Reformulations, Relaxations and Cutting Planes for Linear Generalized Disjunctive Programming

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Discrete/Continuous Optimization Models

Mixed Integer Program (MIP)

- Most common non-linear discrete/continuous optimization model.
- Purely equation-based.
- If all functions in MIP are linear → MILP *(nonlinear → MINLP)*.

Disjunctive Programming (DP)

- Developed by Balas E. (1979, 1985, 1998 (1974))
- Linear programming (LP) with disjunctive constraints.

Generalized Disjunctive Program (GDP)

- Linear: Raman, Grossmann (1994); Nonlinear: Turkay, Grossmann (1996)
- Combination of algebraic equations, disjunctions and logic propositions.
- Natural representation of engineering problems.

Mixed-Logic Linear Programming *Hooker and Osorio (1999)*

Constraint Programming *Van Hentenryck (1989)*

Motivation Hybrid Systems

Examples of work based on discrete/continuous optimization:

Benmporad and Morari (1999) MILP (HYSDEL software)
Verification of Hybrid Systems via Mathematical Programming

Strusberg and Panek (2002) GDP
Control of Switched Hybrid Systems Based on Disjunctive Formulations

Barton, Lee (2004) MINLP
Design of process operations using hybrid dynamic optimization

Goal:

To unify GDP with DP in order to develop MILP reformulations with improved relaxations for linear GDP

- **To unify linear GDP with DP in order to develop**
  - **A hierarchy of LP relaxations for linear GDP**
  - **A family of disjunctive cutting planes for linear GDP**

- **Brief extension to Nonlinear and Bilinear GDPs**
  - **Approximation of convex hull and cutting plane algorithm**
  - **Tightening bounds of bilinear GDPs through basic steps**

Linear Generalized Disjunctive Programming
LGDP Model

Raman R. and Grossmann I.E. (1994) **(LGDP)**

$$Min\ Z = \sum_{k \in K} c_k + d^T x$$ ***Objective function***

$$s.t. \qquad Bx \ge b$$ ***Common constraints***

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ A^{jk} x \ge a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \qquad k \in K$$ ***Disjunctive constraints***

$$\underline{\vee}_{j \in J_k} Y_{jk} \qquad k \in K$$

$$\Omega(Y) = True$$ ***Logic constraints***

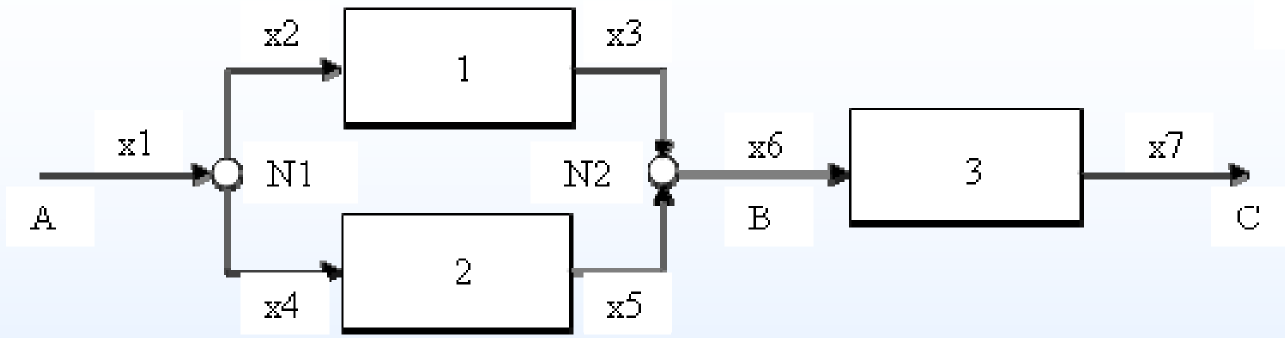
$$x^L \le x \le x^U$$ ***Continuous variables***

$$Y_{jk} \in \{True, False\} \qquad j \in J_k, k \in K$$ ***Boolean variables***

$$c_k \in \mathbb{R}^1 \qquad k \in K$$

Logical OR operator

Process Network with fixed charges



GDP model

$$\begin{aligned}
&Min\ Z = c_1 + c_2 + c_3 + d^T x \\
&s.t. \\
&x_1 = x_2 + x_4 \\
&x_6 = x_3 + x_5 \\
&\begin{bmatrix} Y_{11} \\ x_3 = p_1 x_2 \\ c_1 = \gamma_1 \end{bmatrix} \vee \begin{bmatrix} Y_{21} \\ x_3 = x_2 = 0 \\ c_1 = 0 \end{bmatrix} \\
&\begin{bmatrix} Y_{12} \\ x_5 = p_2 x_4 \\ c_2 = \gamma_2 \end{bmatrix} \vee \begin{bmatrix} Y_{22} \\ x_5 = x_4 = 0 \\ c_2 = 0 \end{bmatrix} \\
&\begin{bmatrix} Y_{13} \\ x_7 = p_3 x_6 \\ c_3 = \gamma_3 \end{bmatrix} \vee \begin{bmatrix} Y_{23} \\ x_7 = x_6 = 0 \\ c_3 = 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&Y_{11} \veebar Y_{21} \\
&Y_{12} \veebar Y_{22} \\
&Y_{13} \veebar Y_{23} \\
&Y_{11} \vee Y_{12} \Rightarrow Y_{13} \\
&Y_{13} \Rightarrow Y_{11} \vee Y_{12} \\
&Y_{21} \vee Y_{22} \\
&0 \le x \le x^U \\
&Y_{11}, Y_{21}, Y_{12}, Y_{22}, Y_{13}, Y_{23} \in \{True, False\} \\
&c_1, c_2, c_3 \in \mathbb{R}^1
\end{aligned}$$

LGDP to MILP Reformulations

Big-M Reformulation

$$Min\ Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + h^T x$$

Big-M parameters

$$
\begin{array}{lll}
s.t. & Bx \le b & \\
 & A_{jk}\,x - a_{jk} \le M_{jk}\,(1-\lambda_{jk}) & j \in J_k,\ k \in K \\
 & \sum_{j \in J_k} \lambda_{jk} = 1 & k \in K \\
 & D\lambda \le d & \\
 & x \in R^n,\ \lambda_{jk} \in \{0,1\} & j \in J_k,\ k \in K
\end{array}
\qquad \text{(BM)}
$$

Note: $(M_{jk} = max_{LB \le x \le UB}\,(A_{jk}x - a_{jk}),\ j \in J_k \text{ and } k \in K).$

Relaxation : $\lambda_{jk} \in \{0,1\}$ becomes $0 \le \lambda_{jk} \le 1$ in (BM)

LGDP to MILP Reformulations

Convex Hull Reformulation

$$Min\ Z = \sum_{k\in K}\sum_{j\in J_k} \gamma_{jk}\lambda_{jk} + h^T x$$

Disaggregated variables

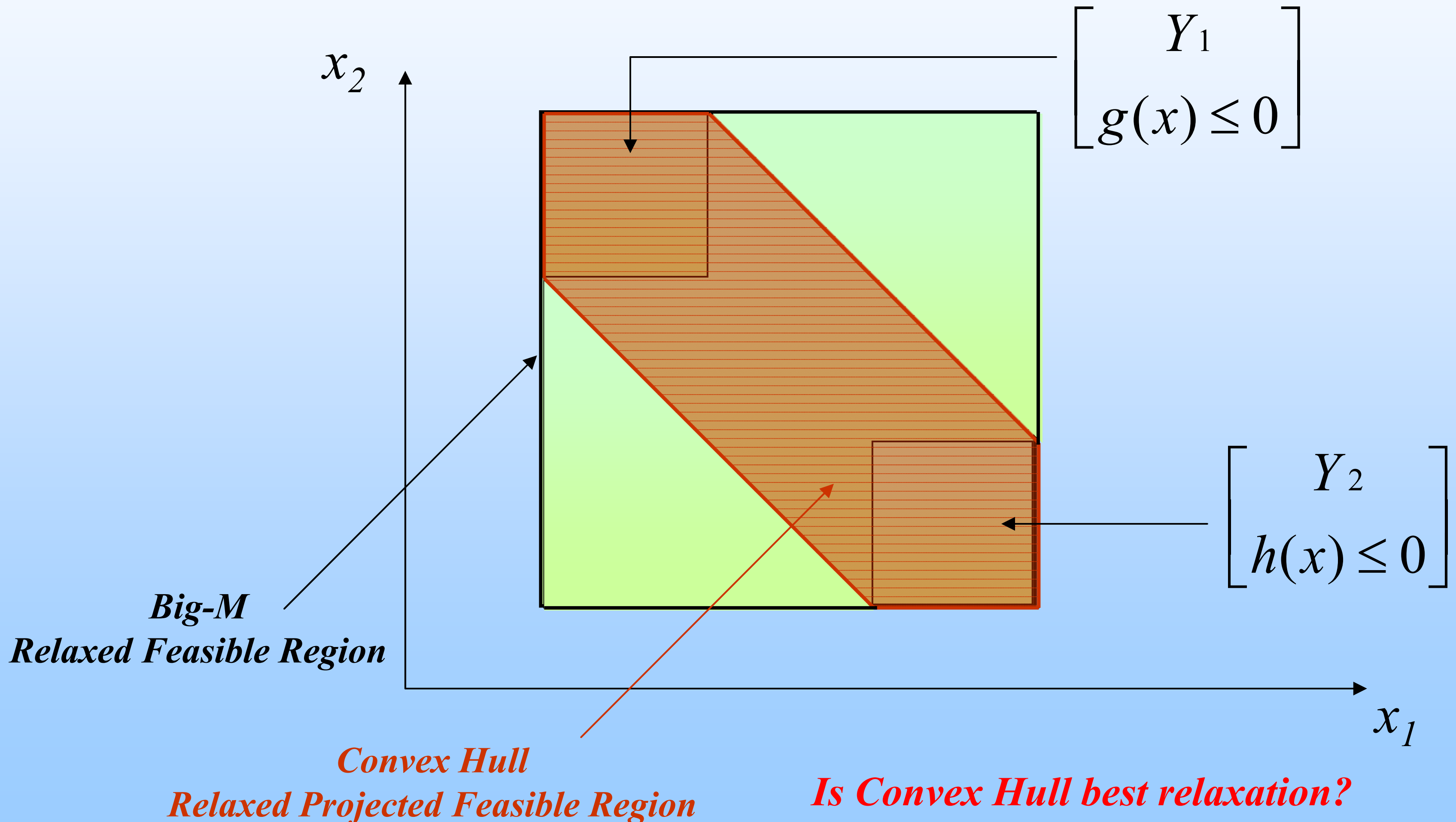
$$
\begin{aligned}
s.t.\quad & Bx \le b \\
& A_{jk}\, \nu^{jk} - a_{jk}\lambda_{jk} \le 0 && j\in J_k,\ k\in K \\
& x = \sum_{j\in J_k} \nu^{jk} && k\in K \\
& 0 \le \nu^{jk} \le \lambda_{jk} U_{jk} && j\in J_k,\ k\in K \\
& \sum_{j\in J_k} \lambda_{jk} = 1 && k\in K \\
& D\lambda \le d \\
& x\in R^n,\ \nu^{jk}\in R^n_+,\ \lambda_{jk}\in\{0,1\} && j\in J_k,\ k\in K
\end{aligned}
\qquad \text{(CH)}
$$

Relaxation : $\lambda_{jk} \in \{0,1\}$ becomes $0 \le \lambda_{jk} \le 1$ in (CH)

Proposition:

The Convex Hull of a set of disjunctions is the smallest convex set that includes that set of disjunctions. Furthermore, the projected relaxation of (CH) onto the space of (BM) is always as tight or tighter than that of (BM) ***(Grossmann & Lee , 2002)***

1. **Tighter feasible region/lower bound → less nodes → decrease in computational solution time.**
2. **More variables and constraints → more iterations → increase in computational solution time.**



Is Convex Hull best relaxation?

Disjunctive Programming

Disjunction: **A set of constraints connected to one another through the logical OR operator $\vee$**

Conjunction: **A set of constraints connected to one another through the logical AND operator $\wedge$**

Constraint set of a DP can be expressed in two <u>equivalent</u> extreme forms

- **Disjunctive Normal Form (DNF)**
  - **A disjunction whose terms do not contain further disjunctions**

$$F = \left\{ x \in \mathbb{R}^n : \underset{i \in Q}{\vee} (A^i x \geq a^i) \right\}$$

- **Conjunctive Normal Form (CNF)**
  - **A conjunction whose terms do not contain further conjunctions**

$$F = \left\{ x \in \mathbb{R}^n : \widehat{A} x \geq \widehat{a}, \quad \underset{h \in Q_j}{\vee} (d^h x \geq d_0{}^h), \; j = 1, \ldots, t \right\}$$

Linear Generalized Disjunctive Programming
LGDP Model

Raman R. and Grossmann I.E. (1994) **(LGDP)**

$$\begin{aligned}
&Min\ Z = \sum_{k \in K} c_k + d^T x && \textit{Objective function}\\
&s.t. \quad Bx \geq b && \textit{Common constraints}\\
&\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ A^{jk} x \geq a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K && \textit{Disjunctive constraints}\\
&\underline{\vee}_{j \in J_k} Y_{jk} \quad k \in K && \textit{Logic constraints}\\
&\Omega(Y) = True \\
&x^L \leq x \leq x^U \\
&Y_{jk} \in \{True, False\} \quad j \in J_k, k \in K && \textit{Boolean variables}\\
&c_k \in \mathbb{R}^1 \quad k \in K
\end{aligned}$$

How to deal with Boolean and logic constraints in Disjunctive Programming?

Reformulating LGDP into Disjunctive Programming Formulation

Sawaya N.W. and Grossmann I.E. (2007)

$$\begin{aligned}
&Min\ Z = \sum_{k\in K} c_k + d^T x \\
&s.t. \qquad Bx \ge b \\
&\bigvee_{j\in J_k} \begin{bmatrix} Y_{jk} \\ A^{jk}x \ge a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \qquad k \in K \\
&\underset{j\in J_k}{\veebar} Y_{jk} \qquad k \in K \\
&\Omega(Y) = True \\
&x^L \le x \le x^U \\
&Y_{jk} \in \{True, False\} \qquad j \in J_k, k \in K \\
&c_k \in \mathbb{R}^1 \qquad k \in K
\end{aligned}$$

$$\begin{aligned}
&Min\ Z = \sum_{k\in K} c_k + d^T x \\
&s.t. \qquad Bx \ge b \\
&\bigvee_{j\in J_k} \begin{bmatrix} \lambda_{jk} = 1 \\ A^{jk}x \ge a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \qquad k \in K \\
&\sum_{j\in J_k} \lambda_{jk} = 1 \qquad k \in K \\
&H\lambda \ge h \\
&x^L \le x \le x^U \\
&0 \le \lambda_{jk} \le 1 \qquad j \in J_k, k \in K \\
&c_k \in \mathbb{R}^1 \qquad k \in K
\end{aligned}$$

LGDP

LDP *=> Integrality λ guaranteed*

Theorem. LGDP and LDP have equivalent solutions.

Equivalent Forms in DP Through Basic Steps

There are many forms between CNF and DNF that are equivalent

Regular Form (RF): form represented by intersection of unions of polyhedra

Thus the RF is:

$$F = \bigcap_{t \in T} S_t$$

where for $t \in T, \quad S_t = \bigcup_{i \in Q_t} P_i, \quad P_i$ a polyhedron, $i \in Q_t$.

Proposition 1 (Theorem 2.1 in Balas (1979)). *Let F be a disjunctive set in RF. Then F can be brought to DNF by* $\underline{|T|-1}$ *<u>recursive applications of the following basic steps,</u> which preserve regularity:*

For some $r, s \in T, r \neq s,$ *bring* $S_r \cap S_s$ *to DNF, by replacing it with:*

$$S_{rs} = \bigcup_{\substack{i \in Q_r \\ t \in Q_s}} (P_i \cap P_t).$$

Illustrative Example: Basic Steps

$$F = S_1 \cap S_2 \cap S_3$$

$$S_1 = (P_{11} \cup P_{21}) \qquad S_2 = (P_{12} \cup P_{22}) \qquad S_3 = (P_{13} \cup P_{23})$$

Then F can be brought to DNF through 2 basic steps.

Apply Basic Step to:

$$S_1 \cap S_2 = (P_{11} \cup P_{21}) \cap (P_{12} \cup P_{22})$$

$$S_{12} = (P_{11} \cap P_{12}) \cup (P_{11} \cap P_{22}) \cup (P_{21} \cap P_{12}) \cup (P_{21} \cap P_{22})$$

We can then rewrite

$$F = S_1 \cap S_2 \cap S_3 \qquad \text{as } F = S_{12} \cap S_3$$

Apply Basic Step to:

$$S_{12} \cap S_3 = ((P_{11} \cap P_{12}) \cup (P_{11} \cap P_{22}) \cup (P_{21} \cap P_{12}) \cup (P_{21} \cap P_{22})) \cap (P_{13} \cup P_{23})$$

$$S_{123} = \begin{pmatrix} (P_{11} \cap P_{12} \cap P_{13}) \cup (P_{11} \cap P_{22} \cap P_{13}) \cup (P_{21} \cap P_{12} \cap P_{13}) \cup (P_{21} \cap P_{22} \cap P_{13}) \\ \cup (P_{11} \cap P_{12} \cap P_{23}) \cup (P_{11} \cap P_{22} \cap P_{23}) \cup (P_{21} \cap P_{12} \cap P_{23}) \cup (P_{21} \cap P_{22} \cap P_{23}) \end{pmatrix}$$

We can then rewrite

$$F = S_{12} \cap S_3 \qquad \text{as } F = S_{123}$$

which is its equivalent DNF

Equivalent Forms for GDP

$$
\begin{aligned}
& Min\ Z = \sum_{k\in K} c_k + d^T x \\
& s.t. \quad Bx \ge b \\
& \bigvee_{j\in J_k} \begin{bmatrix} Y_{jk} \\ A^{jk} x \ge a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K \\
& \underset{j\in J_k}{\underline{\vee}}\ Y_{jk} \quad k \in K \\
& \Omega(Y) = True \\
& x^L \le x \le x^U \\
& Y_{jk} \in \{True, False\} \quad j \in J_k, k \in K \\
& c_k \in \mathbb{R}^1 \quad k \in K
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
& Min\ Z = \sum_{k\in K} c_k + d^T x \\
& s.t. \quad Bx \ge b \\
& \bigvee_{j\in J_k} \begin{bmatrix} \lambda_{jk} = 1 \\ A^{jk} x \ge a^{jk} \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K \\
& \sum_{j\in J_k} \lambda_{jk} = 1 \quad k \in K \\
& H\lambda \ge h \\
& x^L \le x \le x^U \\
& 0 \le \lambda_{jk} \le 1 \quad j \in J_k, k \in K \\
& c_k \in \mathbb{R}^1 \quad k \in K
\end{aligned}
$$

LGDP **LDP**

$$F = \left\{ z := (x,\lambda,c) \in \mathbb{R}^{n+\sum_{k\in K}|J_k|+|K|} : \bigcap_{i\in T} \bar{b}^i z \ge \bar{b}_0^i \bigcap_{k\in K} \bigcup_{j\in J_k} (\bar{A}^{jk} z \ge \bar{a}^{jk}) \right\}$$

All possible equivalent forms for GDP, obtained through any number of basic steps, are represented by:

LDP'

$$F = \left\{ z := (x,\lambda,c) \in \mathbb{R}^{n+\sum_{k\in K}|J_k|+|K|} : \bigcap_{i\in \hat{T}} \bar{b}^i z \ge \bar{b}_0^i \bigcap_{k\in \tilde{K}} \bigcup_{j\in J_k} (\tilde{A}^{jk} z \ge \tilde{a}^{jk}) \bigcap_{n\in \hat{K}} \bigcup_{m\in J_n} (\hat{A}^{mn} z \ge \hat{a}^{mn}) \right\}$$

Converting LDP to MIP reformulations

Proposition 2 (Theorem 3.3 combined with Corollary 3.5 in Balas (1979)). Let $F = \bigcup_{i \in Q} P_i$, $P_i = \left\{ x \in \mathbb{R}^n : \tilde{A}^i x \geq \tilde{a}_0{}^i \right\}$, $i \in Q$, where Q is an arbitrary set and each $(\tilde{A}^i, \tilde{a}_0{}^i)$ is an $m_i \times (n+1)$ matrix such that every P_i is a *bounded non-empty* polyhedron.

Furthermore, let $\zeta(Q)$ be the set of all those $x \in \mathbb{R}^n$ such that there exist vectors $(v^i, y_i) \in \mathbb{R}^{n+1}$, $i \in Q$, satisfying

$$
\begin{aligned}
& x - \sum_{i \in Q} v^i = 0 \longrightarrow v^i \quad \textit{disaggregated variables} \\
& \tilde{A}^i v^i - \tilde{a}_0{}^i y_i \geq 0 \qquad i \in Q \\
& y_i \geq 0 \qquad i \in Q \\
& \sum_{i \in Q} y_i = 1 \qquad i \in Q
\end{aligned}
$$

=> Convex Hull

Then $cl\ conv\ F = \zeta(Q)$.

Proposition 3 (Corollary 3.7 in Balas (1979).

Let $\zeta_I(Q) := \left\{ x \in \zeta(Q) : y_i \in \{0,1\},\ i \in Q \right\}$. $\boxed{\textit{=> MIP representation}}$

Then $\zeta_I(Q) = F$.

Family of MIP Reformulations For GDP

$$F=\left\{ z:=(x,\lambda,c)\in\mathbb{R}^{n+\sum_{k\in K}|J_k|+|K|} : \bigcap_{i\in\hat{T}} \bar{b}^i z\ge \bar{b}_0^{\,i} \bigcap_{k\in\tilde{K}} \bigcup_{j\in J_k} (\tilde{A}^{jk} z\ge \tilde{a}^{jk}) \bigcap_{n\in\hat{K}} \bigcup_{m\in J_n} (\hat{A}^{mn} z\ge \hat{a}^{mn}) \right\}$$

LDP'

$\Downarrow$

General template for any MILP reformulation

$$Min\ Z=\sum_{k\in K}\sum_{j\in J_k}\gamma_{jk} y_{jk} + d^T x$$

$s.t.$

$$\begin{array}{ll}
b^i x\ge b_0^{\,i} & i\in I_{B_1} \\
h^i y\ge h_0^{\,i} & i\in I_{H_1} \\
x_i^{\,L}\le x_i\le x_i^{\,U} & i\in I_{X_1} \\
0\le y_{jk}\le 1 & (j,k)\in L_1 \\
y_{jk}=\sum_{\widehat{j}\in J_{\widehat{k}}} \tilde{u}_{jk}^{\ \widehat{j}\widehat{k}} & (j,k)\in L_{2_{\widehat{k}}} \cup K'_{S_{2_{\widehat{k}}}} \cup I'_{H_{2_{\widehat{k}}}}, \widehat{k}\in\tilde{K} \\
y_{jk}=\sum_{m\in J_n} \hat{u}_{jk}^{\ mn} & (j,k)\in L_{3_n} \cup K'_{S_{3_n}} \cup I'_{H_{3_n}}, n\in\hat{K} \\
x=\sum_{j\in J_k}\tilde{v}^{jk} & k\in\tilde{K} \\
x=\sum_{m\in J_n}\hat{v}^{mn} & n\in\hat{K} \\
b^i\tilde{v}^{jk}\ge b_0^{\,i} y_{jk} & i\in I_{B_{2_k}}, j\in J_k, k\in\tilde{K} \\
b^i\hat{v}^{mn}\ge b_0^{\,i}\hat{y}_{mn} & i\in I_{B_{3_n}}, m\in J_n, n\in\hat{K} \\
\sum_{j\in J_k}\tilde{u}_{jk}^{\ \widehat{j}\widehat{k}} = y_{\widehat{j}\widehat{k}} & k\in K_{S_{2_{\widehat{k}}}}, \widehat{j}\in J_{\widehat{k}}, \widehat{k}\in\tilde{K} \\
\sum_{j\in J_k}\hat{u}_{jk}^{\ mn} = \hat{y}_{mn} & k\in K_{S_{3_{\widehat{n}}}}, m\in J_n, n\in\hat{K} \\
h^i\tilde{u}^{\widehat{j}\widehat{k}}\ge h_0^{\,i} y_{\widehat{j}\widehat{k}} & i\in I_{H_{2_{\widehat{k}}}}, \widehat{j}\in J_{\widehat{k}}, \widehat{k}\in\tilde{K} \\
h^i\hat{u}^{mn}\ge h_0^{\,i}\hat{y}_{mn} & i\in I_{H_{3_n}}, m\in J_n, n\in\hat{K}
\end{array}$$

$$\begin{array}{ll}
\tilde{u}_{jk}^{\ \widehat{j}\widehat{k}} = y_{\widehat{j}\widehat{k}} & \widehat{j}\in J_k, j=\widehat{j}\in J_k, \widehat{k}\in\tilde{K}, k=\widehat{k}\in\tilde{K} \\
\hat{u}_{jk}^{\ mn}=\hat{y}_{mn} & (j,k)\in M_{mn}, m\in J_n, n\in\hat{K} \\
A^{jk}\tilde{v}^{jk}\ge a^{jk} y_{jk} & j\in J_k, k\in\tilde{K} \\
A^{jk}\hat{v}^{mn}\ge a^{jk}\hat{y}_{mn} & (j,k)\in M_{mn}, m\in J_n, n\in\hat{K} \\
0\le\tilde{v}^{jk}\le x^U y_{jk} & j\in J_k, k\in\tilde{K} \\
0\le\hat{v}^{mn}\le x^U\hat{y}_{mn} & m\in J_n, n\in\hat{K} \\
0\le\tilde{u}_{jk}^{\ \widehat{j}\widehat{k}}\le y_{\widehat{j}\widehat{k}} & (j,k)\in L_{2_{\widehat{k}}}, \widehat{j}\in J_{\widehat{k}}, \widehat{k}\in\tilde{K} \\
0\le\hat{u}_{jk}^{\ mn}\le\hat{y}_{mn} & (j,k)\in L_{3_n}, m\in J_n, n\in\hat{K} \\
\sum_{\widehat{j}\in J_{\widehat{k}}} y_{\widehat{j}\widehat{k}}=1 & \widehat{k}\in\tilde{K} \\
\sum_{m\in J_n}\hat{y}_{mn}=1 & n\in\hat{K} \\
\sum_{m\in Q_{n_{\widehat{j}\widehat{k}}}}\hat{y}_{mn}=y_{\widehat{j}\widehat{k}} & n=\hat{n}_{\widehat{j}\widehat{k}}\in\hat{K}, \widehat{j}\in J_{\widehat{k}}, \widehat{k}\in K\setminus\tilde{K} \\
\sum_{\widehat{j}\in J_{\widehat{k}}} y_{\widehat{j}\widehat{k}}=1 & \widehat{k}\in K\setminus\tilde{K} \\
\hat{y}_{mn}\ge 0 & m\in J_n, n\in\hat{K} \\
y_{jk}\in\{0,1\} & j\in J_k, k\in K
\end{array}$$

MIP'

Particular case: Convex Hull Reformulation of LGDP

Lee S. and Grossmann I.E. (2000) **(CH)**

Disaggregated variables

$$\begin{aligned}
&Min\ Z = \sum_{k\in K}\sum_{j\in J_k} \gamma_{jk} y_{jk} + d^T x \\
&s.t. \qquad Bx \geq b \\
&x = \sum_{j\in J_k} v^{jk} && k \in K \\
&A^{jk} v^{jk} \geq a^{jk} y_{jk} && j \in J_k, k \in K \\
&x^L y_{jk} \leq v^{jk} \leq x^U y_{jk} && j \in J_k, k \in K \\
&\sum_{j\in J_k} y_{jk} = 1 && k \in K \\
&Hy \geq h \\
&y_{jk} \in \{0,1\} && j \in J_k, k \in K
\end{aligned}$$

While this MILP formulation has stronger relaxation than big-M, it is not strongest!!

A Hierarchy of Relaxations for DP

Hull Relaxation (Balas, 1985):

Let us take the following disjunctive set:

$$F = \bigcap_{j \in T} S_j$$

Then the hull-relaxation corresponds to:

$$h-rel\ F := \bigcap_{j \in T} clconv\ S_j .$$

Proposition 3 (Theorem 4.3 in Balas (1979)): *For* $i = 0,1,\ldots,t,$ *let* $F_i = \bigcap_{j \in T_i} S_j$ *be a sequence of regular forms of a disjunctive set, such that*

i) F_0 *is in CNF, with* $P_0 = \bigcap_{j \in T_0} S_j$ *;*

ii) F_t *is in DNF;*

iii) for $i = 1,\ldots,t,$ F_i *is obtained from* F_{i-1} *by a <u>basic step</u>.*

Then,

$$P_0 = h-rel\ F_0 \supseteq h-rel\ F_1 \supseteq \cdots \supseteq h-rel\ F_t = clconv\ F_t .$$ ***(true convex hull)***

Chemical ENGINEERING

A Hierarchy of Relaxations for GDP

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Proposition 4. *For* $i \in \overline{T}\,| + |\,K\,| - 1$ *let* F_{GDP_i} *be a sequence of regular forms of the disjunctive set:*

$$F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n + \sum_{k \in K} |J_k| + |K|} : \bigcap_{i \in \hat{T}} \overline{b}^i z \geq \overline{b}_0^{\,i} \bigcap_{k \in \tilde{K}} \bigcup_{j \in J_k} (\tilde{A}^{jk} z \geq \tilde{a}^{jk}) \bigcap_{n \in \hat{K}} \bigcup_{m \in J_n} (\hat{A}^{mn} z \geq \hat{a}^{mn}) \right\},\ \textit{such that}$$

i) F_{GDP_0} *corresponds to the disjunctive form:*

$$F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n + \sum_{k \in K} |J_k| + |K|} : \bigcap_{i \in \overline{T}} \overline{b}^i z \geq \overline{b}_0^{\,i} \bigcap_{k \in K} \bigcup_{j \in J_k} (\overline{A}^{jk} z \geq \overline{a}^{jk}) \right\};$$

ii) $F_{GDP_{|\overline{T}|+|K|-1}} := F_t$ *is in DNF;*

iii) for $i = 1, \ldots, t$, F_{GDP_i} *is obtained from* $F_{GDP_{i-1}}$ *by a basic step.*

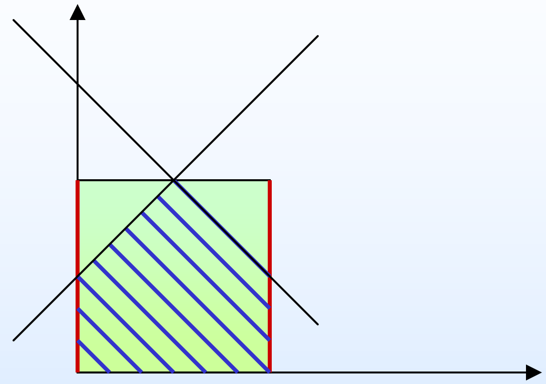
Then,

$$h-rel\ F_{GDP_0} \supseteq h-rel\ F_{GDP_1} \supseteq \cdots \supseteq h-rel\ F_{GDP_{|\overline{T}|+|K|-1}} = clconv\ F_{GDP_{|\overline{T}|+|K|-1}} = clconv\ F_t.$$ ***(true convex hull)***

Illustrative Example: Hierarchy of Relaxations

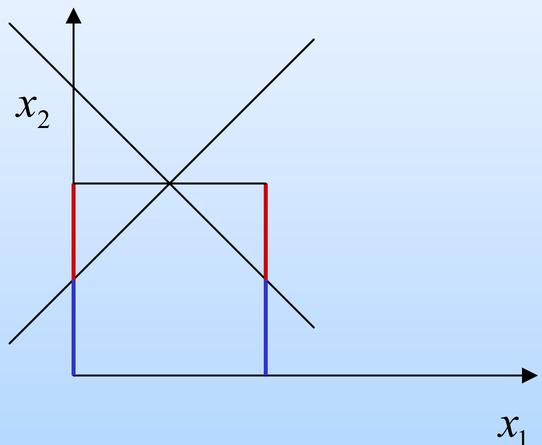
$$x_1 - x_2 + 0.5 \geq 0$$
$$-x_1 - x_2 + 1 \geq 0$$
$$\begin{bmatrix} x_1 = 0 \\ 0 \leq x_2 \leq 1 \end{bmatrix} \vee \begin{bmatrix} x_1 = 1 \\ 0 \leq x_2 \leq 1 \end{bmatrix}$$

Convex Hull of disjunction



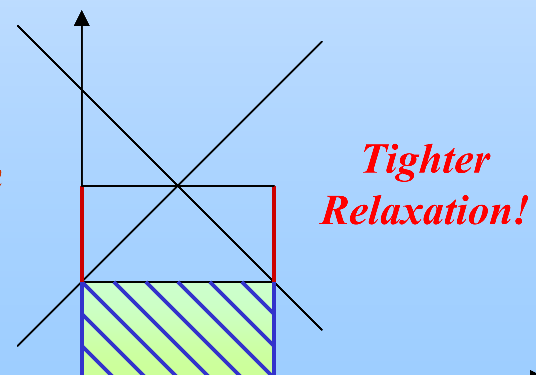
LP Relaxation

Application of 2 Basic Steps



$$\begin{bmatrix} x_1 - x_2 + 0.5 \geq 0 \\ -x_1 - x_2 + 1 \geq 0 \\ x_1 = 0 \\ 0 \leq x_2 \leq 1 \end{bmatrix} \vee \begin{bmatrix} x_1 - x_2 + 0.5 \geq 0 \\ -x_1 - x_2 + 1 \geq 0 \\ x_1 = 1 \\ 0 \leq x_2 \leq 1 \end{bmatrix}$$

Convex Hull of disjunction

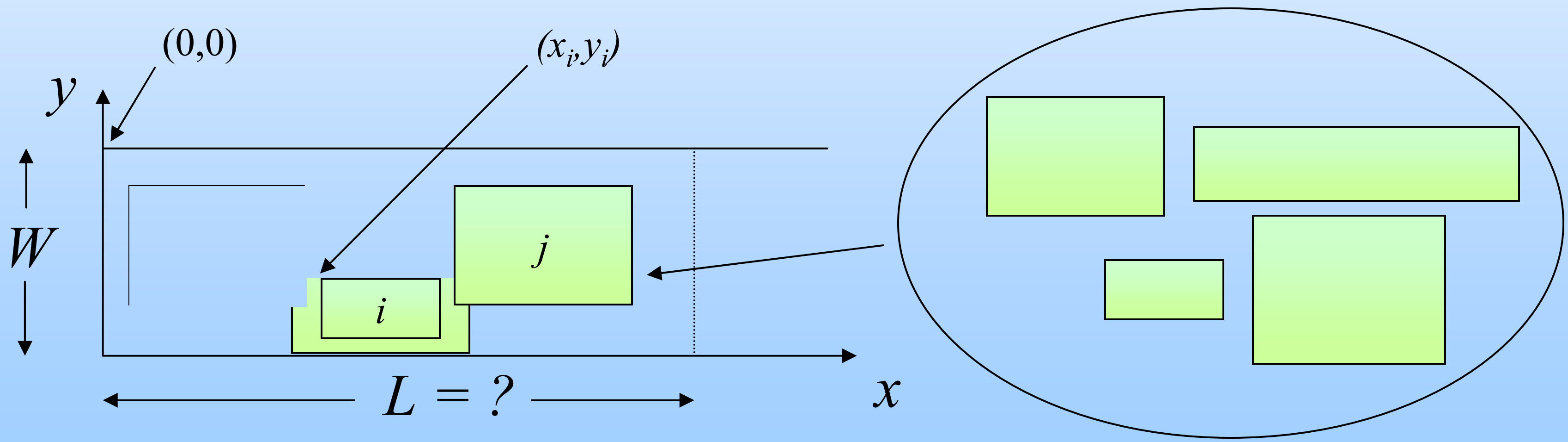


Tighter Relaxation!

Numerical Example: Strip-packing problem

Problem statement: *Hifi (1998)*

- Given a set of small rectangles with width H_i and length L_i.
- Large rectangular strip of fixed width W and unknown length L.
- Objective is to fit small rectangles onto strip without overlap and rotation while **minimizing length L of the strip.**



Set of small rectangles

GDP/DP Model for Strip-packing problem

$$
\begin{aligned}
&Min \quad lt \\
&s.t. \quad lt \ge x_i + L_i && \forall i \in N \\
&\begin{bmatrix} Y^1_{ij} \\ x_i + L_i \le x_j \end{bmatrix} \vee \begin{bmatrix} Y^2_{ij} \\ x_j + L_j \le x_i \end{bmatrix} \vee \begin{bmatrix} Y^3_{ij} \\ y_i - H_i \ge y_j \end{bmatrix} \vee \begin{bmatrix} Y^4_{ij} \\ y_j - H_j \ge y_i \end{bmatrix} && \forall i,j \in N, i<j \\
&x_i \le UB_i - L_i && \forall i \in N \\
&H_i \le y_i \le W && \forall i \in N \\
&lt, x_i, y_i \in \mathbb{R}^1_+ \,,\; Y^1_{ij}, Y^2_{ij}, Y^3_{ij}, Y^4_{ij} \in \{True, False\} && \forall i,j \in N, i<j
\end{aligned}
$$

Objective function
Minimize length

Disjunctive constraints
No overlap between rectangles

Bounds on variables

⇩

$$
\begin{aligned}
&Min \quad lt \\
&s.t. \quad lt \ge x_i + L_i && \forall i \in N \\
&\begin{bmatrix} \lambda^1_{ij} = 1 \\ x_i + L_i \le x_j \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{ij} = 1 \\ x_j + L_j \le x_i \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{ij} = 1 \\ y_i - H_i \ge y_j \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{ij} = 1 \\ y_j - H_j \ge y_i \end{bmatrix} && \forall i,j \in N, i<j \\
&\lambda^1_{ij} + \lambda^2_{ij} + \lambda^3_{ij} + \lambda^4_{ij} = 1 \quad \forall i,j \in N, i<j \\
&x_i \le UB_i - L_i && \forall i \in N \\
&H_i \le y_i \le W && \forall i \in N \\
&lt, x_i, y_i \in \mathbb{R}^1_+ \,,\; 0 \le \lambda^1_{ij}, \lambda^2_{ij}, \lambda^3_{ij}, \lambda^4_{ij} \le 1 && \forall i,j \in N, i<j
\end{aligned}
$$

DP Model For
4 Rectangle Strip-packing Problem

$$
\begin{aligned}
&\min \quad lt \\
&s.t. \quad lt \ge x_i + L_i \qquad \forall i \in N \\
&\begin{bmatrix} \lambda^1_{12}=1 \\ x_1+L_1 \le x_2 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{12}=1 \\ x_2+L_2 \le x_1 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{12}=1 \\ y_1-H_1 \ge y_2 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{12}=1 \\ y_2-H_2 \ge y_1 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{13}=1 \\ x_1+L_1 \le x_3 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{13}=1 \\ x_3+L_3 \le x_1 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{13}=1 \\ y_1-H_1 \ge y_3 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{13}=1 \\ y_3-H_3 \ge y_1 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{14}=1 \\ x_1+L_1 \le x_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{14}=1 \\ x_4+L_4 \le x_1 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{14}=1 \\ y_1-H_1 \ge y_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{14}=1 \\ y_4-H_4 \ge y_1 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{23}=1 \\ x_2+L_2 \le x_3 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{23}=1 \\ x_3+L_3 \le x_2 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{23}=1 \\ y_2-H_2 \ge y_3 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{23}=1 \\ y_3-H_3 \ge y_2 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{24}=1 \\ x_2+L_2 \le x_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{24}=1 \\ x_4+L_4 \le x_2 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{24}=1 \\ y_2-H_2 \ge y_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{24}=1 \\ y_4-H_4 \ge y_2 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{34}=1 \\ x_3+L_3 \le x_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{34}=1 \\ x_4+L_4 \le x_3 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{34}=1 \\ y_3-H_3 \ge y_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{34}=1 \\ y_4-H_4 \ge y_3 \end{bmatrix} \\
&H_1 \lambda \ge h_1 \\
&x_i \le UB_i - L_i \qquad \forall i \in N \\
&H_i \le y_i \le W \qquad \forall i \in N \\
&lt, x_i, y_i \in \mathbb{R}^1_+,\ 0 \le \lambda^1_{ij}, \lambda^2_{ij}, \lambda^3_{ij}, \lambda^4_{ij} \le 1 \qquad \forall i,j \in N, i<j
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
&\min \quad lt \\
&s.t. \quad lt \ge x_i + L_i \qquad \forall i \in N \\
&\begin{bmatrix} \lambda^1_{12}=1 \\ \lambda^1_{13}=1 \\ \lambda^1_{23}=1 \\ x_1+L_1 \le x_2 \\ x_1+L_1 \le x_3 \\ x_2+L_2 \le x_3 \end{bmatrix} \vee \ldots \vee \begin{bmatrix} \lambda^4_{12}=1 \\ \lambda^4_{13}=1 \\ \lambda^4_{23}=1 \\ y_2-H_2 \ge y_1 \\ y_3-H_3 \ge y_1 \\ y_3-H_3 \ge y_2 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{14}=1 \\ x_1+L_1 \le x_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{14}=1 \\ x_4+L_4 \le x_1 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{14}=1 \\ y_1-H_1 \ge y_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{14}=1 \\ y_4-H_4 \ge y_1 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{24}=1 \\ x_2+L_2 \le x_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{24}=1 \\ x_4+L_4 \le x_2 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{24}=1 \\ y_2-H_2 \ge y_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{24}=1 \\ y_4-H_4 \ge y_2 \end{bmatrix} \\
&\begin{bmatrix} \lambda^1_{34}=1 \\ x_3+L_3 \le x_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^2_{34}=1 \\ x_4+L_4 \le x_3 \end{bmatrix} \vee \begin{bmatrix} \lambda^3_{34}=1 \\ y_3-H_3 \ge y_4 \end{bmatrix} \vee \begin{bmatrix} \lambda^4_{34}=1 \\ y_4-H_4 \ge y_3 \end{bmatrix} \\
&H_2 \lambda \ge h_2 \\
&x_i \le UB_i - L_i \qquad \forall i \in N \\
&H_i \le y_i \le W \qquad \forall i \in N \\
&lt, x_i, y_i \in \mathbb{R}^1_+,\ 0 \le \lambda^1_{ij}, \lambda^2_{ij}, \lambda^3_{ij}, \lambda^4_{ij} \le 1 \qquad \forall i,j \in N, i<j
\end{aligned}
$$

Original CH formulation

Strengthened formulation

102 variables (24 0-1), 143 constraints
LP relaxation = 4, No.nodes=45
Optimum min L=8

170 variables (60 0-1), 347 constraints
LP relaxation = 8, No.nodes=0
Optimum min L = 8

25 Rectangle Problem Optimal solution= 31

Original CH		*Strengthened*
1,112 0-1 variables	=>	1,112 0-1 variables
4,940 cont vars		5,783 cont vars
7,526 constraints		8,232 constraints
LP relaxation = 9		LP relaxation = 27!

31 Rectangle Problem Optimal solution= 38

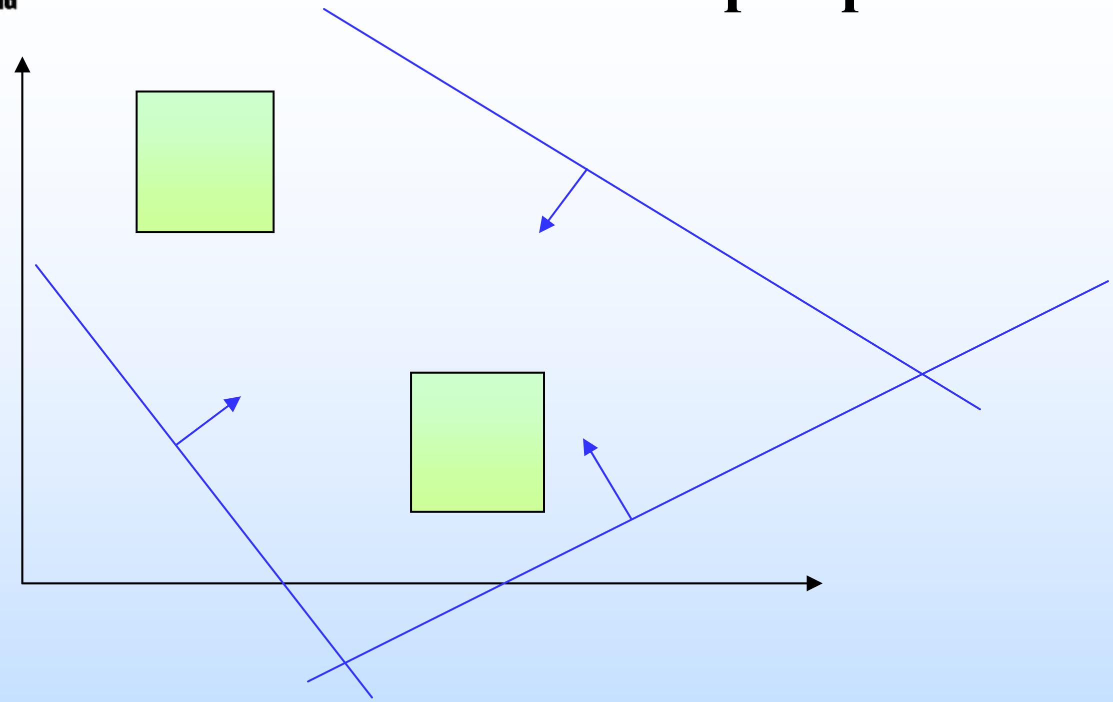
Original CH		*Strengthened*
2,256 0-1 variables	=>	2,256 0-1 variables
9,716 cont vars		11,452 cont vars
14,911 constraints		15,624 constraints
LP relaxation = 10.64		LP relaxation = 33!

Motivation for Cutting Plane Method

1. Tighter feasible region/lower bound → less nodes → decrease in computational solution time.
2. More variables and constraints → more iterations → increase in computational solution time.

Feasible region as tight as full space AND fewer variables and constraints

DERIVATION OF CUTTING PLANES:
Dual perspective



Proposition 5. The inequality $\alpha z \geq \alpha_0$ is a consequence of

$$F = \left\{ z := (x, \lambda, c) \in \mathbb{R}^{n+\sum_{k \in K} |J_k| + |K|} : \bigcap_{i \in \bar{T}} \bar{b}^i z \geq \bar{b}_0^i \bigcap_{k \in \tilde{K}} \bigcup_{j \in J_k} (\tilde{A}^{jk} z \geq \tilde{a}^{jk}) \bigcap_{n \in \hat{K}} \bigcup_{m \in J_n} (\hat{A}^{mn} z \geq \hat{a}^{mn}) \right\},$$

if and only if there exists a set of

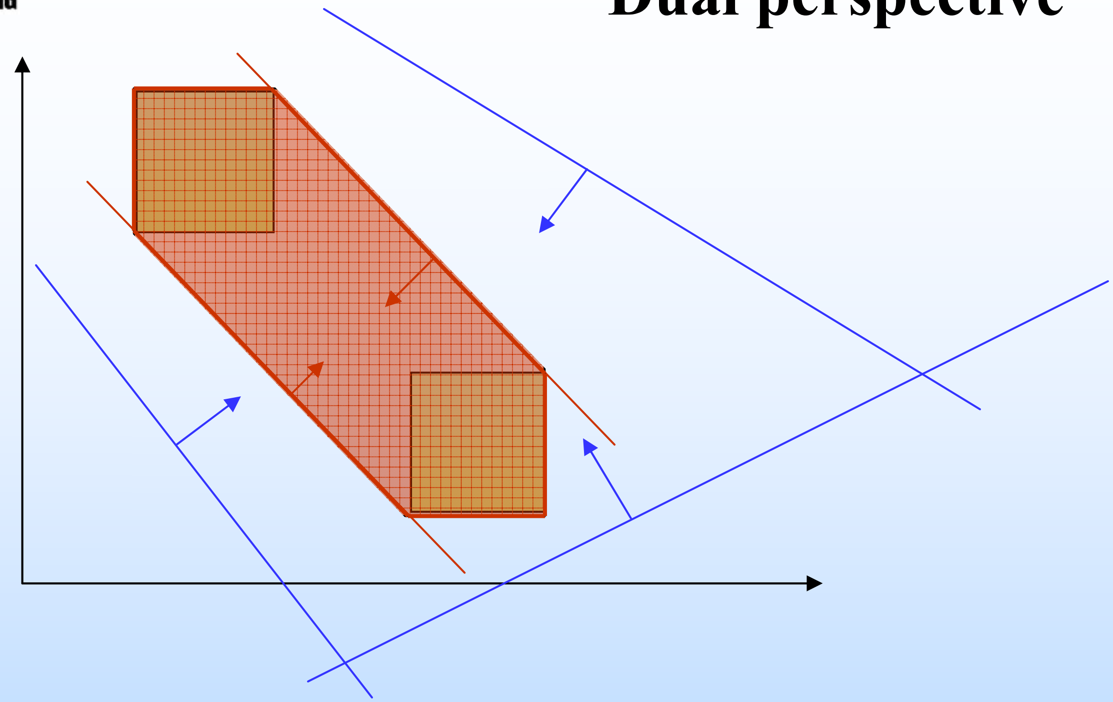
$\tilde{\theta}^{jk} \geq 0,\ j \in J_k, k \in \tilde{K};\ \hat{\theta}^{mn} \geq 0,\ m \in J_n, n \in \hat{K};$ and $\Gamma^i,\ i \in \bar{T}$, satisfying

$$\alpha \geq \sum_{i \in \bar{T}} \Gamma^i \bar{b}^i + \sum_{k \in \tilde{K}} \tilde{\theta}^{jk} \tilde{A}^{jk} + \sum_{n \in \hat{K}} \hat{\theta}^{mn} \hat{A}^{mn},\ j \in J_k, m \in J_n$$

$$\alpha_0 \leq \sum_{i \in \bar{T}} \Gamma^i \bar{b}_0^i + \sum_{k \in \tilde{K}} \tilde{\theta}^{jk} \tilde{a}^{jk} + \sum_{n \in \hat{K}} \hat{\theta}^{mn} \hat{a}^{mn},\ j \in J_k, m \in J_n$$

Reverse Polar Cone

DERIVATION OF CUTTING PLANES:
Dual perspective



Proposition 7. $\psi z \geq \alpha_0$ with $\alpha_0 \neq 0$ is a facet of the $h-rel\ F_{GDP_i}$, $i = 0,1,\ldots,|\bar{T}|+|K|-1$

if and only if $\alpha \neq 0$ is a vertex of the polyhedron

$$\left\{ \psi \in \mathbb{R}^{n+\sum_{k\in K}|J_k|+|K|} \;\middle|\; \begin{array}{l} \psi \geq \sum_{i\in\hat{T}} \Gamma^i \bar{b}^i + \sum_{k\in\tilde{K}} \tilde{\theta}^{jk}\tilde{A}^{jk} + \sum_{n\in\hat{K}} \hat{\theta}^{mn}\hat{A}^{mn},\ j\in J_k, m\in J_n \\ \alpha_0 \leq \sum_{i\in\hat{T}} \Gamma^i \bar{b}_0^i + \sum_{k\in\tilde{K}} \tilde{\theta}^{jk}\tilde{a}^{jk} + \sum_{n\in\hat{K}} \hat{\theta}^{mn}\hat{a}^{mn},\ j\in J_k, m\in J_n \\ \tilde{\theta}^{jk} \geq 0,\ j\in J_k, k\in\tilde{K} \\ \hat{\theta}^{mn} \geq 0,\ m\in J_n, n\in\hat{K} \\ \Gamma^i,\ i\in\hat{T} \end{array} \right\}.$$

Reverse Polar Cone

CUT GENERATION PROBLEM: Dual perspective

$$Max \;\; -\alpha z^{LP} + \alpha_0$$

$$s.t.$$

$$\alpha = \alpha^+ - \alpha^-$$

$$\sum_{i=1}^{n+\sum_{k\in K}|J_k|+|K|} (\alpha_i^+ - \alpha_i^-) \le 1$$

$$\alpha \ge \sum_{i\in \widehat{T}} \Gamma^i \overline{b}^i + \sum_{k\in \tilde{K}} \tilde{\theta}^{jk} \tilde{A}^{jk} + \sum_{n\in \hat{K}} \hat{\theta}^{mn} \hat{A}^{mn}, \; j \in J_k, m \in J_n$$

$$\alpha_0 \le \sum_{i\in \widehat{T}} \Gamma^i \overline{b}_0^i + \sum_{k\in \tilde{K}} \tilde{\theta}^{jk} \tilde{a}^{jk} + \sum_{n\in \hat{K}} \hat{\theta}^{mn} \hat{a}^{mn}, \; j \in J_k, m \in J_n$$

$$\tilde{\theta}^{jk} \ge 0, \; j \in J_k, k \in \tilde{K}$$

$$\hat{\theta}^{mn} \ge 0, \; m \in J_n, n \in \hat{K}$$

$$\Gamma^i, \; i \in \widehat{T}$$

$$\alpha^+, \alpha^- \ge 0$$

Normalization set

Balas, Ceria, Cornuejols (1993)

Reverse Polar Cone

CUT GENERATION PROBLEM: Primal perspective

$$
\begin{array}{ll}
Min \quad \eta & \\
\eta \geq z - z^{LP} & \\
\eta \geq z^{LP} - z & \\
\bar{b}^i z \geq \bar{b}_0^i & i \in \hat{T} \\
z - \sum_{j \in J_k} \tilde{\sigma}^{jk} = 0 & k \in \tilde{K} \\
z - \sum_{m \in J_n} \hat{\sigma}^{mn} = 0 & n \in \hat{K} \\
\tilde{A}^{jk} \tilde{\sigma}^{jk} - \tilde{a}^{jk} y_{jk} \geq 0 & j \in J_k, k \in \tilde{K} \\
\hat{A}^{mn} \hat{\sigma}^{mn} - \hat{a}^{mn} \hat{y}_{mn} \geq 0 & m \in J_n, n \in \hat{K} \\
\sum_{j \in J_k} y_{jk} = 1 & k \in \tilde{K} \\
\sum_{m \in J_n} \hat{y}_{mn} = 1 & n \in \hat{K} \\
y_{jk}, \tilde{\sigma}^{jk} \geq 0 & j \in J_k, k \in \tilde{K} \\
\hat{y}_{mn}, \hat{\sigma}^{mn} \geq 0 & m \in J_n, n \in \hat{K}
\end{array}
$$

Infinity Norm

Hull-relaxation

Cut Generation Problem for Lee & Grossmann

(Separation Problem)

Primal perspective separation problem ***Sawaya, Grossmann (2006)***

$$Min\ \ \phi(z) = \ \| z - z^{bm} \| \qquad \text{(SEP)}$$

$$\begin{aligned}
s.t. \qquad & Bx \le b \\
& A_{jk} v_{jk} \le a_{jk} y_{jk} && \forall j \in J_k, \forall k \in K \\
& x = \sum_{\forall j \in J_k} v_{jk} && \forall k \in K \\
& v_{jk} \le y_{jk} U_{jk} && \forall j \in J_k, \forall k \in K \\
& \sum_{\forall j \in J_k} y_{jk} = 1 && \forall k \in K \\
& Dy \le d \\
& 0 \le y_{jk} \le 1 && \forall j \in J_k, \forall k \in K \\
& x, v \in \mathbb{R}^n{}_+, z \equiv [x, y] \in \mathbb{R}^n{}_+ \times \mathbb{R}_+^{\sum_{\forall k \in K} |J_k|}
\end{aligned}$$

Hull Relaxation for Lee & Grossmann

Note that $\phi(z) = \| z - z^{bm} \|_1$ or $\phi(z) = \| z - z^{bm} \|_2$ or $\phi(z) = \| z - z^{bm} \|_\infty$ can be used.

Derivation of Cutting Planes: Primal perspective

Proposition 10: Let (FR-SEP) be the feasible region of the separation problem (SEP), and let (FRP-SEP) represent the projection of (FR-SEP) onto the z-space. Then,(FR-SEP) $\subseteq$ (FR-BM), where (FR-BM) represents the feasible region of (BM). Furthermore, (FRP-SEP) is a convex set.

Proposition 11: Let z^{bm} be the optimal solution of (BM) and z^{sep} be an optimal solution to (SEP). If $z^{bm} \notin$ (FRP-SEP), then $\exists\ \xi$ such that $\xi^T(z - z^{sep}) \geq 0$ is a valid linear inequality in z that cuts away z^{bm}, and such that ξ is a subgradient of $\phi(z)$ at z^{sep}.

Cutting Plane Method
Derivation of Cutting Planes

Propositions 12, 13, 14: (1) Let $\Phi(z) \equiv \|z - z^{bm}\|_2 \equiv (z - z^{bm})^T(z - z^{bm})$. Then,

$$\xi \equiv \nabla \Phi = (z - z^{bm})$$

(2) Let $\Phi(z) \equiv \|z - z^{bm}\|_{inf} \equiv max_i |z_i - z_i^{bm}|$. Then,

$$\xi \equiv (\mu^+ - \mu^-)$$

$$\begin{array}{llll} \textit{Min } u & & & \textit{Lagrange Multipliers} \\ s.t & u \geq z_i - z_i^{bm} & i \in I & \longleftarrow \mu^+ \\ & u \geq z_i^{bm} - z_i & i \in I & \longleftarrow \mu^- \end{array}$$

Feasible region of (SEP)

(3) Let $\Phi(z) \equiv \|z - z^{bm}\|_1 \equiv \Sigma |z_i - z_i^{bm}|$. Then,

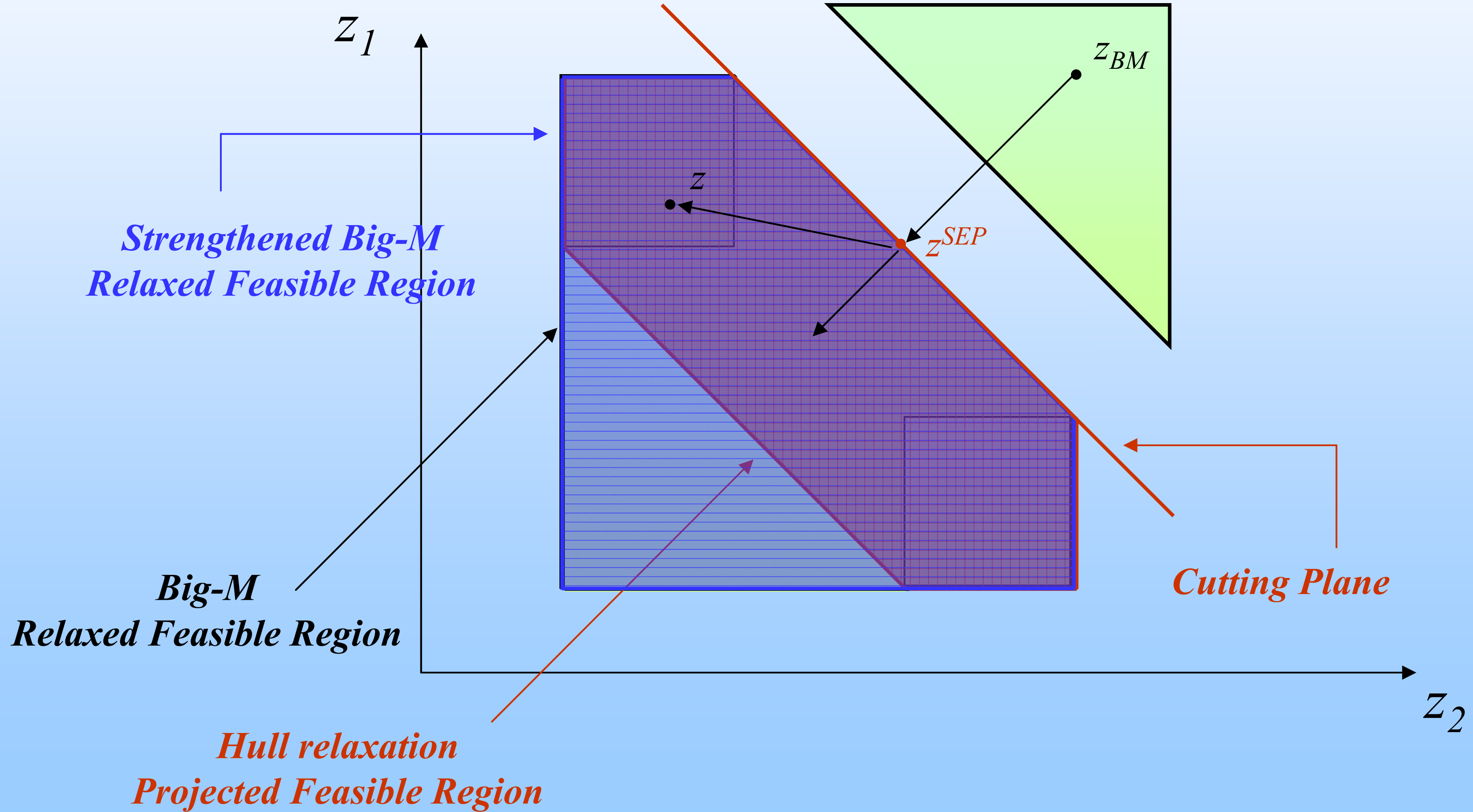
$$\xi \equiv (\mu^+ - \mu^-)$$

$$\begin{array}{llll} \textit{Min } u & & & \textit{Lagrange Multipliers} \\ s.t & u_i \geq z_i - z_i^{bm} & i \in I & \longleftarrow \mu^+ \\ & u_i \geq z_i^{bm} - z_i & i \in I & \longleftarrow \mu^- \end{array}$$

Feasible region of (SEP)

CUTTING PLANE METHOD

1. Solve relaxed Big-M MILP. This yields z^{BM}.
2. Solve separation problem.
   Feasible region corresponds to relaxed hull relaxation.
   Objective corresponds to finding point in hull relaxation closest to z^{BM}. This yields z^{SEP}.
3. Cutting plane is generated and added to relaxed big-M MILP.
4. Solve strengthened relaxed Big-M MILP. Go to 2.



NUMERICAL RESULTS
21-RECTANGLE STRIP PACKING PROBLEM

Table 3
Results for twenty one-rectangle strip-packing problem (CPLEX v. 8.1, default MIP options turned on)

	Relaxation	Optimal Solution	Gap (%)	Total Nodes in MIP	Solution Time for Cut Generation (sec)	*Total Solution Time (sec)	Number of Nodes per sec
Convex Hull	9.1786	---	---	968 652	0	>10 800	89.69
Big-M	9	24	62.5	1 416 137	0	4 093.39	345.95

NUMERICAL RESULTS
21-RECTANGLE STRIP PACKING PROBLEM

Table 3
Results for twenty one-rectangle strip-packing problem (CPLEX v. 8.1, default MIP options turned on)

	Relaxation	Optimal Solution	Gap (%)	Total Nodes in MIP	Solution Time for Cut Generation (sec)	*Total Solution Time (sec)	Number of Nodes per sec
Convex Hull	9.1786	---	---	968 652	0	>10 800	89.69
Big-M	9	24	62.5	1 416 137	0	4 093.39	345.95
Big-M + 20 cuts	9.1786	24	61.75	306 029	3.74	917.79	334.80
Big-M + 40 cuts	9.1786	24	61.75	547 828	7.48	1 063.51	518.76
Big-M + 60 cuts	9.1786	24	61.75	28 611	11.22	79.44	419.32
Big-M + 62 cuts	9.1786	24	61.75	32 185	11.59	91.4	403.27

* Total solution time includes times for relaxed MIP(s) + LP(s) from separation problem + MIP

Non-linear Discrete/Continuous Optimization: GDP Model

Lee & Grossmann I.E. (2000) (GDP)

$$Min\ Z = \sum_{k \in K} c_k + f(x)$$ *Objective function*

$$s.t. \quad r(x) \le 0$$ *Common constraints*

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \le 0 \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K$$ *Disjunctive constraints*

$$\underline{\vee}_{j \in J_k} Y_{jk} \quad k \in K$$

$$\Omega(Y) = True$$ *Logic constraints*

$$x^L \le x \le x^U$$

$$Y_{jk} \in \{True, False\} \quad j \in J_k, k \in K$$

$$c_k \in \mathbb{R}^1 \quad k \in K$$

$$\boxed{f(x), r(x), g_{jk}(x)}$$

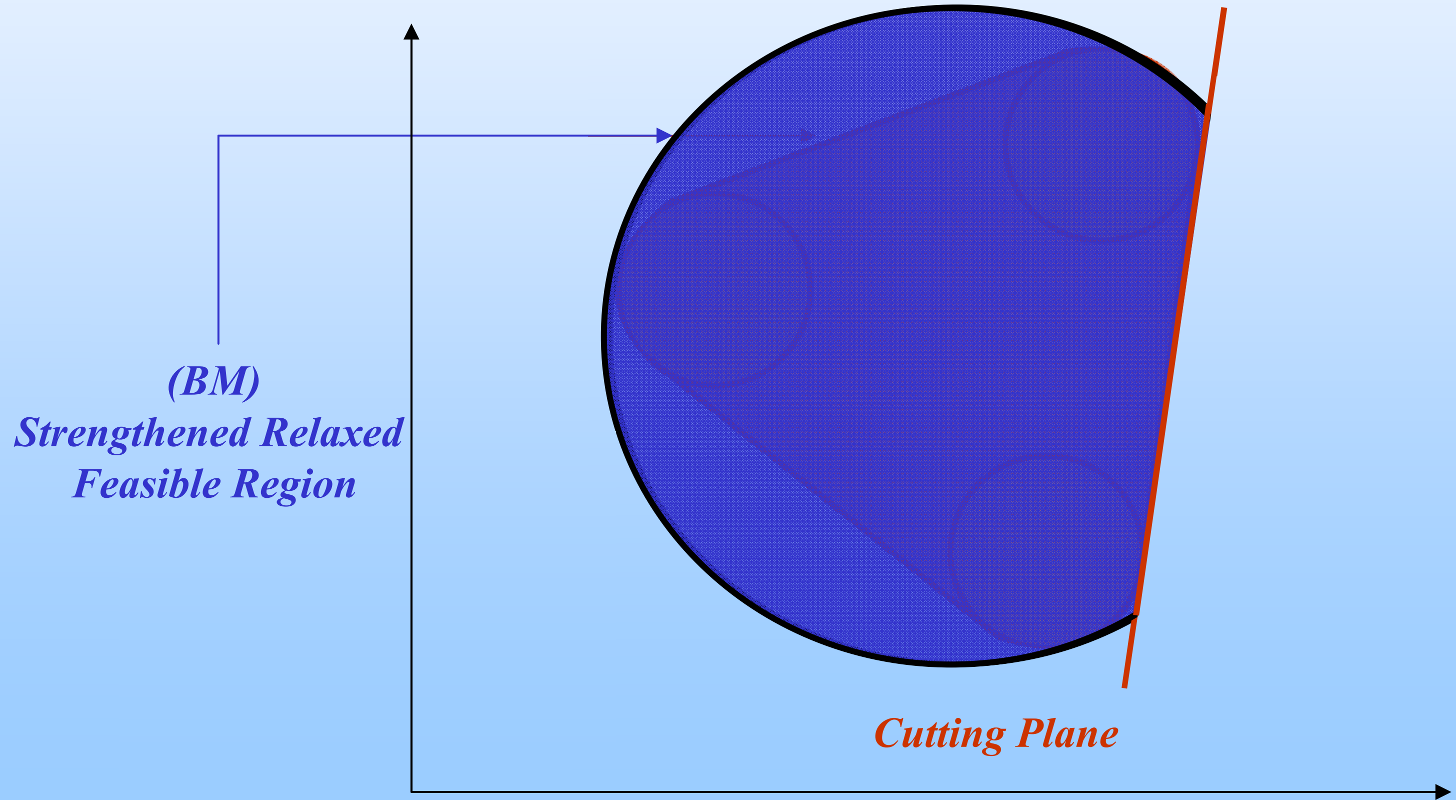
convex functions

Logical OR operator

Boolean variables

Cutting Plane Method

1. Solve relaxed (BM) problem. This yields $z^{bm} \equiv [x,y]^{bm}$.
2. Solve separation problem.
   Feasible region corresponds to relaxed (CH) problem.
   Objective corresponds to finding point in relaxed projected (CH) problem closest to z^{bm}.
   This yields z^{sep}.
3. Cutting plane is generated and added to relaxed (BM) problem.
4. Solve strengthened relaxed (BM) problem. Go to 2.



Convex Hull Formulation

- Consider **Disjunction** $k \in K$

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \le 0 \\ c = \gamma_{jk} \end{bmatrix}$$

- **Theorem**: Convex Hull of Disjunction k *(Lee, Grossmann, 2000)*
  - Disaggregated variables ν^j

$$\{(x,c) \mid x = \sum_{j \in J_k} v^{jk}, \quad c = \sum_{j \in J} \lambda_{jk} \gamma_{jk},$$

$$0 \le v^{jk} \le \lambda_{jk} U_{jk}, \; j \in J_k$$

=> Convex Constraints

$$\sum_{j \in J_k} \lambda_{jk} = 1, \quad 0 < \lambda_{jk} \le 1,$$

$$\lambda_{jk} g_{jk}(v^{jk} / \lambda_{jk}) \le 0, \; j \in J_k \}$$

  - λ_j - weights for linear combination

- Generalization of Stubbs and Mehrotra (1999)

Carnegie Mellon

Remarks

1. $h(\nu, \lambda) = \lambda g(\nu / \lambda)$

   If g(x) is a bounded convex function,
   $h(\nu, \lambda)$ is a bounded convex function

   Hiriart-Urruty and Lemaréchal (1993)

2. $h(\nu, 0) = 0$ for bounded g(x)

3. For linear constraints convex hull reduces to result by *Balas (1985)*

Cutting Plane Method: Separation Problem

$$\text{Min } \phi(z) = \| z - z^{bm} \| \qquad \text{(SEP)}$$

$$\begin{aligned}
&s.t. \quad r(x) \le 0 \\
&y_{jk} g_{jk}(v_{jk} / y_{jk}) \le 0 && \forall j \in J_k, \forall k \in K \\
&x = \sum_{\forall j \in J_k} v_{jk} && \forall k \in K \\
&v_{jk} \le y_{jk} U_{jk} && \forall j \in J_k, \forall k \in K \\
&\sum_{\forall j \in J_k} y_{jk} = 1 && \forall k \in K \\
&Dy \le d \\
&0 \le y_{jk} \le 1 && \forall j \in J_k, \forall k \in K \\
&x, v \in \mathbb{R}^n{}_+, z \equiv [x, y] \in \mathbb{R}^n{}_+ \times \mathbb{R}_+^{\sum_{\forall k \in K} |J_k|}
\end{aligned}$$

CONVEX NLP

How do we Implement this?

(CH) Relaxed Feasible Region

$For\ \phi(z) = \| z - z^{bm} \|_2\ , we\ get\ (z^{sep} - z^{bm})(z - z^{sep}) \ge 0$

$For\ \phi(z) = \| z - z^{bm} \|_1\ , we\ get\ (\mu^+ - \mu^-)(z - z^{sep}) \ge 0$

$For\ \phi(z) = \| z - z^{bm} \|_\infty\ , we\ get\ (\mu^+ - \mu^-)(z - z^{sep}) \ge 0$

Different Norms Different Cuts

Computational Implementation of Separation Problem

Furman, Sawaya & Grossmann (2007)

Replace $y_{jk} g_{jk}(\nu_{jk} / y_{jk}) \leq 0$ by: where $0 \leq \nu_{jk} \leq U y_{jk}$

$$\boxed{((1-\varepsilon) y_{jk} + \varepsilon)(g_{jk}(\nu_{jk} /((1-\varepsilon) y_{jk} + \varepsilon))) - \varepsilon g_{jk}(0)(1 - y_{jk}) \leq 0}$$

1. The divisibility by 0 problem is avoided.

2. The new constraints are an exact approximation of the original constraints as ε → 0.

3. The new constraints are an exact approximation of the original constraints at $y_{jk} = 0$ and at $y_{jk} = 1$ regardless of value of ε.

$$if\ y_{jk} = 0, \Rightarrow (\varepsilon)(g_{jk}(0)) - \varepsilon g_{jk}(0) = 0 \leq 0$$

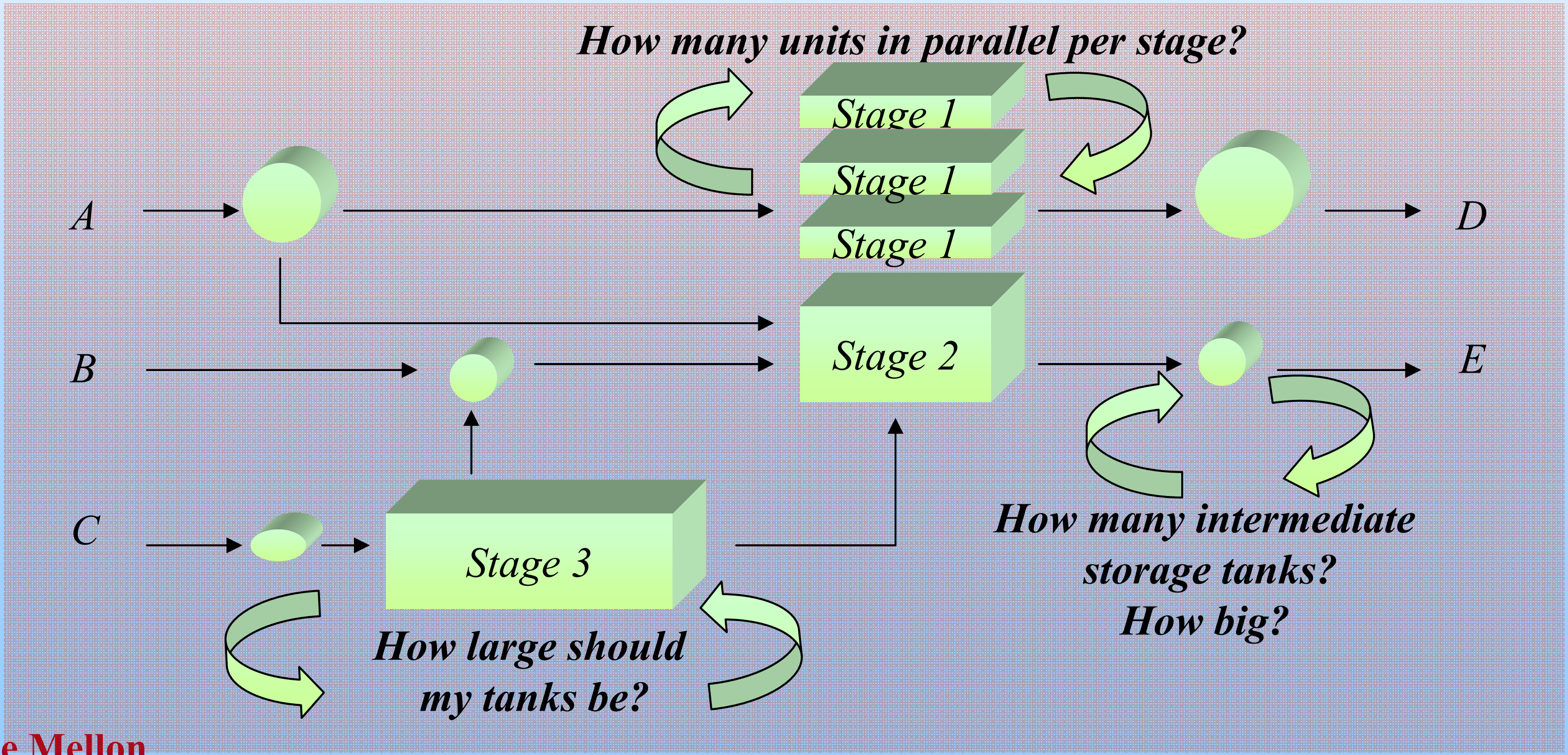
$$if\ y_{jk} = 1, \Rightarrow ((1)(g_{jk}(\nu_{jk} /(1)) - \varepsilon g_{jk}(0)(0) = (1) g_{jk}(\nu_{jk} /(1) \leq 0$$

4. The LHS of the new constraints are convex.

Numerical Example
Design of Multi-product Batch Plant

Problem statement: *Ravemark (1995)*

- Design of batch plant with multiple units in parallel and intermediate storage tanks.
- Problem consists of determining volume of equipment, number of units in parallel and volume and location of intermediate storage tanks while minimizing investment cost.



Numerical Results
Design of 10 Unit/Product Batch Plant

Problem Size

	# of constraints	# of variables	# of discrete variables
Convex Hull	1296	637	89
Big-M	800	239	89

Table 1: Results for design of 10 stage/product batch plant using traditional B&B (SBB)

	Relaxation	Optimal Solution	Gap (%)	Total Nodes in MINLP	Solution Time for Cut Generation (sec)	*Total Solution Time (sec)	Number of Nodes per sec
Convex Hull	650 401. 14	729 948.49	10.9	5 359	0	711.76	7.53
Big-M	641 763.19	729 948.49	12.1	12 449	0	787.98	15.80
Big-M + 58 cuts	650 401. 14	729 948.49	10.9	7 528	8.7	610.00	12.52

improvement: Total Nodes in MINLP 40%; *Total Solution Time 23%

Global Optimization of Bilinear Generalized Disjunctive Programs

Juan Ruiz

$$Min \quad Z = \boxed{f(x)} + \sum_{k \in K} c_k$$ → **Objective Function**

$$s.t. \quad \boxed{g(x)} \leq 0$$ → **Global Constraints**

$$\bigvee_{i \in D_k} \begin{bmatrix} Y_{ik} \\ \boxed{r_{ik}(x)} \leq 0 \\ c_k = \gamma_{ik} \end{bmatrix}$$ → **Disjunctions** $k \in K$

$$\Omega(Y) = True$$ → **Logic Propositions**

$$x \in R^n, c_k \in R, Y_{ik} \in \{True, False\} \qquad i \in D_k \,, \; k \in K$$

Bilinearities

Bilinearities may lead to multiple local minima → **Global Optimization techniques are required**

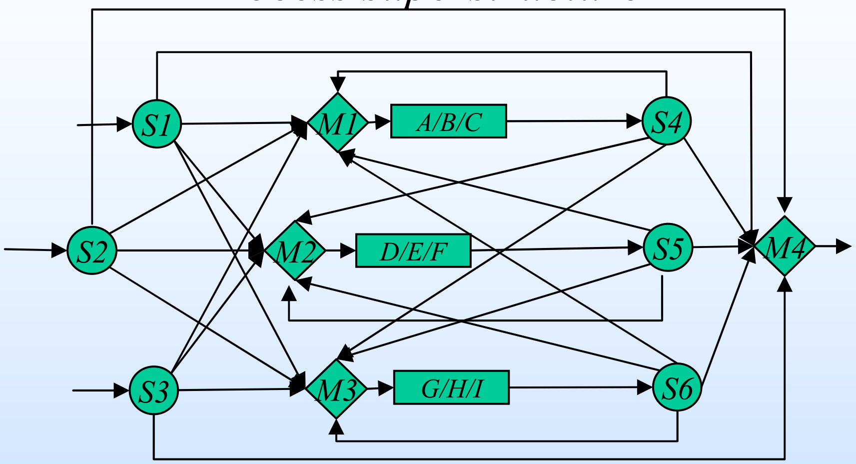
Relaxation of Bilinear terms using McCormick envelopes leads to a LGDP → ***Improved relaxations for <u>Linear GDP</u> has recently been obtained (Sawaya & Grossmann, 2007)***

Guidelines for applying basic steps in Bilinear GDP

- Replace bilinear terms in GDP by McCormick convex envelopes (LGDP)
- Apply basic steps between those disjunctions with at least one variable in common.
- The more variables in common two disjunctions have the more the tightening can be expected
- If bilinearities are outside the disjunctions apply basic steps by introducing them in the disjunctions previous to the relaxation.
- If bilinearities are inside the disjunctions a smaller tightening effect is expected.
- A smaller increase in the size of the formulation is expected when basic steps are applied between improper disjunctions and proper disjunctions.

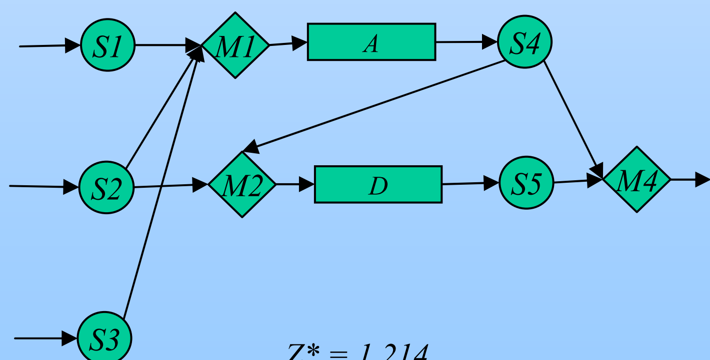
Case Study I: Water treatment network design

Process superstructure



N of cont. vars. : 114
N of disc. vars. : 9
N of bilinear terms: 36

Optimal structure



$Z^* = 1.214$

Generalized Disjunctive Program

$$Min\ Z = \sum_{k \in PU} CP_k$$

$s.t.$

$$f_k^{\,j} = \sum_{i \in M_k} f_i^{\,j} \qquad \forall j \quad k \in MU$$

$$\sum_{i \in S_k} f_i^{\,j} = f_k^{\,j} \qquad \forall j \quad k \in SU$$

$$\sum_{i \in S_k} \zeta_i^{\,k} = 1 \qquad k \in SU$$

$$\boxed{f_i^{\,j} = \zeta_i^{\,k} f_k^{\,j}} \qquad \forall j \quad i \in S_k \qquad k \in SU$$

$$\bigvee_{h \in D_k} \left[\begin{array}{c} YP_k^{\,h} \\ f_i^{\,j} = \beta_k^{\,jh} f_{i'}^{\,j}, i \in OPU_k, i' \in IPU_k, \forall j \\ F_k = \sum_j f_i^{\,j}, i \in OPU_k \\ CP_k = \partial_{ik} F_k \end{array} \right] \qquad k \in PU$$

$$0 \le \zeta_i^{\,k} \le 1 \qquad \forall j, k$$

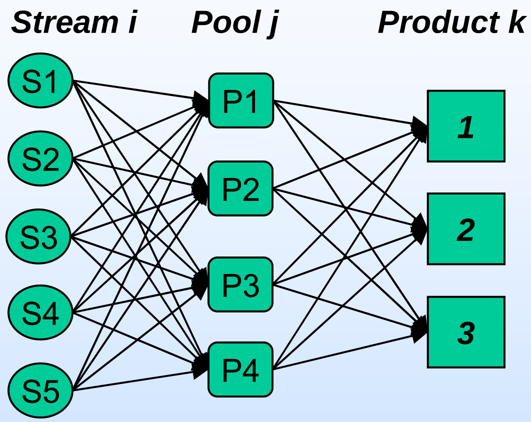
$$0 \le f_i^{\,j}, f_k^{\,j} \qquad \forall i, j, k$$

$$0 \le CP_k \qquad \forall k$$

$$YP_k^{\,h} \in \{true, false\} \qquad \forall h \in D_k \qquad \forall k \in PU$$

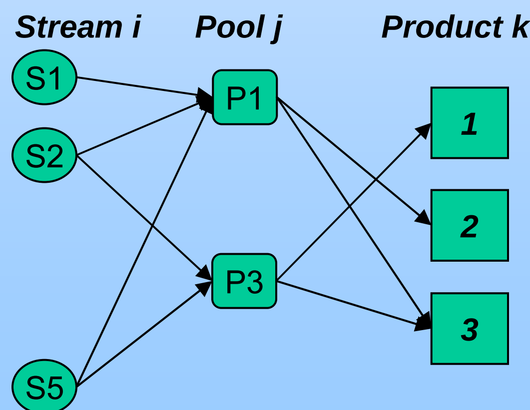
Case Study II: Pooling network design

Process superstructure



N of cont. vars. : 76
N of disc. vars. : 9
N of bilinear terms: 24

Optimal structure



Z = -4.640*

Generalized Disjunctive Program

$$Min\ Z = \sum_{j\in J} CP_j + \sum_{i\in I} CST_i + \sum_{j\in J}\sum_{i\in I} c_{ij} \sum_{w\in W} f_{ijw} - \sum_{k\in K} d_k \sum_{j\in J}\sum_{w\in W} f_{jkw}$$

$s.t.$

$$\sum_{i\in I}\sum_{w\in W} f_{ijw} = \sum_{k\in K}\sum_{w\in W} f_{jkw} \qquad \forall j\in J$$

$$\sum_{j\in J}\sum_{w\in W} f_{jkw} - S_k = 0 \qquad \forall k\in K$$

$$f_{ijw} = \lambda_{iw} \sum_{w'\in W} f_{ijw'} \qquad \forall i\in I, \forall j\in J, \forall w\in W$$

$$\sum_{j\in J} f_{jkw} - Z_{kw} \sum_{j\in J}\sum_{w'\in W} f_{jkw'} = 0 \qquad \forall k\in K, \forall w\in W$$

$$\begin{bmatrix} YST_i \\ f^{lo} \le \sum_{j\in J}\sum_{w\in W} f_{ijw} \\ CST_i = \alpha_i \end{bmatrix} \vee \begin{bmatrix} \neg YST_i \\ f_{ijw} = 0 \\ CST_i = 0 \end{bmatrix} \qquad \forall i\in I$$

$$\begin{bmatrix} YP_j \\ f^{lo} \le \sum_{i\in I}\sum_{w\in W} f_{ijw} \\ \sum_{k\in K} f_{jkw} = \sum_{i\in I} f_{ijw}, \forall w\in W \\ \boxed{f_{jkw} = \zeta_j^{\,k} \sum_{i\in I} f_{ijw}}, \forall w\in W, k\in K \\ \sum_{k\in K} \zeta_j^{\,k} = 1 \\ CP_j = \gamma_j \end{bmatrix} \vee \begin{bmatrix} \neg YP_j \\ f_{ijw} = 0, \forall i\in I, w\in W \\ f_{jkw} = 0, \forall k\in K, w\in W \\ CP_j = 0 \end{bmatrix} \qquad \forall j\in J$$

$$0 \le \zeta_j^{\,k} \le 1; 0 \le f_{jkw}, f_{ijw} \le f^{up}$$

$$0 \le CST_i, CP_j; YST_i, YP_j \in \{true, false\}$$

Performance

		Global Optimization Technique using Lee & Grossmann relaxation	*Global Optimization Technique using proposed relaxation*	*Relative Improvement*
Example 1	**Initial Lower Bound**	**400.66**	**499.86**	**24.90%**
	Bound contraction			**99.7%**
	Nodes	**399**	**204**	**51%**

		Global Optimization Technique using Lee & Grossmann relaxation	*Global Optimization Technique using proposed relaxation*	*Relative Improvement*
Example 2	**Initial Lower Bound**	**-5515**	**-5468**	**0.90%**
	Bound contraction			**8%**
	Nodes	**748**	**683**	**9%**

Conclusions

Unified GDP with Disjunctive Programming

- Developed DP equivalent formulation for GDP
- Developed a family of MIP reformulations for GDP
- Developed a hierarchy of relaxations for GDP

Developed framework for obtaining improved LP relaxations

- Demonstrated improved relaxations can be obtained compared to convex hull formulation Lee & Grossmann (2000)
- Numerical results have shown great improvement in lower bound for strip packing problem

Cutting Planes

- Showed equivalence dual and primal cut-generation problems.
- Developed a primal cut-and-branch algorithm where cutting planes were generated from the primal separation problem

Nonlinear GDPs

- Cutting planes can be readily extended
- Concept basic steps improves relaxation in bilinear GDPs