Differential Variational Inequalities and friends

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Introduction

Differential variational inequalities (DVI's) have the form

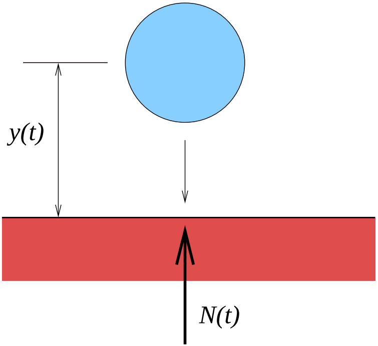
$$\begin{aligned} \frac{dx}{dt} &= f(t, x(t), u(t)), \qquad x(t_0) = x_0, \\ 0 &\leq \langle w - u(t), G(x(t), u(t)\rangle \qquad \text{for all } w \in K, \\ u(t) &\in K \qquad \text{for (almost) all } t \geq t_0. \end{aligned}$$

If K is a closed convex cone this is equivalent to the differential complementarity problem (DCP)

$$\begin{aligned} \frac{dx}{dt} &= f(t, x(t), z(t)), \qquad x(t_0) = x_0, \\ K \ni z(t) &\perp G(x(t), z(t)) \in K^*. \end{aligned}$$

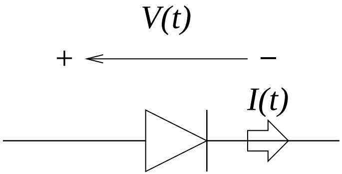
Examples

Contact:



$$m\frac{d^2y}{dt^2} = -mg + N(t)$$
$$0 \le y(t) - r \perp N(t) \ge 0 \qquad \text{for all } t.$$

Electronic circuits:



current flow

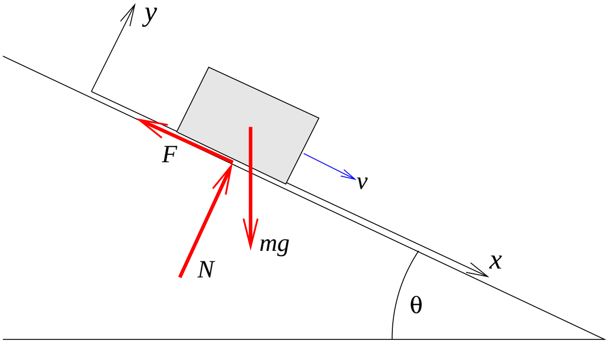
$$0 \leq I(t) \quad \perp \quad V(t) \leq 0$$

(Networks of) Queues:

Length $= \ell(t)$, service rate $r(t)$, max service rate r^*.

$$\begin{aligned} 0 \leq \ell(t) \quad &\perp \quad r^* - r(t) \geq 0 \\ \frac{d\ell}{dt} \quad &= \quad g(t) - r(t) \end{aligned}$$

Coulomb friction:



$$F \in [-\mu N, +\mu N] \quad \& \quad (\widetilde{F} - F)v \geq 0 \qquad \text{for all } \widetilde{F} \in [-\mu N, +\mu N]$$

$$m\frac{dv}{dt} = F - mg\sin\theta, \qquad N = mg\cos\theta$$

Index

A crucial quantity in a DVI is its *index*. This is the number of times that we need to differentiate $G(x, z)$ with respect to t to explicitly obtain z. This is the number of derivatives separating the two quantities in the VI or complementarity problem.

For example, for the contact problem, the index is *two*.

For the queuing and friction examples, the index is *one*.

If the index is *zero*, then the VI can be solved for $z = z(t, x)$ and we can substitute this into the right-hand side of the differential equation.

Structure

For index zero DVI's, it is sufficient to have

$$\begin{aligned} \frac{dx}{dt} &= f(t,\, x(t),\, z(t)), \qquad x(t_0) = x_0, \\ z(t) \in K \quad \& \quad \langle \widetilde{z} - z(t),\, G(x(t), z(t)\rangle \geq 0 &\qquad \text{for all } \widetilde{z} \in K \end{aligned}$$

with K a closed convex set provided (for example) that $G(x,z)$ is strongly monotone in z. That is, there is an $\eta_G > 0$ where

$$\langle G(x,z_1) - G(x,z_2),\, z_1 - z_2\rangle \;\geq\; \eta_G\, \|z_1 - z_2\|^2 \qquad \text{for all } z_1,\, z_2.$$

For index one DVI's, we need some more structure as the solutions tend to be less regular:

$$\begin{aligned} \frac{dx}{dt} &= f(t,\,x(t)) + B(x(t))\,z(t) \\ z(t) \in K \quad \& \quad & \langle \widetilde{z} - z(t),\, G(x(t)) \rangle \geq 0 \qquad \text{for all } \widetilde{z} \in K, \end{aligned}$$

with $\nabla G(x)\,B(x)$ to be positive definite. We also may need K to be a *cone*.

For index two DVI's we need even more structure, such as for (frictionless) mechanical impact problems:

$$\begin{aligned} \frac{dq}{dt} &= v \qquad \text{or } G(q)v, \\ M(q)\frac{dv}{dt} &= \psi(t,\,q,\,v) + \nabla g(q)^T \lambda \\ 0 \leq \lambda &\perp g(q) \geq 0. \end{aligned}$$

Here $q =$ position, $v =$ velocity, $M =$ mass matrix, $\lambda =$ contact forces.

Basic issues

- **Existence:** We need the functions that make up the problem to be smooth (at least Lipschitz). The VI/CP used should also be nicely behaved.
  - **Index zero:** Strongly monotone $G(x, \cdot)$; Lipschitz $f(t, \cdot, \cdot)$.
  - **Index one:** Strongly monotone $\nabla G(x)\, B(x)$ for all x; f, ∇G and B Lipschitz.
  - **Index two:** $M(q)$ positive definite; $\nabla g(q)$ full rank; ψ, ∇g, M Lipschitz.
- **Uniqueness:**
  - **Index zero:** Comes for free.
  - **Index one:** Requires $\nabla G(x)\, B(x)$ to be *symmetric* in addition.
  - **Index two:** Can't get it.

Friends

- **Differential algebraic equations:**

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, z), \qquad x(t_0) = x_0, \\ 0 &= G(x(t), z(t)) \qquad \text{for all } t. \end{aligned}$$

These don't have a complementarity problem or variational inequality attached, but they give a certain reference point for understanding DVI's, such as the importance of the index.

- **Sweeping processes:** This allows solutions that are bounded variation, so that the derivative is a *measure*:

$$\frac{dx}{dt}(t) \quad \in \quad -N_{C(t)}(x(t)) + \psi(t, x(t)), \qquad x(t_0) = x_0.$$

The set $C(t)$ "sweeps" the solution $x(t)$. If $x(t) \in \operatorname{int} C(t)$ then $N_{C(t)}(x(t)) = \{0\}$, and $dx/dt(t) = \psi(t, x(t))$ which is an ordinary differential equation, until $x(t)$ hits the boundary of $C(t)$.

These were invented by J.-J. Moreau for understanding impact problems. Existence is shown by a time-stepping algorithm: $x^{k+1} = \operatorname{Proj}_{C(t_{k+1})}(x^k + h\,\psi(t_k, x^k))$ provided $\delta(C(t), C(s)) \le r(s) - r(t)$ for a function $r(\cdot)$ of bounded variation.

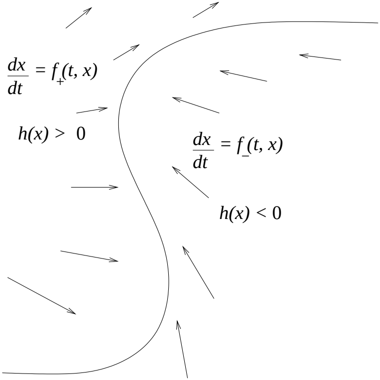
- **Differential inclusions:**

$$\begin{aligned} \frac{dx}{dt} &\in F(t,\, x) \\ &= \{\, f(t, x, z) \mid z \text{ solves } \mathrm{VI}(K,\ G(x, \cdot))\,\} \end{aligned}$$

Special case: $F(t,\, x) = -\Phi(x)$ where Φ *maximal monotone*:

$$\langle z_1 - z_2,\, x_1 - x_2 \rangle \geq 0 \qquad \text{for } z_i \in \Phi(x_i).$$

Existence can be shown in $F(t, \cdot)$ is bounded, closed, convex in $\mathbb{R}^n$ and its graph is a closed set, ***or*** $F(t, x) = -\Phi(x)$ and Φ is maximal monotone. The theory of differential inclusions starts with A.F. Filippov in the 1960's, and developed with Brezis in the 1970's, and others later. Uniqueness holds if $F(t, x) = -\Phi(x) + G(t, x)$ with $G(t, \cdot)$ Lipschitz.



Example of discontinuous ODE which can be treated as differential inclusion $dx/dt \in F(t, x)$ where

$$F(t,\, x) \;=\; \begin{cases} \{f_+(t,x)\}\,, & h(x) > 0, \\ \mathrm{co}\,\{f_+(t,x),\, f_-(t,x)\}\,, & h(x) = 0, \\ \{f_-(t,x)\}\,, & h(x) < 0. \end{cases}$$

- **Projected Dynamical Systems (PDS's):**

$$\frac{dx}{dt} \in f(t, x(t)) - N_K(x(t)), \qquad x(t_0) = x_0 \qquad \text{or}$$
$$\frac{dx}{dt} = \mathrm{Proj}_{T_K(x(t))}(f(t, x(t))), \qquad x(t_0) = x_0.$$

These are systems that project the right-hand side of the (ordinary) differential equation onto the set of feasible directions from $x(t)$:
$T_K(x) = \{ \lim_{k \to \infty} (x_k - x)/t_k \mid t_k \downarrow 0,\ x_k \in K \}$.

- **Convolution Complementarity Problems (CCP's):** the dynamics are given by a convolution rather than a differential equation:

$$\begin{aligned} x(t) &= x_0 + \int_0^t k(t-\tau)\, z(\tau)\, d\tau, \\ K^* \ni x(t) &\perp z(t) \in K \qquad \text{for all } t. \end{aligned}$$

These are useful for describing some PDE situations (e.g., Routh's rod). This has index zero if $k(t) = k_0\, \delta(t) + \cdots$, index one if $k(t)$ has a jump discontinuity at $t = 0$, index two if $k(t) = k_0\, t + \cdots$, etc. for small $t > 0$. This can be extended to fractional index problems with $k(t) = k_0\, t^{\alpha-1} + \cdots$ with $0 < \alpha < 2$.

- **Linear Complementarity Systems (LCS's):**

$$\begin{aligned} \frac{dx}{dt}(t) &= A\,x(t) + B\,z(t), \qquad x(t_0) = x_0, \\ w(t) &= C\,x(t) + D\,z(t) \qquad \text{for all } t, \\ 0 \le z(t) &\perp w(t) \ge 0 \qquad \text{for all } t. \end{aligned}$$

Theory has been developed for arbitrarily high index for this family of problems, but this is restricted to linear time-invariant cases with polyhedral cones.

How to prove it. . .

Existence for index one in finite dimensions:

Implicit time-stepping, perhaps with a slight modification to simplify the proof:

$$\begin{aligned} x^{k+1} &= x^k + h\left[f(x^k) + B(x^k)z^k\right] \\ z^k \in K \quad \& \quad & \left(\widetilde{z} - z^k\right)^T\left[G(x^k) + \nabla G(x^k)(x^{k+1} - x^k)\right] \geq 0 \\ & \text{for all } \widetilde{z} \in K. \end{aligned}$$

This gives an affine VI for z^k using matrix $\nabla G(x^k)\, B(x^k)$.

If $\nabla G(x)\, B(x)$ uniformly positive definite, we get a bound on z^k independent of $h > 0$ & step number provided $kh \le T$.

Numerical trajectories $x_h(\cdot)$ and $z_h(\cdot)$, and $x_h(\cdot)$ equicontinuous, as $h \to 0$; therefore in a suitable subsequence we have $x_h \to \widehat{x}$ uniformly, and $z_h \rightharpoonup \widehat{z}$ weak* in $L^\infty(0, T)$. Final step is to show that $\widehat{x}$ and $\widehat{z}$ are indeed solutions. To get complementarity in limit, use

$$\lim_{h\to 0} \int_0^T \left(\widetilde{z}(t) - z_h(t)\right)^T G(x_h(t))\, dt \;=\; \int_0^T \left(\widetilde{z}(t) - \widehat{z}(t)\right)^T G(\widehat{x}(t))\, dt.$$

Uniqueness for index one in finite dimensions:

This needs $\nabla G(x)\, B(x)$ *symmetric* as well as positive definite. This means that there is a positive definite and symmetric $Q(x)$ where

$$Q(x)\, B(x) \quad = \quad \nabla G(x)^T.$$

Then for two solutions (x_1, z_1) and (x_2, z_2) we have the error $e(t) = x_1(t) - x_2(t)$, $\zeta(t) = z_1(t) - z_2(t)$, and

$$\begin{aligned}
\frac{d}{dt}\left[e(t)^T Q(t) e(t)\right] &= 2\, e(t)^T Q(t) e'(t) \\
&= \mathcal{O}\left(\|e(t)\|^2\right) \qquad \text{using} \\
e(t)^T \nabla G(x_1(t))^T \zeta(t) &\approx (G(x_1(t)) - G(x_2(t)))^T (z_1(t) - z_2(t)) \le 0.
\end{aligned}$$

Some tools

Differentiation lemmas: Complementarity version: Suppose that

$$K \ni a(t) \quad \perp \quad b(t) \in K^* \qquad \text{for all } t,$$

with $a' \in L^p(0,T;X)$ and $b \in L^q(0,T;X')$ where $1/p + 1/q = 1$. Then

$$\langle a'(t), b(t) \rangle \;=\; 0 \qquad \text{for (almost) all } t.$$

This is just one of many lemmas of this kind: others conclude that $\langle a'(t), b'(t)\rangle \le 0$ or that $\langle a''(t), b(t)\rangle \ge 0$; others suppose that the Fourier transforms of a and b decay at the right rates. With all these "differentiation lemmas" no compactness/finite dimensionality is needed, and the results are as sharp as you can expect.

These results fail to apply for a and b having bounded variation (a' understood in distributional sense).

Index reduction: Approximate a DVI/CCP/... with a similar thing with lower index. This is like using a penalty method to approximate hard constraints.

Example; Schatzman and Petrov's approach to impact for a visco-elastic rod:

$$\begin{aligned} u_{tt} &= u_{xx} + \beta\, u_{txx}, \qquad t > 0, \quad 0 < x < L, \\ N(t) &= u_x(t,0) + \beta\, u_{tx}(t,0), \\ 0 &= u_x(t,L) + \beta\, u_{tx}(t,L), \\ 0 \leq u(t,0) &\perp N(t) \geq 0 \qquad \text{for all } t. \end{aligned}$$

This can be expressed as a CCP:

$$\begin{aligned} u(t,0) &= q(t) + \int_0^t m(t-\tau)\, N(\tau)\, d\tau \;=\; q(t) + (m * N)(t) \\ 0 \leq u(t,0) &\perp N(t) \geq 0. \end{aligned}$$

Problem: $m(t) \sim m_0\, t^{1/2}$ for $t > 0$ small. This is index $1\frac{1}{2}$.

Strategy to show existence: Put $m_\epsilon(t) = m(t) + \epsilon$. Then the CCP for m_ϵ is index one, so solutions $N_\epsilon(t)$ exist. We want bounds on N_ϵ as $\epsilon \to 0$. Fourier transforms can't do this directly, so we use the first differentiation lemma:

$$0 \;=\; \int_0^T N_\epsilon(t)\,\left[q'(t) + (m'_\epsilon * N_\epsilon)(t)\right]\,dt.$$

But $\mathcal{F}[m'_\epsilon](\omega) = \mathcal{F}[m'](\omega) + \epsilon = m_0\,(i\omega)^{-1/2} + \epsilon$ (principal argument) which has positive real part. This means that convolution with m' is a positive definite operator! This gives uniform bounds on N_ϵ in $H^{-1/4}(0,T)$; but $\psi \mapsto m * \psi$ maps $H^{-1/4}(0,T)$ to $H^{5/4}(0,T)$ with plenty of room to obtain compactness and show that limits satisfy the complementarity condition.

Elastic bodies in impact

Elastic bodies are governed by PDE's, so we have to deal with infinite-dimensional problems. Elasticity operator:

$$\begin{aligned} -A &= \lambda \operatorname{grad}\operatorname{div} + \mu\nabla^2 \\ \sigma[\mathbf{u}] &= \mathcal{A}\varepsilon[\mathbf{u}] + \mathcal{B}\varepsilon[\dot{\mathbf{u}}] \qquad \text{stress tensor,} \\ \varepsilon[\mathbf{u}] &= \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \qquad \text{strain tensor.} \end{aligned}$$

Frictionless contact conditions:

$$\begin{aligned} \sigma[\mathbf{u}](t,\mathbf{x})\,\mathbf{n}(\mathbf{x}) &= -N(t,\mathbf{x})\,\mathbf{n}(\mathbf{x}), \\ 0 \le N(t,\mathbf{x}) &\perp \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}(t,\mathbf{x}) \qquad \text{on } \Gamma_C. \end{aligned}$$

Abstract formulation

This is formally an index two DVI:

$$\begin{aligned} \mathbf{u}_{tt} &= -A\mathbf{u} - B\dot{\mathbf{u}} + \beta^* N(t) + \mathbf{f}(t) \\ K^* \ni N(t) &\perp \beta\mathbf{u}(t) - \varphi \in K. \end{aligned}$$

Here $\beta\mathbf{u} = \mathbf{u}\cdot\mathbf{n}$ restricted to the contact boundary $\Gamma_C \subset \partial\Omega$, and β^* is its adjoint. Assume A (and B) elliptic.

Existence of solutions is based on energy:

$$E[\mathbf{u}, \dot{\mathbf{u}}] \;=\; \frac{1}{2}\langle \dot{\mathbf{u}},\, \dot{\mathbf{u}} \rangle + \frac{1}{2}\langle \mathbf{u},\, A\mathbf{u} \rangle \,.$$

Discrete time trajectories can get uniform bound on energy $E[\mathbf{u}_h,\, \dot{\mathbf{u}}_h] \leq E_{bound}$. For B elliptic also get $\int_0^T \|\dot{\mathbf{u}}_h\|^2 \, dt \leq C$. This gives convergence of a subsequence solving the problem.

Uniqueness is an open problem.

Energy balance

Most papers on elastic impact problems only consider existence questions. . .

In rigid body dynamics we need to have a *coefficient of restitution* e describing the amount of "bounce" in the impacting object(s): $\mathbf{n}^T\mathbf{v}(t^+) = -e\,\mathbf{n}^T\mathbf{v}(t^-)$.

This is difficult to define for general elastic bodies: $\mathbf{v} \in L^2(\Omega)$ and restricting $\mathbf{v}$ to $\partial\Omega$ doesn't make sense. So we don't have a coefficient of restitution.

$$
\begin{aligned}
\frac{d}{dt}E[\mathbf{u}, \dot{\mathbf{u}}] &= \langle \ddot{\mathbf{u}}, \dot{\mathbf{u}} \rangle + \langle \dot{\mathbf{u}}, A\mathbf{u} \rangle \\
&= \langle -A\mathbf{u} - B\dot{\mathbf{u}} + \beta^* N(t) + \mathbf{f}(t), \dot{\mathbf{u}} \rangle + \langle \dot{\mathbf{u}}, A\mathbf{u} \rangle \\
&= -\langle \dot{\mathbf{u}}, B\dot{\mathbf{u}} \rangle + \langle \beta^* N(t), \dot{\mathbf{u}} \rangle + \langle \mathbf{f}(t), \dot{\mathbf{u}} \rangle
\end{aligned}
$$

Note:

$$
\begin{aligned}
\langle \beta^* N(t), \dot{\mathbf{u}} \rangle &= \langle N(t), \beta\dot{\mathbf{u}} \rangle \\
&= \left\langle N(t), \frac{d}{dt}\left[\beta\mathbf{u} - \varphi\right] \right\rangle, \qquad \text{but} \\
K^* \ni N(t) &\perp \beta\mathbf{u}(t) - \varphi \in K,
\end{aligned}
$$

So by a differentiation lemma provided we have sufficient regularity, we get

$$
\frac{d}{dt}E[\mathbf{u}, \dot{\mathbf{u}}] = -\langle \dot{\mathbf{u}}, B\dot{\mathbf{u}} \rangle + \langle \mathbf{f}(t), \dot{\mathbf{u}} \rangle .
$$