An Introduction of Hybrid Systems with Memory

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Hybrid Systems

- Formed by interacting continuous and discrete variables. Many systems in technology are hybrid by construction.
- Switching dynamical systems in genomics seismology, ecology, electrophysiology, etc. are best modeled by hybrid systems.
- They are very efficient in constructing analytically tractable approximations of more sophisticated non-linear dynamical systems.

A Formal Description of Hybrid I/O Systems

A Hybrid I/O system H is a collection $H = (Q, X, U, T, Init, f, Inv, E, G, R)$ consisting of

- A set of discrete states $Q = \{q_1, ..., q_m\}$ also called locations,
- A space of continuous variables $X = \mathbb{R}^n$
- A set of initial conditions $Init \subseteq Q \times X$
- A space of inputs $U = \mathbb{R}^m$ (control, disturbance or both)
- A space of independent variables $T = \mathbb{R}^k$, typically the time $T = [t_0, \infty)$
- A vector field $f, : Q \times Y \times U \rightarrow Y$, governing the continuous evolution

- An invariant set (domain, subspace) for each $q \in Q$, $Inv : Q \to P(Y)$, where $P(\cdot)$ denotes the power set. Each state's governing dynamics is valid within its invariant set.
- A set of edges (state transitions) $E \subset Q \times Q$,
- Guard conditions for each edge $G : E \to P(X)$,
- A reset map for each egde $R : E \times X \times U \to P(X)$.
  - For verifiability analysis we will consider $R : E \times G \to X$

Lygeros et. al., Koutsoukos et. al., Automata: von Neumann

Piecewise Linear Case

A typical subclass is the piecewise linear Hybrid I/O System with a state space description:

$$\frac{dx}{dt} = A_{q(t)}x(t) + B_{q(t)}u(t) + k_{q(t)} \quad i : 1, 2, ..., n \quad x(0) = x_0, q(0) = q_0$$

$$q(t) = q_j \quad \text{if} \quad x(t) \in \mathbf{X}_j$$

Hybrid Systems and Nontrivial Behavior

- Multi-stationarity (existence of multiple stationary steady states in a dynamical system) is a feature underlying differentiation and specialization (adoption, learning, memorization)
- Abstractions (Boolean Networks, Boolean delay equations / Kauffman et. al, Ghill at. al. Thomas et. al) of Hybrid systems are proven in modeling of multi-stationary dynamics.

- Subclasses (Piecewise linear / Glass et. al.) of hybrid systems are also proven in modeling of such systems.
- The facts: each circuit of the graph of a hybrid system represents a possible cycle, the state space can be divided into invariant sets, threshold phenomena in biological systems.

Motivations for Functional Extension

- Co-existence of multi-stationarity with delays are known to lead history dependent responses to perturbations or future inputs. Demonstrated by BDE.
- Dependence of the stationary steady state to the initial state in a multi-stationary system can be treated as "*exhibited stationary state represent some information from the initialization*" and the information content of patterns (history) is much more than single values (initial state). Demonstrated by BDE.

- There exist systems which are known to be history dependent: associative memory (neural system), permanent memory (via genomic regulation), immunity response, training forecast models by existing past information and using it for intervention or decision making.

Functional Differential Equations

FDE's represent a class of dynamical systems where the derivative of the variables are determined by a functional (a function acting on a function) of the initial function. A concrete case is the Delay Differential Equation

$$\frac{dx}{dt} = f_1(x(t-\tau_1), u(t))dt + f_2(x(t-\tau_2), u(t)) + ...$$

Also various types of delay, integral and integro-differential equations are typical functional systems simplest being the convolution equation

$$x(t) = \int_{-\infty}^{t} c(\tau)u(t-\tau)d\tau$$

- They can grab history dependent behaviour observed in biology, biophysics, climatology, engineering and finance.
- They are generally developed by considering naturally occuring delays in dynamical systems.

Motivations for Hybrid Extension

- First of all implementation of FDE's require abstraction of infinite dimensional initial functions and the notion of guard conditions can directly be utilized for adaptive sampling.
- Especially in regulatory systems more sophisticated responses to initial patterns may arise: selective switching, evolutionary schemes, dilation and replications of patterns, etc.
- Delays in processes involving threshold phenomena.
- Benefits of hybrid abstractions.

Hybrid Systems with Memory

A Hybrid system with memory H is a collection $H = (Q, X, U, T, Init, \mathcal{M}, f, Inv, E, G, R)$ consisting of

- A set of discrete states $Q = \{q_1, ..., q_m\}$ also called locations,
- A space of continuous variables $X = \mathbb{R}^n$
- A set of initial conditions $Init \subseteq Q \times X$
- A space of inputs $U = \mathbb{R}^m$ (control, disturbance or both)
- A space of independent variables $T = \mathbb{R}^k$, typically the time $T = [t_0, \infty)$
- A vector field $f, : Q \times X \times U \times \mathcal{M} \to X$, governing the continuous evolution
- An invariant set (domain, subspace) for each $q \in Q$, $Inv : Q \to P(Y)$, where $P(\cdot)$ denotes the power set. Each state's governing dynamics is valid within its invariant set.

- A set of edges (state transitions) $E \subset Q \times Q$,
- Guard conditions for each edge $G : E \rightarrow P(X)$,
- A reset map for each egde $R : E \times X \times U \rightarrow P(X)$.
  - For verifiability analysis we will consider $R : E \times G \rightarrow X$
- $M(t) \in \mathcal{M}$ is a growing memory of past state transitions such that:
  - $M(0) = \{m_0\} = \{(t_0, x_0, q_0)\}$
  - if $M(t_j-) = \{m_0, m_1, ..., m_i\}$ and $x(t_j) \in g\{(q(t), q \in Q)$ than $M(t_j+) = \{M(t_j-), m_{i+1}\}$
  - $m_{i+1} = \{t_j, x(t_{j-}), x(t_{j-})\}$

Thus the past evolution of the system is sampled at state transitions containing the time and the values of variables before and after the state transition.

Remarks

- To be able to implement HSM verifiability (finite number of transitions in a finite duration) is a necessary condition.
- To bound the size of growing memory, especially for cyclic executions a rule for removing the redundant memory elements can be embedded.
- Once the history is memorized not only the flow but the other terms like Inv, G, R can also be taken as memory dependent.

Piecewise Linear Case

A typical subclass is the piecewise linear Hybrid System with memory with a state space description:

$$\frac{dx}{dt} = A_{q(t),M(t)}x(t) + B_{q(t),M(t)}u(t) + k_{q(t),M(t)} \quad i:1,2,...,n \quad x(0) = x_0, q(0) = q_0$$

$$q(t) = q_j \quad \text{if} \quad x(t) \in \mathbf{X}_j$$

$$M(0) = \{m_0\} = \{(t_0, x_0)\}$$

$$\text{if} \quad x(t_{a-}) \in \mathbf{X}_j \quad \text{and} \quad x(t_{a+}) \notin \mathbf{X}_j \quad \text{and} \quad M(t_{a-}) = \{m_1, ..., m_k\}, \quad k = 1, 2, ...$$

$$\text{then} \quad M(t_{a+}) = \{M(t_{a-}), m_{k+1}\}$$

$$m_{k+1} = \{t_a, x(t_{a-}), x(t_{a+})\}.$$

Verifiability of the Solutions

- The space of the independent variable (time) can be divided into hybrid time sets formed by the intervals spent in each state. Thus the memory *M(t)* and the state *q(t)* are constant in each interval (piecewise constant).
- If there exist finite number of intervals within finite time (non-Zeno execution) than whether a solution exists, whether it is unique and whether the local solutions in an interval are diverging can be checked by finite number of algorithmic or computational steps.

Assume a cycle $C = \{e_{12}, e_{23}, ..., e_{k-1,k}, e_{k,1}\}$ is traversing the states $(q_1, q_2, ..., q_k)$ Than the Zeno set $Z(C_{12})$ of C on g_{12} can be expressed as

$$Z(C_{12}) \subseteq \mathrm{Re}\, ach(Init) \cap g_{12} \cap r_{12}^{-1}(g_{23}) \cap ... \cap r_{12}^{-1}(r_{23}^{-1} ... r_{k-1,k}^{-1}(g_{k1})...)$$

$$Z(C_{23}) = r_{12}(Z(C_{12})),\ r_{23}(Z(C_{23})),\ ...$$

if reset maps are identity

$$Z(C) \subseteq \mathrm{Re}\, ach(Init) \cap g_{12} \cap g_{23} \cap ... \cap g_{k1}$$

if $Z(C) = \emptyset$ than C is Zeno free and if all cycles of H are Zeno free than H is verifiable

Discussions and Conclusions

- We introduced a class of Hybrid Systems. It can be considered either as introducing memory into HSs or introducing switching behaviour into FDEs
- Many systems with pattern memorization capability and subject to external inputs can be modelled.