the prime $p$, then we can restrict in order to obtain a $p$-adic representation $M_{i,p}$ for the Galois group of the completion $E_x$ at $x$. We will show that $M_{i,p}$ is always semistable. Furthermore, $M_{i,p}$ is crystalline if and only if $M$ is unramified at $x$, i.e. $M \in MT(O_x)$ (Theorem 2.2.3). In fact, $p$-adic representations attached to mixed Tate motives are contained in an abelian subcategory which admits a fibre functor $\tau$ similar to $\omega$. Denoting by $H_\tau$ the corresponding Tannaka group over $\mathbb{Q}_p$, $p$-adic realization yields a group homomorphism

$$H_\tau \to G_\omega \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$ 

The main purpose of this paper is to construct an $E_{x,st}$-valued point $\eta_{st}$ of $H_\tau$, where $\text{Spec}(E_{x,st})$ is a 1-dimensional affine space over the field $E_x$. The $E_x$-valued points of $\text{Spec}(E_{x,st})$ correspond naturally to the extensions of the canonical logarithm $\log : O^\times_{E_x} \to E$ to $E^\times$. Therefore, any choice of such an extension induces via $\eta_{st}$ an $E_x$-valued point of $H_\tau$ and $\mathbb{G}_m$. For the Tannaka subcategory of crystalline representations the picture is simpler: if $H_{\tau,cris}$ denotes their Tannaka group and $\pi : H_\tau \to H_{\tau,cris}$ is the projection, then $\pi \circ \eta_{st}$ factors through $\text{Spec}(E_x)$ and we obtain an $E_x$-valued point $\eta$ of $H_{\tau,cris}$. We denote by $\eta^{ur}_x$ the image of $\eta$ in $G_x$.

To state our main theorem, we need to recall how extensions $M$ of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ in $MT(O_x)$ give rise to functions on $G_x$ for $n \geq 1$. The natural isomorphisms $\alpha : \mathbb{Q} \to \text{Hom}(\mathbb{Q}(n), \text{gr}_{-2n}^W M)$ and $\beta : \text{Hom}(\mathbb{Q}(0), \text{gr}_0^W M) \to \mathbb{Q}$ induce elements $\alpha^{-1} \in \omega(M)^\vee$ and $\beta^{-1} \in \omega(M)$; we set $M(\eta^{ur}_x) = \alpha^{-1}(\eta^{ur}_x : \beta^{-1}(1))$.

**Theorem** (Theorem 2.3.3). For all $n \geq 1$, the map

$$\text{Ext}^1_{MT(O_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to E_x, \quad M \mapsto M(\eta^{ur}_x),$$

is the composition of the $p$-adic realization

$$\text{Ext}^1_{MT(O_x)}(\mathbb{Q}(0), \mathbb{Q}(n)) \to \text{Ext}^1_{\text{crisys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

and the inverse of the Bloch-Kato exponential map $\mut(\mathbb{Q}_p)$. 

1. **FILTERED $\phi$-MODULES AND MIXED TATE FILTERED $\phi$-MODULES**

1.1. **Mixed Tate filtered $\phi$-modules.**

1.1.1. Let $K$ be a $p$-adic field with residue field $k$, i.e. $\text{char}(K) = 0$, $K$ is complete with respect to a fixed discrete valuation and the residue field $k$ is perfect of characteristic $p$. Let $W(k)$ be the ring of Witt vectors of $k$, $\sigma : W(k) \to W(k)$ the Frobenius lift and $K_0$ the field of fractions of $W(k).

1.1.2. We denote by $MF_K^\phi$ the category of filtered $\phi$-modules, i.e. the objects are triples $(M, \phi, F)$, where $(M, \phi)$ is an isocrystal over $K_0$ and $F$ is a descending, exhaustive and separated filtration on $M_K = M \otimes_{K_0} K$. We denote by $MF_K^{\phi, N}$ the category of filtered $(\phi, N)$-modules, i.e. objects are tuples $(M, \phi, N, F)$ with $(M, \phi, F) \in MF_K^\phi$ and $N : M \to M$ is a $K_0$-linear endomorphism such that $N\phi = p\phi N$. We consider $MF_K^\phi$ as full subcategory of $MF_K^{\phi, N}$ via the functor $(M, \phi, F) \mapsto (M, \phi, 0, F)$. 


The Dieudonné-Manin classification [Ma63, II, § 4.1] implies, by descent, that every isocrystal \((M, \phi)\) over \(K_0\) admits a slope decomposition 

\[ M = \bigoplus_{\lambda \in \mathbb{Q}} M_{\lambda}, \]

with \(\phi(M_{\lambda}) = M_{\lambda}\) and \((M_{\lambda}, \phi|_{M_{\lambda}})\) is isoclynic of slope \(\lambda\). From the relation \(N\phi = p\phi N\), it follows that \(N(M_{\lambda}) \subseteq M_{\lambda-1}\). In the following, we will use the notation:

\[ M_{\leq \lambda} := \bigoplus_{\lambda' \leq \lambda} M_{\lambda'}, \quad M_{\geq \lambda} := \bigoplus_{\lambda' \geq \lambda} M_{\lambda'} . \]

**Definition 1.1.3.** We say that an object \((M, \phi, F) \in MF^\phi_K\) is a *mixed Tate filtered \(\phi\)-module* if the following properties are satisfied:

1. There is an isomorphism of \(\phi\)-modules 
   \[ (M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{n_i} \sigma), \]
   for some index set \(I\), and \(n_i \in \mathbb{Z}\).

2. For all \(i \in \mathbb{Z}\) the natural map 
   \[ F^i M_K \to M_{\geq i} \otimes_{K_0} K \]
   is an isomorphism.

We say that \((M, \phi, N, F) \in MF^{\phi,N}_K\) is a *mixed Tate filtered \((\phi, N)\)-module* if \((M, \phi, F)\) is a mixed Tate filtered \(\phi\)-module.

We denote by \(MT^\phi_K\) (resp. \(MT^{\phi,N}_K\)) the full subcategory of \(MF^\phi_K\) (resp. \(MF^{\phi,N}_K\)) with mixed Tate filtered \(\phi\)-modules (resp. \((\phi, N)\)-modules) as objects. The categories \(MT^{\phi}_K\) and \(MT^{\phi,N}_K\) are additive. Again, we consider \(MT^{\phi}_K\) as full subcategory of \(MT^{\phi,N}_K\).

For \((M, \phi, N, F) \in MT^{\phi,N}_K\), it follows from Property (1) that all the slopes of \((M, \phi)\) are integers. From Property (2) we conclude that the Hodge polygon of \((M, \phi)\) equals the Newton polygon of \((M, \phi)\).

**Definition 1.1.4.** (Tate objects) Let \(n \in \mathbb{Z}\) be an integer. We define the *Tate object* \(K(n) \in MT^{\phi}_K\) by 

\[ K(n) := (K_0, p^{-n} \sigma, F), \]

with \(F\) defined by

\[ F^j = \begin{cases} K & \text{if } j \leq -n, \\ 0 & \text{if } j > -n. \end{cases} \]

**Definition 1.1.5.** (Weight filtration) Let \((M, \phi, N, F) \in MT^{\phi,N}_K\). Let \(i \in \mathbb{Z}\) be an integer. We define an object \(W_i(M, \phi, N, F) \in MF^{\phi}_K\) by 

\[ W_i(M, \phi, N, F) := (M_{\leq i}, \phi|_{M_{\leq i}}, N|_{M_{\leq i}}, F \cap M_{\leq i}) \]

We define an object \(\text{gr}^{W}_i(M, \phi, N, F) \in MT^{\phi}_K\) by

\[ \text{gr}^{W}_i(M, \phi, N, F) := (M_i, \phi|_{M_i}, \tilde{F}), \]

where \(\tilde{F}\) is defined as follows:

\[ \tilde{F}^i M_i = M_i, \quad \tilde{F}^{i+1} M_i = 0. \]
Note that \( N(M_{\leq i}) \subset M_{\leq i-1} \) and \( N |_{M_{\leq i}} \) is well-defined.

**Proposition 1.1.6.** Let \((M, \phi, N, F) \in MT^{\phi,N}_K\) and \(i \in \mathbb{Z}\). The following statements hold.

1. The object \(W_2(M, \phi, N, F)\) is contained in \(MT^{\phi,N}_K\).
2. There is an exact sequence
   \[
   0 \to W_2((i-1))(M, \phi, N, F) \to W_2(M, \phi, N, F) \to gr^{W}_{2i}(M, \phi, N, F) \to 0.
   \]

**Proof.** It is sufficient to prove the statement for \((M, \phi, 0, F)\), i.e. for objects in \(MT^{\phi}_K\).

For (1). It is obvious that
\[
W_2(W_2((i+1))(M, \phi, F)) = W_2(M, \phi, F),
\]
for all \((M, \phi, F)\). Therefore we may reduce to the case \(W_2((i+1))(M, \phi, F) = (M, \phi, F)\).

In this case \(M = M_{\leq i} \oplus M_{i+1}\), and we have to prove that for all \(j \in \mathbb{Z}\) the map
\[
F^j \cap (M_{\leq i} \otimes_{K_0} K) \to (M_{\leq i})_{\geq j} \otimes_{K_0} K
\]
is an isomorphism. Since \((M, \phi, F)\) is an object in \(MT^{\phi}_K\), the map is injective. In particular, the map is an isomorphism for all \(j \geq i + 1\).

We need to show the surjectivity for \(j \leq i\). By assumption, for every \(m \in (M_{\leq i})_{\geq j} \otimes_{K_0} K\) there exists a preimage \(m' \in F^j M_K\). By definition, the projection of \(m'\) to \(M_{i+1} \otimes_{K_0} K\) vanishes, thus \(m' \in F^j \cap (M_{\leq i} \otimes_{K_0} K)\).

For (2). There is an obvious morphism \(W_2((i-1))(M, \phi, F) \to W_2(M, \phi, F)\) in \(MT^{\phi}_K\). The morphism \(W_2(M, \phi, F) \to gr^{W}_{2i}(M, \phi, N, F)\) is defined by the projection \(M_{\leq i} \to M_i\). Since \(F^{i+1} \cap (M_{\leq i} \otimes_{K_0} K) = 0\), the projection is compatible with the filtrations. Therefore the sequence \((1.1.1)\) is well-defined.

In order to prove that the sequence is exact we need to show that it is an exact sequence of \(\phi\)-modules and an exact sequence of filtered \(K\)-vector spaces. The first statement is obvious. For the second statement we note that all members in the sequence \((1.1.1)\) are objects in \(MT^{\phi}_K\), thus the Hodge polygons equal the Newton polygons. In particular,
\[
\dim(F^j \cap M_{\leq i}) = \dim(F^j \cap M_{\leq i-1}) + \dim \tilde{F}^j,
\]
for all \(j \in \mathbb{Z}\). This immediately implies the claim.

**Corollary 1.1.7.** The category \(MT^{\phi,N}_K\) is contained in the category of weakly admissible filtered \((\phi, N)\)-modules.

**Proof.** We use the fact that weakly admissible filtered \((\phi, N)\)-modules are stable under extensions. Therefore the claim follows from Proposition 1.1.6 provided we prove that \(gr^{W}_{2i}(M, \phi, N, F)\) is weakly admissible for all \((M, \phi, N, F) \in MT^{\phi,N}_K\) and all \(i \in \mathbb{Z}\). By Definition 1.1.5, \(gr^{W}_{2i}(M, \phi, N, F)\) is isomorphic to a direct sum of Tate objects \(K(-i)\). Since Tate objects are (weakly) admissible, we are done.

In contrast to the category \(MF^{\phi,N}_K\), the category of weakly admissible filtered \((\phi, N)\)-modules \(MF^{\phi,N,wa}_K\) is an abelian category.
Proposition 1.1.8. Let $f : (M, \phi_M, N_M, F_M) \to (M', \phi_{M'}, N_{M'}, F_{M'})$ be a morphism in $MT^\phi_{K}$. We denote by $\ker(f)$ and $\coker(f)$ the kernel of $f$ and the cokernel of $f$ in $MT^\phi_{K}$, respectively. Then $\ker(f)$ and $\coker(f)$ are contained in $MT^\phi_{K}$. In particular, $MT^\phi_{K}$ is an abelian category.

Proof. First, consider the full subcategory $C$ of isocrystals over $K_0$ with objects $(M, \phi)$ such that there exists an isomorphism $(M, \phi) \cong \bigoplus_{i \in I}(K_0, p^m \sigma)$.

It is easy to see that $C$, as subcategory of the category of isocrystals, contains all the kernels and cokernels of morphisms in $C$.

We denote by $f_0$ the induced morphism $(M, \phi_M) \to (M', \phi_{M'})$. Then

$$\ker(f) = (\ker(f_0), \phi|_{\ker(f_0)}, N|_{\ker(f_0)}, F \cap (\ker(f_0) \otimes K_0))$$

We know that $\ker(f_0) \in C$ and thus satisfies Property (1) of Definition 1.1.3. It remains to show that $F_M^i \cap (\ker(f_0) \otimes K_0) \to \ker(f_0)_{\geq i} \otimes K_0$ is an isomorphism. We have a commutative diagram

$$
\begin{array}{ccc}
0 & \to & F_M^i \cap (\ker(f_0) \otimes K_0) \\
\downarrow & & \downarrow \cong \\
0 & \to & \ker(f_0)_{\geq i} \otimes K_0 \\
\end{array}
\begin{array}{ccc}
& & \to \\
\cong & & \cong \\
& & \\
\end{array}
\begin{array}{ccc}
0 & \to & M_{\geq i} \otimes K_0 \\
\to & & \to \\
M_{\geq i} \otimes K_0 & \to & M_{\geq i} \otimes K_0.
\end{array}
$$

Moreover, both rows are exact, which implies Property (2) of Definition 1.1.3.

The claim for the cokernel follows dually. □

1.1.9. The categories $MT^\phi_{K}$ and $MT^\phi_{K}$ are $\mathbb{Q}_p$-linear rigid $\otimes$-categories.

Lemma 1.1.10. The functor

$$(1.1.2) \quad \tilde{\omega} : MT^\phi_{K} \to (\mathbb{Q}_p\text{-vector spaces}), \quad (M, \phi, N, F) \mapsto \bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_n(M, \phi, F),$$

with

$$\tilde{\omega}_n(M, \phi, F) = \text{Hom}_{MT^\phi_{K}}(K(n), \text{gr}^W_{-2n}(M, \phi, F)),$$

is a fibre functor. In particular, $(MT^\phi_{K}, \tilde{\omega})$ and $(MT^\phi_{K}, \tilde{\omega})$ are Tannaka categories.

Proof. It is easy to see that $\tilde{\omega}$ is a $\otimes$-functor. In order to see that $\tilde{\omega}$ is exact and faithful we will prove the existence of an isomorphism

$$(1.1.3) \quad \tilde{\omega}_K \cong (\gamma : (M, \phi, N, F) \mapsto M),$$

where $\tilde{\omega}_K(M, \phi, N, F) = \tilde{\omega}(M, \phi, N, F) \otimes_{\mathbb{Q}_p} K_0$ and $\gamma$ forgets about $\phi$, $N$ and $F$. Since $\gamma$ is exact and faithful, this will imply the claim.

In order to construct (1.1.3), we observe that there is a functorial isomorphism

$$(1.1.4) \quad \text{Hom}_{MT^\phi_{K}}(K(n), \text{gr}^W_{-2n}(M, \phi, N, F)) \otimes_{\mathbb{Q}_p} K_0 \to M_{-n}, \quad \phi \otimes a \mapsto a \cdot \phi(1).$$

□
Proposition 1.1.11. An object \((M, \phi, N, F) \in MF^{\phi,N,wa}_K\) belongs to \(MT^{\phi,N}_K\) if and only if there exists an increasing exhaustive separated filtration \(W\) by subobjects of \((M, \phi, N, F) \in MF^{\phi,N,wa}_K\) such that \(W_i/W_{i-1}\) vanishes if \(i\) is odd, and is a sum of Tate objects \(K(-\frac{i}{2})\) if \(i\) is even.

Proof. For \((M, \phi, N, F) \in MT^{\phi,N}_K\), such a filtration exists by Definition 1.1.5, Proposition 1.1.6, and the fact that \(\text{gr}_{i}^{W}(M, \phi, N, F)\) is a sum of Tate objects \(K(-\frac{i}{2})\).

Suppose now that \((M, \phi, N, F) \in MF^{\phi,N,wa}_K\) admits a filtration \(W\) satisfying the assumptions. It is easy to see that \((M, \phi)\) satisfies Property (1) of Definition 1.1.3.

In general, if \(0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0\) is an exact sequence in \(MF^{\phi,N,wa}_K\), and \(M_1, M_2\) satisfy Property (2), then \(M\) satisfies Property (2). By induction on \(i\) we conclude that \(W_i \in MT^{\phi,N}_K\) for all \(i\).

It is clear that any filtration as in Proposition 1.1.11 has to coincide with the weight filtration, and that any morphism between two objects in \(MT^{\phi,N}_K\) has to be strict with respect to the weight filtrations on these objects.

1.2. The crystalline logarithmic point.

1.2.1. Recall from (1.1.2) that we have a fibre functor \(\tilde{\omega}\) equipping \(MT^{\phi}_K\) and \(MT^{\phi,N}_K\) with the structure of Tannaka categories (Lemma 1.1.10). Let \(G_{\omega}\) and \(G_{\omega}^{st}\) denote the pro-algebraic groups which represent tensor automorphisms of \(\tilde{\omega}\) on \(MT^{\phi}_K\) and \(MT^{\phi,N}_K\), respectively. In other words, we have \(G_{\omega} = \text{Aut}^{\otimes}_{MT^{\phi}_K}\tilde{\omega}\) and \(G_{\omega}^{st} = \text{Aut}^{\otimes}_{MT^{\phi,N}_K}\tilde{\omega}\). The goal of this section is to construct a non-trivial \(K\)-valued point \(\eta\) of \(G_{\omega}\).

Definition 1.2.2. For \((M, \phi, F) \in MT^{\phi}_K\) we define

\[\eta(M, \phi, F) : M_K \rightarrow M_K\]

to be the unique endomorphism rendering the following diagram commutative:

\[
\begin{array}{cccc}
M_K &=& \bigoplus_{i \in \mathbb{Z}} M_i \otimes_{K_0} K & \rightarrow \bigoplus_{i \in \mathbb{Z}} M_{\geq i} \otimes_{K_0} K \\
&\downarrow \eta(M, \phi, F) & & \downarrow (\oplus_{i \in \mathbb{Z}} \pi_i)^{-1} \\
M_K &\rightarrow & \sum_{i \in \mathbb{Z}} F_i M_K,
\end{array}
\]

where \(\iota_i : M_i \rightarrow M_{\geq i}\) is the obvious inclusion, \(\pi_i : F_i M_K \rightarrow M_{\geq i} \otimes_{K_0} K\) is the projection and therefore by definition an isomorphism (Definition 1.1.3(2)), and \(\sum_{i \in \mathbb{Z}}\) is the sum over the obvious inclusions.

Lemma 1.2.3. The morphisms \(\eta\) from Definition 1.2.2 define a tensor automorphism of the fibre functor \(\tilde{\omega}_K = \tilde{\omega} \otimes_{Q_p} K\).

Proof. Via the \(\otimes\)-isomorphism (1.1.3) we may identify \(\tilde{\omega} \otimes_{Q_p} K_0\) with the forgetful functor \((M, \phi, F) \rightarrow M\). After tensoring with \(K\) we obtain \(\tilde{\omega}_K(M, \phi, F) = M_K\).

First, let us prove that \(\eta(M, \phi, F)\) is an automorphism. We denote by

\[\eta(M, \phi, F)[i, j] : M_j \otimes_{K_0} K \rightarrow M_i \otimes_{K_0} K\]
the composition with the inclusion $M_i \otimes_K K \to M_K$ and the projection $M_K \to M_i \otimes_K K$. It is easy to see from the definitions that

$$\eta(M, \phi, F)[i, j] = \begin{cases} 0 & \text{if } i > j, \\ id_{M_i} & \text{if } i = j. \end{cases}$$

(1.2.2)

Therefore $\eta(M, \phi, F)$ is an automorphism.

Since the diagram (1.2.1) is functorial, $\eta$ defines a natural transformation. The compatibility with the tensor product is obvious.

1.2.4. Let us explain the construction of $\eta$ in the formalism of [Del94]. For $(M, \phi, F) \in MT^p_K$ there are three filtrations on $M_K$:

1. The weight filtration:

$$W_i M_K = \begin{cases} M_{\geq i} \otimes_K K & \text{if } i \text{ is even} \\ M_{\leq i} \otimes_K K & \text{if } i \text{ is odd}. \end{cases}$$

2. The Hodge filtration $F$.
3. The filtration $F^i M_K := M_{\geq i} \otimes_K K$ for all $i \in \mathbb{Z}$.

The three filtrations $W, F, \bar{F}$ satisfy the condition

$$\text{Gr}_F^p \text{Gr}_F^q \text{Gr}_W^n M_K = 0 \quad \text{for } n \neq p + q,$$

of [Del94] §1.1. Induced by $F, \bar{F}$, we obtain maps

$$a_F : M_K = \bigoplus_{i \in \mathbb{Z}} F^i \cap W_{2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}_2^W M_K,$$

$$a_{\bar{F}} : M_K = \bigoplus_{i \in \mathbb{Z}} \bar{F}^i \cap W_{2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}_2^W M_K,$$

where $F^i \cap W_{2i} \to \text{Gr}_2^W M_K$ is the natural map (and similarly for $\bar{F}$). We obtain a unipotent automorphism $d = a_F a_{\bar{F}}^{-1}$ of $\bigoplus_{i \in \mathbb{Z}} \text{Gr}_2^W M_K$.

It is easy to see that we have the equality

$$\eta(M, \phi, F) = a_{\bar{F}}^{-1} \circ d \circ a_F.$$

1.2.5. Let us see in explicit terms how $\eta$ compares the crystalline structure with the Hodge filtration. For $(M, \phi, F) \in MT^p_K$ we say that $v_1, \ldots, v_d \in \tilde{\omega}(M, \phi, F) \otimes_{Q_p} K$ is a homogeneous basis if it is a basis of $\tilde{\omega}(M, \phi, F) \otimes_{Q_p} K$, and for every $v_i$ there is an integer $n_i$ with $v_i \in \tilde{\omega}_{n_i}(M, \phi, F) \otimes_{Q_p} K$; we set $\deg(v_i) = n_i$.

Recall that for all integers $i$ we have isomorphisms

$$a_i : M_{-i} \otimes_K K \xrightarrow{\cong} \tilde{\omega}_i(M, \phi, F) \otimes_{Q_p} K,$$

$$b_i : F^{-i} \cap W_{-2i} M_K \xrightarrow{\cong} \tilde{\omega}_i(M, \phi, F) \otimes_{Q_p} K.$$

The first map is the inverse of (1.1.4) and the second map is given by the composition

$$b_i : F^{-i} \cap W_{-2i} M_K \xrightarrow{\text{projection}} M_{-i} \otimes_K K \xrightarrow{a_i} \tilde{\omega}_i(M, \phi, F) \otimes_{Q_p} K.$$

For a homogeneous basis $\{v_j\}$ we set

$$v_j^{\text{cryst}} := a_{\deg(v_j)}^{-1}(v_j), \quad v_j^{\text{Hodge}} := b_{\deg(v_j)}^{-1}(v_j).$$
We denote by \( \{ v_j^{\cryst, \vee} \} \) the basis dual to \( \{ v_j^{\cryst, \vee} \} \). By definition of \( \eta \) we have 
\[
v_j^{\cryst, \vee}(\eta(v_j)) = v_j^{\cryst, \vee}(v_j^{\Hodge}).
\]

1.2.6. By Lemma 1.2.3 we obtain a \( K \)-valued point \( \eta \in G_{\mathcal{C}}(K) \); we call this point the logarithmic point. Let us check that \( \eta \) is not the identity.

**Proposition 1.2.7.** Let \( n \in \mathbb{Z} \) be an integer. We have
\[
\Ext^1_{MT_K}(K(0), K(n)) \cong \begin{cases} K & \text{if } n > 0 \\ 0 & \text{if } n \leq 0. \end{cases}
\]

Let
\[
0 \to K(n) \xrightarrow{\eta} (E, \phi, F) \xrightarrow{\eta} K(0) \to 0
\]
be an extension. For \( n \neq 0 \), there are unique sections \( f : E \to K_0 \) and \( v : K_0 \to E \) of the underlying maps of \( K_0 \)-isocrystals of \( \iota \) and \( \pi \), respectively. The isomorphism (1.2.3), for \( n \neq 0 \), is given by the formula
\[
E \mapsto f(\eta(E, \phi, F)(v(1))).
\]

**Proof.** First, we consider the case \( n = 0 \). Let \( (E, \phi, F) \) be as in (1.2.3). We have \( F^0(E_K) = E_K \) and \( F^1(E_K) = 0 \) by Definition 1.1.3(2). In view of Definition 1.1.3(2) there is an isomorphism \( (E, \phi) \cong (K_0, \sigma) \oplus (K_0, \sigma) \), thus there is a section of \( \pi \) in \( MT_K^\phi \).

For \( n \neq 0 \): From the slope decomposition we obtain natural sections \( f, v \) as \( \phi \)-modules. If \( n < 0 \) then \( F^1E_K = \iota(K) \) which means \( (E, \phi, F) = K(0) \oplus K(n) \).

For \( n > 0 \), we can uniquely write \( F^{n+1}E_K = K(a \cdot \iota(1) + v(1)) \) with \( a \in K \). Obviously,
\[
f(\eta(E, \phi, F)(v(1))) = a.
\]
It is clear that \( F^{n+1}E_K \) is the only invariant for extensions. \( \square \)

1.2.8. Recall that we have a fibre functor \( \tilde{\omega} \) to the category of \( \mathbb{Q}_p \)-vector spaces. In the obvious way \( \tilde{\omega} \) factors through the category of graded \( \mathbb{Q}_p \)-vector spaces. Furthermore, we have an automorphism \( \eta \) of \( \tilde{\omega}_K \) (Lemma 1.2.3).

**Definition 1.2.9.** We define \( \mathcal{C}_\eta \) to be the category of pairs \( (V, \eta) \), where \( V \) is a finite dimensional graded \( \mathbb{Q}_p \)-vector space and \( \eta : V \otimes K \to V \otimes K \) is a \( K \)-linear map such that for all \( n \in \mathbb{Z} \):
\[
(\eta - id)(V_n \otimes K) \subset \bigoplus_{i>0} V_i \otimes K.
\]
Morphisms \( (V_1, \eta_1) \to (V_2, \eta_2) \) are \( \mathbb{Q}_p \)-linear morphisms \( \tau : V_1 \to V_2 \) which respect the grading and commute with the endomorphisms \( \eta_i \), i.e. \( \eta_2 \circ (\tau \otimes id_K) = (\tau \otimes id_K) \circ \eta_1 \).

The category \( \mathcal{C}_\eta \) is a \( \otimes \)-category with
\[
(V_1, \eta_1) \otimes (V_2, \eta_2) = (V_1 \otimes V_2, \eta_1 \otimes \eta_2).
\]

**Proposition 1.2.10.** The functor
\[
\Psi : MT_K^\phi \to \mathcal{C}_\eta
\]
\[
(M, \phi, F) \mapsto \left( \bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_n(M, \phi, F), \eta(M, \phi, F) \right)
\]
is an equivalence of $\otimes$-categories.

**Proof.** By Lemma 1.2.3, $\eta$ is functorial and $\Psi$ is a $\otimes$-functor. It follows from (1.2.2) that

$$(\eta - id)(\tilde{\omega}_n \otimes K) \subset \bigoplus_{i > n} \tilde{\omega}_i \otimes K.$$ 

We define a functor

$$\Phi : C_{\eta -} \rightarrow MT_{K}^{\phi}$$

$$(\oplus_{n \in \mathbb{Z}} V_n, \eta) \mapsto (\oplus_{n \in \mathbb{Z}} (V_{-n} \otimes_{\mathbb{Q}_p} K_0, p^{-n} \otimes \sigma), F),$$

with the following filtration:

$$F^i := \eta \left( \bigoplus_{j \geq i} V_{-j} \otimes_{\mathbb{Q}_p} K \right),$$

for all $i$. Property (1.2.3) implies that $\Phi$ is well-defined. From Definition 1.2.2 it easily follows that $\Psi \circ \Phi = id_{C_{\eta -}}$.

On the other hand, we have $\Phi \circ \Psi \simeq id_{MT_{K}^{\phi}}$ via

$$\Phi \circ \Psi(M, \phi, F) \mapsto (M, \phi, F)$$

$$\bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_{-n}(M, \phi, F) \otimes_{\mathbb{Q}_p} K_0 \overset{(1.1.4)}{\longrightarrow} M.$$ 

\[\square\]

**Remark 1.2.11.** Via the dictionary of Section 1.2.4 Proposition 1.2.10 is a variant of [Del94, Proposition 1.2].

1.3. The semistable logarithmic point.

1.3.1. Let $K$ be as in 1.1.1 with residue field $k$. We denote by $\nu_K$ the valuation of $K$.

1.3.2. Recall that we have a homomorphism

$$[\cdot] : k^\times \rightarrow \mathcal{O}_K^\times, \quad x \mapsto [x],$$

by taking the Teichmüller lift. Denoting by $U_K := \{ x \in \mathcal{O}_K^\times ; x \in 1 + m_K \}$ the 1-units, we obtain a decomposition

$$\mathcal{O}_K^\times = k^\times \times U_K.$$

The logarithm

$$(1.3.1) \quad \log : \mathcal{O}_K^\times \rightarrow \mathcal{O}_K$$

is by definition trivial on the factor $k^\times$ and is given by

$$\log(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{(u - 1)^n}{n} \text{ for all } u \in U_K.$$
1.3.3. We consider \( O_{K,\mathbb{Q}}^\times := O_K^\times \otimes_{\mathbb{Z}} \mathbb{Q} \) and \( K_{\mathbb{Q}}^\times := K^\times \otimes_{\mathbb{Z}} \mathbb{Q} \) as \( \mathbb{Q} \)-vector spaces, therefore we may form the symmetric algebras Sym\(_{\mathbb{Q}}(O_{K,\mathbb{Q}}^\times)\) and Sym\(_{\mathbb{Q}}(K_{\mathbb{Q}}^\times)\). The exact sequence
\[
0 \to O_{K,\mathbb{Q}}^\times \to K_{\mathbb{Q}}^\times \overset{\nu_K}{\longrightarrow} \mathbb{Q} \to 0
\]
implies that Spec(Sym\(_{\mathbb{Q}}(K_{\mathbb{Q}}^\times)\)) is a 1-dimensional affine space over the scheme Spec(Sym\(_{\mathbb{Q}}(O_{K,\mathbb{Q}}^\times)\)). In other words, for \( x \in K^\times \) with \( \nu_K(x) \neq 0 \), the map
\[
\text{Sym}_{\mathbb{Q}}(O_{K,\mathbb{Q}}^\times)[X] \to \text{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times), \quad X \mapsto x,
\]
is an isomorphism.

The logarithm (1.3.1) induces a ring homomorphism
\[
\text{Sym}_{\mathbb{Q}}(O_{K,\mathbb{Q}}^\times) \to K, \text{ because } K \text{ is torsion free.}
\]

**Definition 1.3.4.** We define the \( K \)-algebra \( K_{\text{st}} \) by
\[
K_{\text{st}} := \text{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times) \otimes_{\text{Sym}_{\mathbb{Q}}(O_{K,\mathbb{Q}}^\times)} K.
\]
By base change, Spec\((K_{\text{st}})\) is a 1-dimensional affine space over \( K \). We have a natural logarithm
\[
\log_{\text{st}} : K^\times \to K_{\text{st}}, \quad x \mapsto x \otimes 1.
\]
The \( K \)-valued points of Spec\((K_{\text{st}})\) admit the following description:
\[
\text{Spec}(K_{\text{st}})(K) = \{ \text{extensions } \log : K^\times \to K \text{ of (1.3.1)} \} \quad f \mapsto f^* \circ \log_{\text{st}}.
\]
By an extension \( \log : K^\times \to K \) we mean a homomorphism such that the restriction to \( O_{K,\mathbb{Q}}^\times \) equals (1.3.1).

1.3.5. The \( p \)-adic Hodge theory for \( K \) (and fixed valuation \( \nu_K \)) depends for semistable representations on the choice of a logarithm
\[
\log : K^\times \to K.
\]
It will be important for us that our constructions do not depend on a particular choice, and for this we have to recall the basic constructions of \( p \)-adic Hodge theory.

We denote by \( R \) the ring
\[
R := \varprojlim \frac{O_K}{pO_K},
\]
where the maps are given by raising to the \( p \)-th power \( x \mapsto x^p \). Denoting by \( C_K = \hat{K} \) the \( p \)-adic completion of \( K \) we have a multiplicative bijection
\[
\varprojlim O_{C_K} \to R,
\]
where the projective system is defined by raising to the \( p \)-th power again. In other words, we can represent every element \( x \) in \( R \) by \((x^{(0)}, x^{(1)}, \ldots) \) with \( x^{(n)} \in O_{C_K} \) and \( x^{(n-1)} = (x^{(n)})^p \).

Let \( \nu_K \) (resp. \( \nu_{C_K} \)) be the extension of \( \nu_K \) (resp. \( \nu_{C_K} \)) to \( \hat{K} \) (resp. \( C_K \)). The map
\[
\nu_R : R \setminus \{0\} \to \mathbb{Q}, \quad x \mapsto \nu_{C_K}(x^{(0)})
\]
can be extended to a valuation
\[
\nu_R : \text{Frac}(R)^\times \to \mathbb{Q}
\]
with valuation ring $R$.

Let $B_{cris}$ be the crystalline period ring; we define

$$B_{st} = \text{Sym}_Q(\text{Frac}(R)^{\times} \otimes \mathbb{Z} \mathbb{Q}) \otimes \text{Sym}_Q(R^{\times} \otimes \mathbb{Z} \mathbb{Q}) B_{cris},$$

where $\text{Sym}_Q(R^{\times} \otimes \mathbb{Z} \mathbb{Q}) \to B_{cris}$ is induced by the crystalline logarithm

$$\log_{cris} : R^{\times} \to B_{cris}.$$

Again, $R^{\times} = \bar{k}^{\times} \times (1 + m R)$; $\log_{cris}$ is trivial on $\bar{k}^{\times}$ and given by

$$\log_{cris}(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{[u] - 1}{n} n$$

for $u \in 1 + m R$, where $[u]$ denotes the Teichmüller lift of $u$ in the Witt ring $W(R)$ of $R$.

By construction we have a natural logarithm

$$\log_{st} : \text{Frac}(R)^{\times} \to B_{st}, \ x \mapsto x \otimes 1.$$

The ring $B_{st}$ has the following properties.

1. We have a $\text{Gal}(\bar{K}/K)$-action on $B_{st}$ extending the action on $B_{cris}$.
2. We have a Frobenius map $\phi : B_{st} \to B_{st}$ extending the Frobenius map on $B_{cris}$. Moreover,

$$\phi \circ \log_{st} = p \log_{st}.$$

3. We have a $B_{cris}$-linear derivation $N : B_{st} \to B_{st}$ such that

$$N(\log_{st}(x)) = \nu_R(x)$$

for all $x \in \text{Frac}(R)^{\times}$.

After choosing a logarithm

$$\log : K^{\times} \to K,$$

which extends (1.3.1), we obtain a morphism of $B_{cris}$-algebras

$$\gamma_{\log} : B_{st} \to B_{dR}.$$

The morphism depends on the choice of $\log$, and the filtration on $B_{st}$ induced by the filtration on $B_{dR}$ via $\gamma_{\log}$ depends on $\log$. In order to simplify the comparison between different logarithms we will restrict ourselves to logarithms $\log$ such that $\log(K_0^{\times}) \subset K_0$. In other words, we will only consider $K_0$-valued points of $\text{Spec}(K_{0,st})$.

**Proposition 1.3.6.** For $\log, \log' \in \text{Spec}(K_{0,st})(K_0)$ there is a unique ring homomorphism

$$\delta_{\log, \log'} : B_{st} \to B_{st}$$

such that $\gamma_{\log'} \circ \delta_{\log, \log'} = \gamma_{\log}$. The map $\delta_{\log, \log'}$ is given by

$$\delta_{\log, \log'} = \exp \left( \frac{\log(x) - \log'(x)}{\nu_K(x)} N \right)$$

for every $x \in K_0 \setminus \mathcal{O}_{K_0}^{\times}$.

**Proof.** Uniqueness follows from the fact that $\gamma_{\log}$ is injective.

Choose $\hat{p} \in R$ with $\hat{p}^{(0)} = p$. By definition we have

$$\gamma_{\log}(\log_{st}(\hat{p})) = \log_{dR}([\hat{p}] / p) + \log(p),$$
where $\log_{dR}$ is defined by the usual series since $[\tilde{p}]/p$ is a 1-unit in $B_{dR}$. Since Spec($B_{\text{st}}$) is a 1-dimensional affine space over Spec($B_{\text{cris}}$), there exists a unique morphism of $B_{\text{cris}}$-algebras $\delta_{\log, \log'}$ such that

$$\delta_{\log, \log'}(\log_{\text{st}}(\tilde{p})) = \log_{\text{st}}(\tilde{p}) + \log(p) - \log'(p).$$

Obviously, $\delta_{\log, \log'}$ satisfies $\gamma_{\log} \circ \delta_{\log, \log'} = \gamma_{\log}$ and the equality $\text{(1.3.6)}$. \hfill $\square$

By using $\gamma_{\log}$ we obtain a filtration on $B_{\text{st}}$. The $p$-adic Hodge theory [CF00, Thm. 4.1] asserts that the functor

$$(1.3.6) \quad D_{\text{st, log}} : (\text{semistable } \mathbb{Q}_p\text{-representations of } \text{Gal}(\bar{K}/K)) \to MF_{K}^{\phi, N, w.a.}$$

$$V \mapsto \left( B_{\text{st}} \otimes_{\mathbb{Q}_p} V \right)^{\text{Gal}(\bar{K}/K)}$$

is an equivalence of categories. We will use the subscript log in $D_{\text{st, log}}$ to emphasize the dependence on log.

Denoting by $\text{forget}_F$ the functor $\text{forget}_F(M, \phi, N, F) = (M, \phi, N)$, we get

$$\text{forget}_F \circ D_{\text{st, log}} = \text{forget}_F \circ D_{\text{st, log}}',$$

because only the filtration depends on the embedding to $B_{dR}$. Proposition $\text{[1.3.9]}$ implies that for the filtrations we have the following comparison:

$$(1.3.7) \quad F_{D_{\text{st, log}^{'}}}(V) = \exp \left( \log(x) - \log'(x) \otimes N \right) F_{D_{\text{st, log}}}(V),$$

for all $i \in \mathbb{Z}$ and all $x \in K_0 \setminus \mathcal{O}_{K_0}^\times$.

**Definition 1.3.7.** Let $K$ be a $p$-adic field. We denote by $MT_{G_K}$ the full subcategory of $p$-adic representations $V$ of $\text{Gal}(\bar{K}/K)$ which admit an increasing exhaustive separated filtration $W$ by subrepresentations of $V$ such that $W_i/W_{i-1}$ vanishes if $i$ is odd, and is a sum of Tate objects $\mathbb{Q}_p(-\frac{i}{2})$ if $i$ is even. We call an object of $MT_{G_K}$ a **mixed Tate representation** of $\text{Gal}(\bar{K}/K)$.

**Proposition 1.3.8.** Let $\log \in \text{Spec}(K_{0, \text{st}})(K_0)$. Then

$$MT_{G_K} = D_{\text{st, log}}^{-1}(MT_{K}^{\phi, N}).$$

In particular, every mixed Tate representation is semistable.

**Proof.** From Proposition $\text{[1.1.11]}$ it follows that every object in $D_{\text{st, log}}^{-1}(MT_{K}^{\phi, N})$ admits a filtration $W$ satisfying the properties of Definition $\text{[1.3.7]}$.

Now, suppose that $V$ is a $p$-adic representation of $\text{Gal}(\bar{K}/K)$ which admits a filtration $W$ as in Definition $\text{[1.3.7]}$. If we know that $V$ is semistable then clearly $D_{\text{st, log}}(V) \in MT_{K}^{\phi, N}$ by Proposition $\text{[1.1.11]}$ again. Therefore it suffices to prove that $V$ is semistable.

We use induction on the length of the filtration $W$ of $V$. If the filtration $W$ has length $\leq 1$, semistability of $V$ follows from those of $\mathbb{Q}_p(n)$. In general, let $n$ be the smallest integer such that $W_{2n}V = V$. Then we have an exact sequence

$$0 \to (W_{2n} - V) \otimes \mathbb{Q}_p(n) \to V \otimes \mathbb{Q}_p(n) \to (V/W_{2n} - V) \otimes \mathbb{Q}_p(n) \to 0.$$ 

By the induction hypothesis the terms on the left and right are semistable. Moreover, since the weights of the term on the left are $\leq -2$ and the term on the right
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has weight 0, we have

\[ F^0 dR((W_{2n-2}V) \otimes \mathbb{Q}_p(n)) = 0 \]

\[ F^0(dR((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n))) = dR((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n)). \]

Therefore \[ \text{[Nek93, Proposition 1.28]} \] shows that the middle term is also semistable. \[ \square \]

Obviously, \[ (1.3.8) \]

\[ \tau = \tilde{\omega} \circ D_{st, \log} \]

is independent of log, and \((MT_{G_K}, \tau)\) is a Tannaka category (by Lemma \[ 1.1.10 \]).

1.3.9. Recall from Lemma \[ 1.1.10 \] that \((MT_{K_0, N}^\phi, \tilde{\omega})\) is a Tannaka category. We will use the ring \( K_{st} \) (Definition \[ 1.3.4 \]) and \( \log_{st} \) (1.3.3).

**Definition 1.3.10.** For a logarithm \( \log \in \text{Spec}(K_{0, st})(K_0) \) and \((M, \phi, N, F) \in MT_{K_0}^\phi, N \) we define \( \eta_{st, \log}(M, \phi, N, F) \in \text{End}_{K_{st}}(M \otimes K_0 K_{st}) \) by

\[ \eta_{st, \log}(M, \phi, N, F) := \exp \left( \frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N \right) \eta(M, \phi, F), \]

for \( x \in K_0^\times \mathcal{O}_{K_0}^\times \). For the definition of \( \eta(M, \phi, F) \) we refer to Definition \[ 1.2.2 \].

Obviously, \( \eta_{st, \log} \) does not depend on the choice \( x \in K_0^\times \mathcal{O}_{K_0}^\times \), but it depends on \( \log \).

**Lemma 1.3.11.** Let \( \log \in \text{Spec}(K_{0, st})(K_0) \). The morphisms \( \eta_{st, \log} \) from Definition \[ 1.3.10 \] define a tensor automorphism of the fibre functor \( \tilde{\omega}_{K_{st}} = \tilde{\omega} \otimes_{\mathbb{Q}_p} K_{st} \). In other words, \( \eta_{st, \log} \in G_{st}^\otimes(K_{st}) \) with \( G_{st}^\otimes = \text{Aut}^\otimes_{MT_{G_K}} \tilde{\omega} \).

**Proof.** Via the \( \otimes \)-isomorphism \[ 1.1.3 \] we may identify \( \tilde{\omega} \otimes_{\mathbb{Q}_p} K_0 \) with the forgetful functor \((M, \phi, N, F) \mapsto M \). After tensoring with \( K_{st} \) we obtain \( \tilde{\omega}_{K_{st}}(M, \phi, N, F) = M \otimes K_{st} \). Lemma \[ 1.2.3 \] implies that \( \eta(M, \phi, F) \) is a tensor automorphism, thus it suffices to prove the statement for \( \exp \left( \frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N \right) \). The functoriality follows immediately. The compatibility with the \( \otimes \)-structure follows from

\[ N_{M_1 \otimes M_2} = N_{M_1} \otimes 1 + 1 \otimes N_{M_2}. \]

\[ \square \]

**Lemma 1.3.12.** The \( K_{st} \)-valued point

\[ \eta_{st} = \eta_{st, \log} \circ D_{st, \log} \]

of \( \text{Aut}^\otimes_{MT_{G_K}} \tau \) is independent of the choice of \( \log \in \text{Spec}(K_{0, st})(K_0) \).

**Proof.** Let \( \log, \log' \in \text{Spec}(K_{0, st})(K_0) \) and \( V \in MT_{G_K} \). In view of \[ 1.3.7 \] we get

\[ \eta(\text{forget}_N D_{st, \log'}(V)) = \exp \left( \frac{\log(x) - \log'(x)}{\nu_K(x)} N \right) \eta(\text{forget}_N D_{st, \log}(V)), \]

(1.3.9)
for very \( x \in K_0 \backslash \mathcal{O}_K^\times \), and forget\(_N(M, \phi, N, F) = (M, \phi, F) \). Thus
\[
\eta_{st, \log'} D_{st, \log'}(V) = \exp \left( \frac{\log'(x) - \log_{st}(x)}{\nu_K(x)} \right) \eta(\text{forget}_N D_{st, \log'}(V))
\]
\[
= \exp \left( \frac{\log(x) - \log_{st}(x)}{\nu_K(x)} \right) \eta(\text{forget}_N D_{st, \log}(V)) \quad \text{by } \text{(1.3.9)}
\]
\[
= \eta_{st, \log} D_{st, \log}(V).
\]

Example 1.3.13. By Kummer theory any \( q \in K^\times \) defines an extension \( V \) of the Gal(\( \bar{K}/K \))-representation \( \mathbb{Q}_p(0) \) by \( \mathbb{Q}_p(1) \). This in turn gives via \( D_{st, \log} \) an extension of \( K(0) \) by \( K(1) \) in \( MT_{\phi,N}^K \):
\[
0 \to K(1) \to M \to K(0) \to 0,
\]
which may be described as follows. The underlying \( K_0 \)-space of \( M \) has a basis \( e_0, e_1 \) such that the following conditions are satisfied:

1. the action of \( \phi \) is given by \( \phi(e_i) = p^{-i} e_i \) for \( i = 0, 1 \),
2. \( e_1 \) is the image of \( 1 \in K(1) \),
3. \( e_0 \) maps to \( 1 \in K(0) \).

The filtration is given by \( F^{i} M_K = M_K, \ F^0 M_K = K \cdot (\log(q) e_1 + e_0) \) and \( F^1 M_K = 0 \). Finally \( N \) is given by \( Ne_0 = -\nu_K(q) \cdot e_1 \) and \( Ne_1 = 0 \). Then we easily compute
\[
\eta_{st}(V) = \left( \frac{1}{\nu_K(x)} \log_{st}(q) - \log_{st}(x) \right) \cdot \left( \begin{array}{cc} 1 & 0 \\ \log(q) & 1 \end{array} \right)
\]
for the obvious basis of
\[
\tau(V) = \text{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \oplus \text{Hom}(\mathbb{Q}(1), \mathbb{Q}(1)),
\]
and every \( x \in K_0 \backslash \mathcal{O}_K^\times \), or, equivalently, for every \( x \in K \backslash \mathcal{O}_K^\times \). If \( \nu_K(q) \neq 0 \) then we can take \( q = x \) in order to see that
\[
\eta_{st}(V) = \left( \begin{array}{c} 1 \\ \log_{st}(q) \end{array} \right)
\]
holds for all \( q \in K^\times \).

1.4. Tannaka group scheme of mixed Tate filtered \( \phi \)-modules.

Definition 1.4.1. We define \( \mathcal{L} \) to be the graded \( \mathbb{Q}_p \)-Lie algebra freely generated by \( K^\vee \) (i.e. the dual of \( K \), where \( K \) is considered as \( \mathbb{Q}_p \)-vector space) in each degree \( i > 0 \). In other words, \( \mathcal{L} \) is defined by
\[
\text{Hom}_{(\text{graded } \mathbb{Q}_p \text{-Lie algebras})}(\mathcal{L}, \mathcal{T}) = \text{Hom}_{(\text{graded } \mathbb{Q}_p \text{-vector spaces})}(\bigoplus_{i \in \mathbb{Z}_{>0}} K^\vee, \mathcal{T}),
\]
for every graded \( \mathbb{Q}_p \)-Lie algebra \( \mathcal{T} \). Via (1.4.1) we get \( \mathbb{Q}_p \)-linear maps
\[
a_i : K^\vee \to \mathcal{L}_i, \quad \text{for all } i > 0.
\]

Obviously, \( \mathcal{L} \) is concentrated in positive degrees and each \( \mathcal{L}_i \) is a finite dimensional \( \mathbb{Q}_p \)-vector space.
Definition 1.4.2. For all $n > 0$, we define a $\mathbb{Q}_p$-Lie algebra $L_{\leq n}$ by

$$L_{\leq n} = L/(\oplus_{i>n} L_i).$$

We set $\hat{L} = \varprojlim_n L_{\leq n}$. For every field extension $K \supset \mathbb{Q}_p$ we define $\hat{L}_K := \varprojlim_n (L_{\leq n} \otimes_{\mathbb{Q}_p} K)$.

We will be only interested in the finite dimensional graded representations of $L$, which can be identified with the finite dimensional graded representations of $\hat{L}$.

1.4.3. There is natural element $\epsilon \in \hat{L}_K$ defined as follows:

$$\epsilon := \sum_{i>0} (a_i \otimes \text{id}_K)(\text{id}),$$

with $\text{id} \in K^\vee \otimes_{\mathbb{Q}_p} K$ the canonical element. After choosing a $\mathbb{Q}_p$-basis $v_1, \ldots, v_d$ of $K$, we see that

$$\epsilon = \sum_{i>0} \sum_{j=1}^d a_i (v_j^\vee) \otimes v_j.$$

Let $V$ be a finite dimensional graded representation of $L$ then $\exp(\epsilon)$ is a unipotent automorphism of $V \otimes_{\mathbb{Q}_p} K$.

Proposition 1.4.4. The $\otimes$-functor

$$\Psi : (\text{finite dim. graded } L\text{-modules}) \to \mathcal{C}_\eta$$

$$V \mapsto (V, \exp(\epsilon))$$

is an equivalence of categories.

Proof. Note first that $\Psi$ is well-defined, because $(\exp(\epsilon) - \text{id})(V_n) \subset \oplus_{i>n} V_i$, for all $n$, thus (1.2.5) is satisfied.

Let us prove that $\Psi$ is essentially surjective. Let $(\oplus_n V_n, \eta)$ be an object of $\mathcal{C}_\eta$. Since $1 - \eta$ is nilpotent we can define

$$\tilde{\epsilon} = \log(\eta) = \log(1 - (1 - \eta)).$$

For every $i > 0$ we define a $\mathbb{Q}_p$-linear map $\beta_i : K^\vee \to \text{End}(V)_i$ by

$$\beta_i(f) = \sum_n (\text{id}_{V_{n+i}} \otimes f) \circ \text{proj}_{V_{n+i} \otimes K} \circ \tilde{\epsilon} \circ \text{incl}_{V_n},$$

where $\text{incl}_{V_n} : V_n \to V$ and $\text{proj}_{V_{n+i} \otimes K} : V \otimes K \to V_{n+i} \otimes K$ is the inclusion and the projection, respectively. Via (1.4.1) we obtain a graded representation $\rho : L \to \text{End}(V)$. We need to show that $\exp(\rho(\epsilon)) = \eta$, or equivalently $\rho(\epsilon) = \tilde{\epsilon}$. This is a straightforward computation which we leave to the reader.

Next, we need to prove that $\Psi$ is fully faithful. Clearly, $\Psi$ is faithful. Let $V, U$ be two graded representations of $L$, and suppose $\tau : V \to U$ is a morphism which respects the grading and commutes with $\exp(\epsilon)$. Then $\tau$ commutes with $\epsilon$. Fix a
basis $v_1, \ldots, v_d$ of $K$. For $v \in V$ we get
\[
\tau \psi(v) = \tau(\sum_{i > 0} \sum_j a_i(v_j^\vee)(v) \otimes v_j),
\]
\[
= \sum_{i > 0} \sum_j \tau(a_i(v_j^\vee)(v)) \otimes v_j,
\]
\[
\epsilon \tau(v) = \sum_{i > 0} \sum_j a_i(v_j^\vee)(\tau(v)) \otimes v_j.
\]
Therefore $\tau \circ a_i(v_j^\vee) = a_i(v_j^\vee) \circ \tau$ for all $i, j$. Since $L$ is generated by the elements \{a_i(v_j^\vee)\}_{i,j} we see that $\tau$ is a morphism of $L$-representations.

\begin{corollary}
There is an equivalence of $\otimes$-categories
\[\Theta : MT_K^0 \to (\text{finite dim. graded } L\text{-modules})\]
such that
\begin{itemize}
\item $\text{forg} \circ \Theta = \tilde{\omega}$, where $\text{forg}$ forgets about the $L$-action.
\item $\exp(\epsilon) |_{\Theta(M, \phi, F)} = \eta(M, \phi, F)$.
\end{itemize}
\end{corollary}

\begin{proof}
Follows immediately from Proposition 1.2.10 and Proposition 1.4.4.
\end{proof}

\begin{corollary}
Let $U$ be the pro-algebraic group $U = \lim_{\leftarrow n} \exp(L/(\oplus_{i>n} L_i))$. Let $G_{\tilde{\omega}}$ be the Tannaka group attached to the fibre functor $\tilde{\omega}$ (Lemma 1.1.10).

There is an isomorphism
\[G_{\tilde{\omega}} \cong \mathbb{G}_m \rtimes U\]
such that $\eta \in G_{\tilde{\omega}}(K)$ corresponds to $\exp(\epsilon) \in U(K)$.
\end{corollary}

\begin{proof}
In view of Corollary 1.4.5, the statement follows from the fact that $\mathbb{G}_m \rtimes U$ is the Tannaka group of the fibre functor:
\[\text{forg} : (\text{graded finite dimensional } L\text{-modules}) \to (\mathbb{Q}_p\text{-vector spaces}).\]

The action of $\mathbb{G}_m$ on $U$ is induced by the action of $\mathbb{G}_m$ on $L$ given by the grading:
\[\mathbb{G}_m \times L_i \to L_i; \ (t, x) \mapsto t^i \cdot x.\]
\end{proof}

2. Mixed Tate motives over a number field and logarithmic points

2.1. Mixed Tate motives.

Let $E$ be a number field and $S$ a set of finite places. Let $O$ be the ring of integers of $E$, and $|\text{Spec}(O)|$ the maximal spectrum of $O$. We denote by
\[O_S := \bigcap_{x \in \text{Spec}(O) \setminus S} O_x\]
the ring of $S$-integers of $E$; the elements of $O_S$ are integral outside of $S$. We will be mainly interested in two cases for $S$. In the first case, we have $S = |\text{Spec}(O)|$ and $O_S = E$. In the second case, we have $S = |\text{Spec}(O)| \setminus \{x\}$, for a point $x \in |\text{Spec}(O)|$, and $O_S = O_x$ is the local ring at $x$. 
2.1.2. Deligne and Goncharov defined in [DG05, 1.6] an abelian category of mixed Tate motives $MT(\mathcal{O}_S)$. By definition it is the full subcategory of $MT(E)$ consisting of objects which are unramified outside $S$ in the following sense. Let $x \in |\text{Spec}(\mathcal{O})|$ be a point lying over a prime $p$; then we say that $M \in MT(\mathcal{E})$ is unramified at $x$ if for all primes $\ell \neq p$ the corresponding Galois representation $M_\ell$ is unramified at $x$, i.e. the inertia subgroup $I_x$ (which is only well-defined up to conjugation) acts trivially at $M_\ell$ [DG05, Proposition 1.8].

2.1.3. For extensions of Tate objects we know that:

$$\text{Ext}^1_{MT(\mathcal{O}_S)}(\mathcal{Q}(0), \mathcal{Q}(1)) = \mathcal{O}_S^\times \otimes \mathcal{Q},$$

$$\text{Ext}^1_{MT(\mathcal{O}_S)}(\mathcal{Q}(0), \mathcal{Q}(n)) = \begin{cases} 0 & \text{if } n \leq 0, \\ \text{Ext}^1_{MT(E)}(\mathcal{Q}(0), \mathcal{Q}(n)) & \text{if } n \neq 1, \end{cases}$$

$$\text{Ext}^2_{MT(\mathcal{O}_S)}(\mathcal{Q}(0), \mathcal{Q}(n)) = 0 \text{ for all } n \in \mathbb{Z},$$

(see [DG05, Proposition 1.9]).

2.1.4. Every object of $MT(\mathcal{O}_S)$ comes equipped with a finite increasing functorial weight filtration, indexed by even integers. For all $n \in \mathbb{Z}$ the graded pieces $\text{gr}_W^n(M)$ are sums of copies of $\mathcal{Q}(-n)$.

In view of [DG05, 1.1] the $\otimes$-functor

$$(2.1.1) \quad \omega : MT(\mathcal{O}_S) \to (\mathcal{Q}\text{-vector spaces}),$$

$$\omega(M) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}(\mathcal{Q}(n), \text{gr}_W^n(M)),$$

is a fibre functor, therefore $MT(\mathcal{O}_S)$ is a Tannaka category. We denote by $G_{S,\omega}$ the group scheme of $\otimes$-automorphisms of $\omega$. By [DG05, 2.1] we can write $G_{S,\omega}$ as a semi-direct product:

$$G_{S,\omega} = \mathbb{G}_m \rtimes U_{S,\omega},$$

where $U_{S,\omega}$ is a unipotent group and $G_{S,\omega} \to \mathbb{G}_m$ is induced by the obvious grading of $\omega$. If $S = |\text{Spec}(\mathcal{O})|$ then we simply write $G_{\omega} = G_{S,\omega}$.

2.1.5. **Functor to $p$-adic representations.** Let $x \in |\text{Spec}(\mathcal{O})|$ be a point lying over a prime $p$. Let $K = E_x$ be the completion of $E$ at the place $x$. Choose algebraic closures $\bar{E}, \bar{K}$, and an embedding $\iota : \bar{E} \to \bar{K}$.

To $M \in MT(E)$ we can attach a Galois representation $M_p$ of $\text{Gal}(\bar{E}/E)$ with coefficients in $\mathcal{Q}_p$, which is called the $p$-adic realization of $M$. By using $\iota$, we get a continuous homomorphism

$$\text{Gal}(\bar{K}/K) \to \text{Gal}(\bar{E}/E),$$

and we can restrict $M_p$ in order to obtain a $p$-adic representation $M_{x,p}$ of $\text{Gal}(\bar{K}/K)$.

**Proposition 2.1.6.** The assignment $M \mapsto M_{x,p}$ defines a functor

$$(\cdot)_{x,p} : MT(E) \to MT_{G_K}.$$

See Definition 1.3.7 for $MT_{G_K}$.

**Proof.** The $p$-adic realization is functorial. Thus we only need to show that $M_{x,p} \in MT_{G_K}$, which follows immediately from the existence of the weight filtration of $M$ and Definition 1.3.7. □
The set \( \{ \iota : E \rightarrow \bar{E}_x \} \) of embeddings over \( E \) is a torsor under the Galois group \( \text{Gal}( \bar{E}/E) \), and for every \( g \in \text{Gal}( \bar{E}/E) \) there is a natural transformation:

\[
\alpha_g : (\cdot)_{i,p} \rightarrow (\cdot)_{i \circ g,p}.
\]

**Lemma 2.1.7.** For the fibre functor \( \tau \) (defined in (1.3.8)) and the fibre functor \( \omega \) defined in (2.1.1) we have a canonical isomorphism

\[
\tau \circ (\cdot)_{i,p} \cong \omega \otimes \mathbb{Q}_p.
\]

For every \( g \in \text{Gal}( \bar{E}/E) \), the diagram

\[
\begin{array}{ccc}
\tau \circ (\cdot)_{i,p} & \xrightarrow{\cong} & \tau \circ (\cdot)_{i \circ g,p} \\
\downarrow & & \downarrow \\
\omega \otimes \mathbb{Q}_p & & \omega \otimes \mathbb{Q}_p
\end{array}
\]

is commutative.

**Proof.** Straightforward.

2.1.8. Recall that we have constructed a \( K_{st} \)-valued \( \eta_{st} \) of \( \text{Aut}^\otimes \tau \) (Lemma 1.3.12).

**Proposition 2.1.9.** For every embedding \( \iota, \eta_{st} := \eta_{st} \circ (\cdot)_{i,p} \) defines a \( K_{st} \)-valued point of \( \text{Aut}^\otimes_{MT(E)} \omega \) which is independent of the choice of \( \iota \).

**Proof.** Since \( \tau \circ (\cdot)_{i,p} = \omega \otimes \mathbb{Q}_p \) by Lemma 2.1.7, \( \eta_{st} \circ (\cdot)_{i,p} \) is a \( K_{st} \)-valued point of \( \text{Aut}^\otimes \omega \).

The independence of the choice of \( \iota \) follows from the commutative diagram

\[
\begin{array}{ccc}
(\tau \circ (\cdot)_{i,p}) \otimes \mathbb{Q}_p K_{st} & \xrightarrow{\eta_{st}} & (\tau \circ (\cdot)_{i,p}) \otimes \mathbb{Q}_p K_{st} \\
\downarrow & & \downarrow \\
\omega \otimes \mathbb{Q}_p K_{st} & & \omega \otimes \mathbb{Q}_p K_{st} \\
\downarrow & & \downarrow \\
(\tau \circ (\cdot)_{i \circ g,p}) \otimes \mathbb{Q}_p K_{st} & \xrightarrow{\eta_{st}} & (\tau \circ (\cdot)_{i \circ g,p}) \otimes \mathbb{Q}_p K_{st}
\end{array}
\]

where the triangles are commutative by Lemma 2.1.7 and the square is commutative because \( \eta_{st} \) is functorial.

2.2. Crystalline characterization of unramified motives.

2.2.1. Let \( E \) be a number field, and let \( M \) be a mixed Tate motive over \( E \), i.e. an object in \( MT(E) \). Let \( \nu \) be a finite place of \( E \), \( M \) is unramified at \( \nu \) [DG05, Definition 1.4, §1.7] if the coaction [DG05 (1.2.2)]

\[
e_M : \omega(M) \rightarrow \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \otimes \omega(M)
\]

of \( \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = \mathbb{E}^\times \otimes \mathbb{Z} \mathbb{Q} \) on \( \omega(M) \) factors through a coaction of \( \mathcal{O}_E^\times \otimes \mathbb{Z} \mathbb{Q} \).
2.2.2. Recall from Proposition 2.1.6 that $M_{i,p}$ is a mixed Tate Galois representation of $G_K = \text{Gal}(\overline{K}/K)$ for the completion $K = E_\nu$ at $\nu$. In particular, $M_{i,p}$ is semistable (Proposition 1.3.8). In the following we will simply write $M_p = M_{p,i}$.

We call $M_p$ crystalline if the monodromy operator $N$ of $D_{st}(M_p)$ is trivial, or equivalently if

$$(B_{cris} \otimes \mathbb{Q}_p)M_p^{G_K} \rightarrow (B_{st} \otimes \mathbb{Q}_p)M_p^{G_K}$$

is an isomorphism.

**Theorem 2.2.3.** Let $M$ be a mixed Tate motive over $E$ and $\nu$ a finite place of $E$. Then $M$ is unramified at $\nu$ if and only if $M_p$ is crystalline.

**Proof.** First note that the statement that $M$ is unramified at $\nu$ is equivalent to the statement that for every subquotient $N$ of $M$ which is of the form

$$0 \rightarrow \mathbb{Q}(n+1) \rightarrow N \rightarrow \mathbb{Q}(n) \rightarrow 0,$$

for some $n$, the extension class $\text{Ext}^1(\mathbb{Q}(n), \mathbb{Q}(n+1)) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^\times \otimes \mathbb{Q}$ lies in $\mathcal{O}_K^\times \otimes \mathbb{Q}$ [DG05, §1.4].

Also in the category of $p$-adic representations of a $p$-adic field $K$, a representation in $\text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ that is associated to some $q \in K^\times \otimes \mathbb{Q} \subseteq \lim_{\leftarrow n} (K^\times / (K^\times)^{p^n}) \otimes \mathbb{Q} = \text{Ext}^1_{G_K}(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ is crystalline if and only if $q \in \mathcal{O}_K^\times \otimes \mathbb{Q}$ [Lsu02, Example 2.3.2].

First, suppose that $M_p$ is crystalline, then every subquotient of $M_p$ is crystalline. So in order to prove that $M$ is unramified at $\nu$ we may assume that $M = N$, where $N$ is as above with $n = 0$ (after Tate twist). Therefore we have an extension in $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$, defined by some $q \in E^\times \otimes \mathbb{Q}$, whose $p$-adic realization is crystalline at $\nu$. Then the above remark implies that the image of $q$ in $E^\times_\nu \otimes \mathbb{Q}$ lies in $\mathcal{O}_K^\times \otimes \mathbb{Q}$, hence $q \in \mathcal{O}_K^\times \otimes \mathbb{Q}$ and $M$ is unramified at $\nu$.

Suppose conversely that $M$ is unramified at $\nu$. We have to show that the monodromy operator $N$ on $D_{st}(M_p) =: D(M)$ vanishes. Note that $N$ maps the slope $\lambda$ piece of $D(M)$ to the slope $\lambda - 1$ piece. Therefore, if $N$ is nonzero on $D(M)$ then there exists an $n$ such that $N$ is nonzero on $D(W_{2n}M/W_{2n-4}M_p) = D((W_{2n}M/W_{2n-4}M_p) \otimes \mathbb{Q}(n))$ we may assume that $M$ is defined by a class in $\text{Ext}^1(\mathbb{Q}(0)^{\otimes r}, \mathbb{Q}(1)^{\otimes s}) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))^{\otimes rs}$, $M$ is unramified, and $N$ is nonzero on $D(M)$. By passing to a subquotient we may further assume that $r = s = 1$. This gives an extension in $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$ which is unramified at $\nu$ (and hence defined by some $q \in \mathcal{O}_K^\times \otimes \mathbb{Q}$) whose $p$-adic realization is not crystalline at $\nu$. This is a contradiction. \hfill \square

2.2.4. Recall the notation of Section 2.1.1 Let $x \in \text{Spec}(\mathcal{O})$ be a point; in the following we will work with $S = \text{Spec}(\mathcal{O}) \setminus \{x\}$, thus $\mathcal{O}_S = \mathcal{O}_x$.

Let $p$ be the prime lying under $x$. In view of Theorem 2.2.3 we know that $MT(\mathcal{O}_x)$ is the full subcategory of $MT(E)$ consisting of motives $M$ such that the $p$-adic realization $M_p$ is crystalline at $x$.

We denote by $G_x$ the group scheme of $\otimes$-automorphisms of the fibre functor (see (2.1.1))

$$(\mathcal{O}_x) \rightarrow (\mathbb{Q}\text{-vector spaces}).$$

The group scheme $G_x$ is a quotient of $G_\omega = \text{Aut}_{MT(E)}^\otimes \omega$. 

[DG05, §1.4]
[DG05, §1.4]
[Lsu02, Example 2.3.2]
Lemma 2.2.5. The morphism Spec$(E_x, \text{st}) \xrightarrow{\eta_x} G_\omega \to G_x$ factors through the structure morphism Spec$(E_x, \text{st}) \to \text{Spec}(E_x)$ and thus defines a point $\eta^u_x \in G_x(E_x)$.

Proof. The point $\eta_x$ was defined in Proposition 2.1.9. If $M \in MT(O_x)$ then $D_{st}(M_{t,p})$ has vanishing monodromy operator $N$ and $\eta_x(M_{t,p}) = \eta_{\text{st,log}}D_{st}(M_{t,p})$ takes values in $E_x$ by Definition 1.3.10.

2.3. Main theorem.

2.3.1. Let $x \in |\text{Spec}(O)|$ and let $E_x$ be the completion of $E$ at $x$. Bloch and Kato [BK90, Definition 3.10] define an exponential map

$$\exp : E_x \to \text{Ext}^1(Q_p(0), Q_p(n)),$$

for all $n \geq 1$,

where $\text{Ext}^1$ is computed in the category of $p$-adic representation of $\text{Gal}(\bar{E}_x/E_x)$. Note that, in fact, the image of the exponential map lies among the crystalline representations $\text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n))$ [BK90, Example 3.9]. Via $p$-adic Hodge theory, we obtain a map

$$E_x \to \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n)) \cong \text{Ext}^1_{MT_{E_x}}(E_x(0), E_x(n)),$$

which, by abuse of notation, will also be called the Bloch-Kato exponential map.

2.3.2. For an extension $M \in \text{Ext}^1_{MT(O_x)}(Q(0), Q(n))$ with $n \geq 1$, there are natural maps $v_0 : Q \to \omega(M)$ and $f_n : \omega(M) \to Q$ defined as follows. By definition, there are isomorphisms $\alpha : Q(n) \to \text{gr}^W M$ and $\beta : \text{gr}^W_0 M \to Q(0)$; we define

$$v_0 : Q = \text{Hom}(Q(0), Q(0)) \xrightarrow{\beta^{-1}} \omega_0(M) \to \omega(M),$$

$$f_n : \omega(M) \to \omega_n(M) \xrightarrow{\alpha^{-1}} \text{Hom}(Q(n), Q(n)) = Q.$$

Therefore, we can attach to $M$ a function in $A^1(G_x)$, defined by

$$M(t) := f_n(t \cdot v_0),$$

for every point $t : T \to G_x$.

Theorem 2.3.3. Let $E$ be a number field and $O$ be the ring of integers. Let $x \in |\text{Spec}(O)|$ be a closed point over a prime $p$. For the Tannaka category $(MT(O_x), \omega)$ of mixed Tate motives we denote by $G_x$ the group scheme of $\otimes$-automorphisms of $\omega$. For all $n \geq 1$, the map

$$\text{Ext}^1_{MT(O_x)}(Q(0), Q(n)) \to E_x,$$

induced by $\eta^u_x \in G_x(E_x)$ (see Lemma 2.2.5), is the composition of the $p$-adic realization

$$\text{Ext}^1_{MT(O_x)}(Q(0), Q(n)) \to \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n))$$

and the inverse of the Bloch-Kato exponential map 2.3.7.

Proof. Let us prove that evaluation at the point $\eta^u_x$ has the desired compatibility with the Bloch-Kato exponential map (see 2.3.2)

$$\exp : E_x \to \text{Ext}^1_{\text{cryst}}(Q_p(0), Q_p(n)) \xrightarrow{\sim} \text{Ext}^1_{MT_{E_x}}(E_x(0), E_x(n)).$$

For this we need to recall the construction of the exponential map. For the rest of the proof let $K := E_x$. First there is an exact sequence [BK90, Proposition 1.17]:

$$0 \to Q_p \to B^p \oplus B^+_{dR} \to B_{dR} \to 0,$$
where the first map sends $x$ to $(x, x)$ and the second one sends $(x, y)$ to $x - y$.

For $n \geq 1$, the Bloch-Kato construction gives a map
\[ K = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K} \to \text{Ext}^1_{\text{crys}}(\mathbb{Q}_p(0), \mathbb{Q}_p(n)). \]
This map is obtained as follows. First tensor the above exact sequence with $\mathbb{Q}_p(n)$:
\[ 0 \to \mathbb{Q}_p(n) \to (\mathbb{Q}_p(n) \otimes B_{\text{crys}}^{\geq 1}) \oplus (\mathbb{Q}_p(n) \otimes B_{dR}^+) \to \mathbb{Q}_p(n) \otimes B_{dR} \to 0. \]

Then an element $a$ in $K(n) = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K}$ gives a map $\mathbb{Q}_p \to \mathbb{Q}_p(n) \otimes B_{dR}$, pulling back the above exact sequence via this map gives the extension we were looking for.

More explicitly, for $a \in K$ the extension constructed above is:
\[ 0 \to V_0 \to V \to V_0 \to 0, \]
where $V_0 = \mathbb{Q}_p \cdot t^n \otimes at^{-n}$, $V_n = \mathbb{Q}_p \cdot t^n$, and $V$ is a 2-dimensional representation of $G_K$ with basis which can be described as follows. By the exact sequence \[ (2.3.3), \]
there exists $x \in B_{\text{crys}}^{\geq 1}$ and $y \in B_{dR}^+$ such that $at^{-n} = x - y$. Then $V$ has basis \[ \{(t^n \otimes x, t^n \otimes y), (t^n \otimes 1, t^n \otimes 1)\}. \]
For $\sigma \in G_K$,
\[ \sigma(t^n \otimes x, t^n \otimes y) = (t^n \otimes x, t^n \otimes y) + \gamma(\sigma)(t^n \otimes 1, t^n \otimes 1), \]
for some $\gamma(\sigma) \in \mathbb{Q}_p$. Therefore
\[ \chi_{\text{cyc}}(\sigma)^n \sigma(x) = x + \gamma(\sigma) \]
and
\[ \chi_{\text{cyc}}(\sigma)^n \sigma(y) = y + \gamma(\sigma). \]

Let us now try to find what this extension corresponds to after we apply the functor $(\cdot \otimes B_{\text{crys}})^{G_K}$. First note that $(V \otimes B_{\text{crys}})^{G_K}$ has basis
\[ e_n := (t^n \otimes 1, t^n \otimes 1) \otimes t^{-n} \]
and
\[ e_0 := (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes x. \]
That $e_n$ is invariant under the Galois action is clear. In order to see that $e_0$ is $G_K$ invariant let $\sigma \in G_K$. Then
\[ \sigma(e_0) = (t^n \otimes (x + \gamma(\sigma)), t^n \otimes (y + \gamma(\sigma))) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes (x + \gamma(\sigma)) = e_0. \]

Now note that $\varphi(e_n) = t^{-n}e_n$ and $\varphi(e_0) = e_0$. Furthermore $e_n$ is the image of $1 \in K(n)$ and $e_0$ maps to $1 \in K(0)$ in the exact sequence (note that $\mathbb{Q}_p(0)$ is identified with $V_0$ via the map that sends 1 to $t^n \otimes at^{-n}$):
\[ 0 \to K(n) \to (V \otimes B_{\text{crys}})^{G_K} \to K(0) \to 0. \]
Therefore in order to compare Bloch-Kato’s construction we need only compute the filtration on $(V \otimes B_{\text{crys}})^{G_K} \otimes_{K_0} K$. So we need to compute the 0-th piece of the filtration on $(V \otimes B_{dR})^{G_K}$.

We claim that $ae_n + e_0 \in \text{Fil}^0(V \otimes B_{dR})^{G_K}$. This follows immediately from
\[ ae_n + e_0 = (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes y, \]
and the fact that $y \in B_{dR}^+$. Now Proposition \ref{1227} implies the claim. \hfill \Box

2.4. Archimedean places. In this section we recall the story for archimedean places; our reference is \cite{Del94} and \cite{BD94}.
2.4.1. Let $E$ be a number field and $\sigma : E \to \mathbb{C}$ an embedding. To $M \in MT(E)$ we can attach a real mixed Tate Hodge structure $M_\sigma$. Recall that a real mixed Tate Hodge structure $(H, W, F)$ consists of an $\mathbb{R}$-vector space $H$, an increasing filtration $W$ of $H$, and a decreasing filtration $F$ of $H \otimes \mathbb{C}$ such that

$$\text{Gr}_F^p \text{Gr}_F^q \text{Gr}_n^W (H \otimes \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even and } (p, q) \neq \left(\frac{n}{2}, \frac{n}{2}\right). \end{cases}$$

Induced by $F, \bar{F}$, we obtain maps

$$a_F : H \otimes \mathbb{C} = \bigoplus_{i \in \mathbb{Z}} F^{-i} \cap W_{-2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}_W^{-2i} H \otimes \mathbb{C},$$

$$a_{\bar{F}} : H \otimes \mathbb{C} = \bigoplus_{i \in \mathbb{Z}} \bar{F}^{-i} \cap W_{-2i} \to \bigoplus_{i \in \mathbb{Z}} \text{Gr}_W^{-2i} H \otimes \mathbb{C},$$

where $F^i \cap W_{2i} \to \text{Gr}_W^W H \otimes \mathbb{C}$ is the natural map (and similarly for $\bar{F}$). For the automorphism $d = a_{\bar{F}} a_F^{-1}$ of $\bigoplus_{i \in \mathbb{Z}} \text{Gr}_W^{-2i} H \otimes \mathbb{C}$ we know that

$$(d - 1)(\text{Gr}_W^{-2i} H \otimes \mathbb{C}) \subset \bigoplus_{j > i} \text{Gr}_W^{-2j} H \otimes \mathbb{C},$$

by [Del94, p.510], and $\check{d} = d^{-1}$ [Del94, p.513].

2.4.2. Let $\mathcal{C}$ be the category of pairs $(\oplus_i H_i, d)$ where $\oplus_i H_i$ is a graded $\mathbb{R}$-vector space and $d : \oplus_i H_i \otimes \mathbb{C} \to \oplus_i H_i \otimes \mathbb{C}$ is an automorphism satisfying the conditions $\bar{d} = d^{-1}$ and $(d - 1)(H_i) \subset \oplus_{j > i} H_j$, for all $i$.

**Proposition 2.4.3.** [Del94, p.514] The functor

$$(\text{Real mixed Tate Hodge structures}) \to \mathcal{C}$$

$$(H, W, F) \mapsto (\oplus_{i \in \mathbb{Z}} \text{Gr}_W^{-2i} H, d)$$

is an equivalence of categories.

The maps $d$ define a $\mathbb{C}$-valued $\otimes$-automorphism for the fibre functor

$$\check{\omega} : (\text{Real mixed Tate Hodge structures}) \to (\mathbb{R}\text{-vector spaces}),$$

$$\check{\omega}(H, W, F) = \bigoplus_{i \in \mathbb{Z}} \text{Gr}_W^{-2i} H.$$

2.4.4. Recall the definition of $\omega$ in (2.1.1). For the functor

$$\mathcal{R}_{\sigma} : MT(E) \to (\text{Real mixed Tate Hodge structures}), \quad M \mapsto M_\sigma,$$

we have an isomorphism

$$(2.4.1) \quad \omega \otimes_{\mathbb{Q}} \mathbb{R} \cong \check{\omega} \circ \mathcal{R},$$

depending on the choice $(2\pi i)^n$ as a generator for the real vector space underlying $\mathbb{R}(n)$, in other words we have to choose a square root of $-1$ in $\mathbb{C}$. In order to avoid this choice one can define

$$\check{\omega}(H, W, F) = \bigoplus_{n \in \mathbb{Z}} i^n \cdot \text{Gr}_W^{-2n} H,$$

as in [BD94, p.111], but we won’t do that.

Via Equation (2.4.1), $d$ defines a $\mathbb{C}$-valued point of $G_\omega$, the $\otimes$-automorphisms of the fibre functor $\omega$. We define $\epsilon = \log(d)$, $\epsilon$ defines a $\mathbb{C}$-valued point of $\text{Lie}(G_\omega)$. 
The dictionary for the notation of [BD94] p.111 is
\[ d = b^{-1}, \quad \epsilon = -2 \cdot N, \]
and \( N \) is purely imaginary.

2.4.5. For \( z \in E \setminus \{0, 1\} \) there is a polylogarithm motive \( \{z\} \in MT(E) \) (strictly speaking it is a pro-object). The motive \( \{z\} \) is defined as a subquotient of the motivic paths from the tangent vector \( t_0 = z \), in the tangent space at 0, to \( z \) (see [DG05 Theorem 4.4]). The \( \mathbb{Q} \)-Hodge realization of \( \{z\} \) is described in [BD94] p.98 and uniquely determines \( \{z\} \).

For every \( k \in \mathbb{Z}_{\geq 0} \) we have natural isomorphisms
\[ \alpha_k : \text{gr} W_{-2k} \{z\} \xrightarrow{\cong} \mathbb{Q}(k); \]
we define \( v_0 \in \omega(\{z\}) \) and \( f_k \in \omega(\{z\})^\vee \) by
\[
\begin{align*}
  v_0 : \mathbb{Q} = \text{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) &\xrightarrow{\alpha_0^{-1}} \omega_0(\{z\}) \\
  f_k : \omega(\{z\}) &\rightarrow \omega_k(\{z\}) \xrightarrow{\alpha_k} \text{Hom}(\mathbb{Q}(k), \mathbb{Q}(k)) = \mathbb{Q}.
\end{align*}
\]
We denote by \( (v_0, \{z\}, f_k) \in \mathbb{A}^1(\text{Lie } G_\omega) \) the function
\[
X \mapsto f_k(X \cdot v_0).
\]

By [BD94] Proposition 2.7 we have
\[
(v_0, \{z\}, f_k)(\epsilon) = \begin{cases} 
2i \sum_{\ell} b_{\ell} \log(e^{z\epsilon}) \text{Im}(\text{Li}_{k-\ell}(z)) & \text{if } k \text{ is even,} \\
2i \sum_{\ell} b_{\ell} \log(e^{z\epsilon}) \text{Re}(\text{Li}_{k-\ell}(z)) & \text{if } k \text{ is odd.}
\end{cases}
\]
Here, \( \{b_k\} \) are the Bernoulli numbers and \( \text{Li} \) is the polylogarithm. For \( k \) even, the result does not depend on the choice of the square root of \(-1\); for \( k \) odd it is independent up to a sign.

References


