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MATH503 APPLIED MATHEMATICS

Answers to HW Set #1

8.2.15) Force $\vec{P} = \vec{i} + \vec{j} + \vec{k}$.

Initial point $A: (2,2,2)$. Final point $B: (4,0,2)$

Parametric representation of the straight line joining A to B:

$x = 2 + 2t \qquad 0 \le t \le 1$

$y = 2 - 2t$

$z = 2$

$\therefore$ Work done $= \int_0^1 \left(\vec{P} \cdot \frac{d\vec{r}}{dt}\right) dt = 0$

8.2.25) Angle btwn the planes

$P_1: x + y + z = 1$ and $P_2: x + 2y + 3z = 6$

Unit normal vectors

$\vec{n}_1 = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$, $\vec{n}_2 = \frac{1}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}$

$\therefore \cos\gamma = \vec{n}_1 \cdot \vec{n}_2 = \frac{1}{\sqrt{42}}(1+2+3) = \sqrt{\frac{6}{7}}$

gives the angle γ.

8.2.28) Vertices of a triangle are

$A: (2,1)$, $B: (4,-1)$, $C: (6,3)$

Position vectors

$\vec{r}_A = 2\vec{i} + \vec{j}$, $\vec{r}_B = 4\vec{i} - \vec{j}$ $\quad \vec{r}_C = 6\vec{i} + 3\vec{j}$

$\therefore \vec{r}_B - \vec{r}_A = 2\vec{i} - 2\vec{j}$, $\vec{r}_C - \vec{r}_A = 4\vec{i} + 2\vec{j}$, $\vec{r}_C - \vec{r}_B = 2\vec{i} + 4\vec{j}$

Then $\cos\gamma_{\hat{ABC}} = \frac{\vec{r}_A - \vec{r}_B}{|\vec{r}_A - \vec{r}_B|} \cdot \frac{\vec{r}_C - \vec{r}_B}{|\vec{r}_C - \vec{r}_B|} = \left(-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) \cdot \left(\frac{\vec{i}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\vec{j}\right) = \frac{1}{\sqrt{10}}$

etc.

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8.3.30) Given $A: (1,3,0)$, $B: (2,0,8)$, $C: (0,2,2)$

$\vec{r}_A = \vec{i} + 3\vec{j}$, $\vec{r}_B = 2\vec{i} + 8\vec{k}$, $\vec{r}_C = 2\vec{j} + 2\vec{k}$

$\therefore \vec{r}_B - \vec{r}_A = \vec{i} - 3\vec{j} + 8\vec{k}$, $\vec{r}_C - \vec{r}_A = -\vec{i} - \vec{j} + 2\vec{k}$

Normal vector to the plane passing through A, B, C:

$\vec{N} = (\vec{r}_B - \vec{r}_A) \times (\vec{r}_C - \vec{r}_A) = 2\vec{i} - 10\vec{j} - 4\vec{k}$

$|N| = \sqrt{4 + 100 + 16} = 2\sqrt{30}$

$\therefore \vec{n} = \frac{1}{\sqrt{30}}\vec{i} - \frac{5}{\sqrt{30}}\vec{j} - \frac{2}{\sqrt{30}}\vec{k}$

Equation of the plane: $2x - 10y - 4z = c$: const.

In particular $2 \cdot 1 - 10 \cdot 3 - 4 \cdot 0 = c$

$\therefore c = -28$

Finally, $\boxed{2x - 10y - 4z = -28}$

8.4.1-2-3) Pressure function $f(x,y) = 9x^2 + 4y^2$

1) $f(2,4) = 9 \cdot 4 + 4 \cdot 16 = 100$

$f\left(\frac{1}{2}, -\frac{13}{4}\right) = 9 \cdot \frac{1}{4} + 4 \cdot \frac{169}{16} = \frac{89}{2}$

$f\left(\sqrt{17}, \frac{1}{\sqrt{6}}\right) = 9 \cdot 17 + 4 \cdot \frac{1}{6} = \frac{461}{3}$

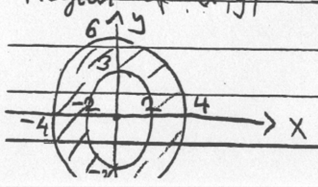
2) Isobars: $9x^2 + 4y^2 = c^2$: positive const.

$\Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \frac{c^2}{36}$

Equation of an ellipse at standard position.

3) Region of (x,y) s.t. $36 \le 9x^2 + 4y^2 \le 144$

or $1 \le \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 4$



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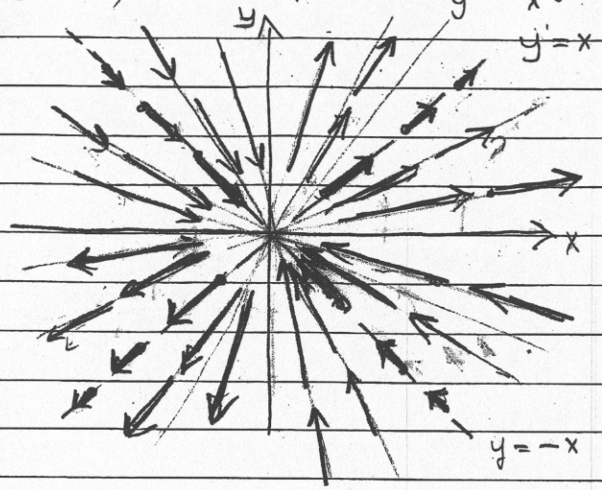
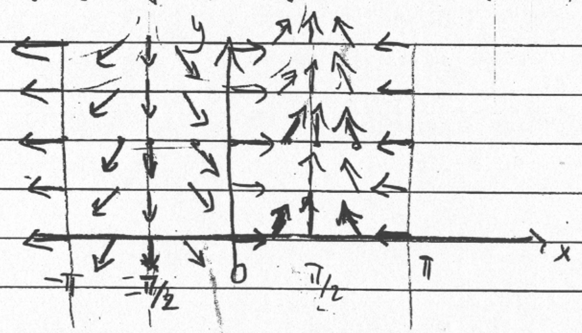
8.4.23) b) Vector field $\vec{V} = \frac{1}{y}\vec{i} + \frac{1}{x}\vec{j}$



e) Vector field $\vec{V} = \cos x\,\vec{i} + \sin x\,\vec{j}$



8.4.25) $\vec{F} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$

$\frac{\partial F_1}{\partial x} = 0$, $\frac{\partial F_1}{\partial y} = 2y$, $\frac{\partial F_1}{\partial z} = 0$

$\frac{\partial F_2}{\partial x} = 0$, $\frac{\partial F_2}{\partial y} = 0$, $\frac{\partial F_2}{\partial z} = 2z$

$\frac{\partial F_3}{\partial x} = 2x$, $\frac{\partial F_3}{\partial y} = 0$, $\frac{\partial F_3}{\partial z} = 0$

since $F_1 = y^2$, $F_2 = z^2$, $F_3 = x^2$

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8.5.5) $A : (2,3,0)$, $B : (5,-1,0)$

$\vec{r}_A = 2\vec{i} + 3\vec{j}$, $\vec{r}_B = 5\vec{i} - \vec{j}$

$\vec{r}_B - \vec{r}_A = 3\vec{i} - 4\vec{j}$

A parametric representation of the straight line from A to B:

$x = 2 + 3t$ $\quad 0 \le t \le 1$

$y = 3 - 4t$

8.5.14) $x = 3 + 6\cos t$, $y = -2 + \sin t$, $z = 4$ $\Rightarrow$ $\frac{(x-3)^2}{36} + (y+2)^2 = 1$

Equation of an ellipse in the $z=4$ plane (that is ∥ to xy-plane) with center at $O : (3,-2,4)$

8.5.15) Orientation is counter-clockwise

Orientation reversing transformation = send t to $-t$.

8.5.23) Curve C : $\vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$

Point P : $(\frac{1}{2}, \sqrt{3}, 0)$

Is on the curve P $(t = \pi/3)$

$\vec{r}'(t) = -\sin t\,\vec{i} + 2\cos t\,\vec{j}$

Tangent vector at P : $\vec{r}'(\pi/3) = -\frac{\sqrt{3}}{2}\vec{i} + \vec{j}$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + 4\cos^2 t} = \sqrt{1 + 3\cos^2 t}$

Unit tangent vector

$\vec{u}(t) = -\frac{\sin t}{\sqrt{1+3\cos^2 t}}\vec{i} + \frac{2\cos t}{\sqrt{1+3\cos^2 t}}\vec{j}$

at P : $\boxed{\vec{u}(\pi/3) = -\sqrt{\frac{3}{7}}\,\vec{i} + \frac{2}{\sqrt{7}}\vec{j}}$

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8.5.27 Catenary: $\vec{r}(t) = t\,\vec{i} + \cosh t\,\vec{j}$ $\quad t \in [0,1]$

Plot the graph of the curve $y = \cosh^{-1} x$

$\sinh t = \frac{e^t - e^{-t}}{2}$, $\cosh t = \frac{e^t + e^{-t}}{2}$ $\quad \therefore \cosh(0) = 1$

$\cosh(1) = \frac{1}{2}\left(e + \frac{1}{e}\right) \approx 1.5$

$\vec{r}\,'(t) = \vec{i} + \sinh t\,\vec{j}$

$\therefore \sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)} = \sqrt{1+\sinh^2 t} = \cosh t$

length of the curve $= \int_0^1 \cosh t \cdot dt = \sinh t \Big|_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right)$

8.5.32 $\rho^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$

plane polar coordinates

$\Rightarrow$ $x = \rho\cos\theta$

$y = \rho\sin\theta$

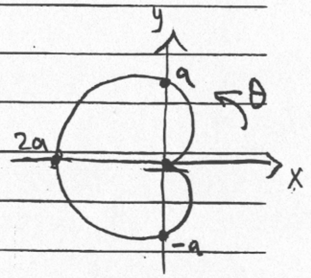
$\therefore$ $dx = \cos\theta\, d\rho - \rho\sin\theta\, d\theta$

$dy = \sin\theta\, d\rho + \rho\cos\theta\, d\theta$

$\Rightarrow$ $(dx)^2 + (dy)^2 = (d\rho)^2 + \rho^2 (d\theta)^2 = (ds)^2$.

$\therefore$ $l = \int_C ds = \int_C \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2}\, d\theta$

Cardioid: $\rho = a(1 - \cos\theta)$



$\frac{d\rho}{d\theta} = a\sin\theta$

$\therefore \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2} = 2a\sin\theta/2$

$\Rightarrow$ $l = 2\int_0^{\pi} 2a\sin\frac{\theta}{2}\, d\theta = 8a$

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8.6.10 Elliptical orbit $\quad \vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$

Velocity: $\vec{r}\,'(t) = -\sin t\,\vec{i} + 2\cos t\,\vec{j}$

$|\vec{r}\,'(t)|^2 = \sin^2 t + 4\cos^2 t \equiv s^2(t)$

Max. speed?

$\frac{ds^2}{dt} = 2\sin t\cos t - 8\cos t\sin t = -6\cos t\sin t$

$\frac{d^2s^2}{dt^2} = 6\sin^2 t - 6\cos^2 t = -6\cos 2t$

$\therefore$ $t = 0, \pi, \ldots$ are local minima

$t = \pi/2, \frac{3\pi}{2}, \ldots$ are local maxima

$\Downarrow$

max. speed attained at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Acceleration:

$\vec{r}\,''(t) = -\cos t\,\vec{i} - 2\sin t\,\vec{j}$

$|\vec{r}\,''(t)|^2 = \cos^2 t + 4\sin^2 t \equiv a^2(t)$

Max. acceleration? $\quad \frac{da^2}{dt} = -2\cos t\sin t + 8\cos t\sin t = 6\cos t\sin t$

$\frac{d^2a^2}{dt^2} = -6\sin^2 t + 6\cos^2 t = 6\cos 2t$

$\therefore$ $t = 0, \pi, \ldots$ are local maxima

$t = \pi/2, \frac{3\pi}{2}, \ldots$ are local minima

$\Downarrow$

max. acceleration attained at $t = 0, \pi, \ldots$

$\vec{a}_{tang} = \frac{(\vec{r}\,'\cdot\vec{r}\,'')}{(\vec{r}\,'\cdot\vec{r}\,')}\vec{r}\,' = \frac{5\sin t\cos t}{4\cos^2 t + \sin^2 t}\left(-\sin t\,\vec{i} + 2\cos t\,\vec{j}\right)$

$\vec{a}_{nor} = \vec{a} - \vec{a}_{tang}$

$= \frac{(-4\cos^2 t + 4\sin^2 t)}{4\cos^2 t + \sin^2 t}\cos t\,\vec{i} - \frac{(18\cos^2 t + 2\sin^2 t)}{4\cos^2 t + \sin^2 t}\sin t\,\vec{j}$

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8.6.16 Satellite orbit. $h = 450$ miles above ground with frequency $f = \frac{1}{100}$ rpm

Change units:

$$h = 450 \cdot \frac{8}{5} = 720 \text{ km.}$$

$$r_E = 3960 \text{ miles} = 6400 \text{ km.} \quad \text{(Earth's average radius)}$$

$$\omega = \frac{1}{100}\,\frac{2\pi}{60} = \frac{2\pi}{3000} \text{ rad/s} \quad \text{: angular frequency.}$$

$\therefore$ Gravitational acceleration

$$g = \omega^2 (r_E + h) \cong 7.78 \text{ m/s}^2.$$

8.9.10 Heat flow: $f(x,y) = \tan^{-1}\frac{y}{x}$ $\quad P: (3,4)$

$$\vec{\nabla} f = \frac{\partial}{\partial x} \tan^{-1}\frac{y}{x}\,\vec{i} + \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x}\,\vec{j}$$

$$= -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$$

Since $|\vec{\nabla} f| = \frac{1}{\sqrt{x^2+y^2}}$

$$\vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = -\frac{y}{\sqrt{x^2+y^2}}\vec{i} + \frac{x}{\sqrt{x^2+y^2}}\vec{j}$$

$\therefore$ Unit normal vector at P:

$$\vec{n}(P) = -\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$$

gives the direction of max. increase in temp.

$\Rightarrow$ Direction of max. decrease in temperature T will be given by

$$\vec{n} = \frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$$

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8.9.14 Elevation above the sea level on a mountain

$$z = 1500 - 3x^2 - 5y^2$$

$\therefore$ Equation of the mountain surface

$$f(x,y,z) = 3x^2 + 5y^2 + z - 1500 = 0$$

$$\Rightarrow \vec{\nabla} f = 6x\,\vec{i} + 10y\,\vec{j} + \vec{k}$$

At the point $P: (-\frac{1}{5}, 1)$

$$\vec{\nabla} f(P) = -\frac{6}{5}\vec{i} + 10\vec{j} + \vec{k}$$

and

$$|\vec{\nabla} f(P)| = \sqrt{\frac{36}{25} + 101}.$$

Direction of steepest ascent at P will be

$$\frac{\vec{\nabla} f(P)}{|\vec{\nabla} f(P)|}$$

8.10.10 Rotational flow of the vector field $\vec{v} = \vec{\omega} \times \vec{r}$

Since $v_1 = \omega_2 z - \omega_3 y$ and $\vec{\omega}$: const.

$$v_2 = \omega_3 x - \omega_1 z$$

$$v_3 = \omega_1 y - \omega_2 x$$

we have $\frac{\partial v_1}{\partial x} = 0 = \frac{\partial v_2}{\partial y} = \frac{\partial v_3}{\partial z}$ $\therefore$ $\boxed{\vec{\nabla}\cdot\vec{v} = 0}$

8.11.10 $\vec{v} = \sec x\,\vec{i} + \operatorname{cosec} x\,\vec{j}$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}\sec x + \frac{\partial}{\partial y}\operatorname{cosec} x = \tan x \sec x \neq 0$$

irrotational

$$\vec{\nabla}\times\vec{v} = -\cot x \operatorname{cosec} x\,\vec{k} \neq 0$$

Incompressible

Path of particles in the flow:

$\frac{d\vec{r}}{dt} = \vec{v}$ $\therefore$ $\frac{dx}{dt} = \sec x \Rightarrow x = \sin^{-1} t$

$\frac{dy}{dt} = \operatorname{cosec} x$ $\quad y = \ln t$

i.e. ... [illegible]

$x = \sin y$ ... $= 0$