MATH 503 APPLIED MATHEMATICS

Answers to HW Set #3

6.2.10 a) Suppose A is $(m\times n)$-matrix.
Then A^T is $(n\times m)$-matrix and $B = AA^T$ is a square $(m\times m)$-matrix.

Let $b_{ij} = \sum_{k=1}^{m} a_{ik}\, a_{kj}^T = \sum_{k=1}^{m} a_{ik} a_{jk}$

$= \sum_{k=1}^{m} a_{jk} a_{ik} = \sum_{k=1}^{m} a_{jk} a_{ki}^T = b_{ji} = b_{ij}^T$

$\therefore\ B^T = B$ i.e. B is a symmetric matrix.

Let $A^T = A$ and $B^T = B$.
Then $(AB)^T = B^T A^T$ by definition
$= BA$ by assumption
$= AB$ provided $BA - AB = 0$

b) A is idempotent if $A^2 = A$
B is nilpotent if $B^k = 0$ for some integer $k > 0$.

Example:

$$A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \quad \text{s.t. } A^2 = A$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{s.t. } B^2 = 0$$

$$C = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{s.t. } C^2 = I$$

c) If U_1, U_2 are upper triangular matrices, then U_1+U_2, U_1U_2, U_1^2, U_2^2 are also upper triangular.
If L_1, L_2 are lower triangular matrices, then L_1+L_2, L_1L_2, L_1^2, L_2^2 are also lower triangular.
U_1L_1, U_1+L_1 are not triangular.

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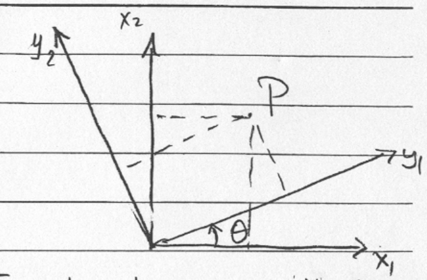
6.2.18

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow$ $y_1 = \cos\theta\, x_1 - \sin\theta\, x_2$
$y_2 = \sin\theta\, x_1 + \cos\theta\, x_2$
coordinate transformation



From the diagram, write down the coordinates of a point P.

b) Write

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Then

$$A^2 = (\cos^2\theta - \sin^2\theta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\,2\cos\theta\sin\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

By induction it can be shown that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \quad \text{for } n \geq 2.$$

c) Similarly

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \left(\cos\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\alpha \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) \cdot \left(\cos\beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$$

$$= (\cos\alpha\cdot\cos\beta - \sin\alpha\sin\beta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

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Elementary row operations:

6.3.24)

$$E_1X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \\ w \end{pmatrix}$$: 2^{nd} and 3^{rd} rows are exchanged

$$E_2X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ -5x+z \\ w \end{pmatrix}$$: (-5) times the 1^{st} row is added to the 3^{rd} row

$$E_3X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 8w \end{pmatrix}.$$: 4th row is multiplied by 8.

$$B = E_1E_2E_3A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} x & p \\ y & q \\ z & r \\ w & s \end{pmatrix}$$

$$= \begin{pmatrix} x & p \\ -5x+z & -5p+r \\ y & q \\ 8w & 8s \end{pmatrix}$$

$$C = E_3E_2E_1A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x & p \\ y & q \\ z & r \\ w & s \end{pmatrix}$$

$$= \begin{pmatrix} x & P \\ z & r \\ -5x+y & -5p+q \\ z & s \end{pmatrix}$$

$\therefore$ $B \neq C$.

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6.6.20) Find the equation of a Line L : $ax+by+c=0$ that passes through points $P_1:(x_1,y_1)$ and $P_2:(x_2,y_2)$.

P_1 is on L means $ax_1+by_1+c=0$

P_2 is on L means $ax_2+by_2+c=0$

Thus we obtain 3 equations in 3 unknowns a,b,c:

$$ax+by+c=0$$
$$ax_1+by_1+c=0$$
$$ax_2+by_2+c=0$$

A non-trivial solution exists if the determinant of the coefficient matrix vanishes:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(y_1-y_2) - y(x_1-x_2) + (x_1y_2 - x_2y_1) = 0$$

$-x_1y_1$ (add) $\qquad +y_1x_1$ (subtract)

$$\Rightarrow \quad (x-x_1)(y_1-y_2) - (y-y_1)(x_1-x_2) = 0$$

$$\therefore \quad \boxed{\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}}$$

This is the equation of L that passes through P_1 and P_2

b) Equation of a plane that passes through P_1, P_2, P_3:

$$ax+by+cz+d=0$$

Write
$$ax_1+by_1+cz_1+d=0$$
$$ax_2+by_2+cz_2+d=0$$
$$ax_3+by_3+cz_3+d=0$$

(cont.)

construct the determinant of the coefficient matrix

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Example: If $P_1:(1,1,1)$, $P_2:(3,2,6)$, $P_3:(5,0,5)$

then $\boxed{3x+4y-2z=5}$.

c) Equation of a circle in the $(x\text{-}y)$-plane that passes from $P_1:(x_1,y_1)$, $P_2:(x_2,y_2)$, $P_3:(x_3,y_3)$

$$(x-a)^2+(y-b)^2=R^2$$

or explicitly

$$x^2-2ax+a^2+y^2-2by+b^2-R^2=0$$

Write also

$$x_1^2-2ax_1+a^2+y_1^2-2by_1+b^2-R^2=0$$
$$x_2^2-2ax_2+a^2+y_2^2-2by_2+b^2-R^2=0$$
$$x_3^2-2ax_3+a^2+y_3^2-2by_3+b^2-R^2=0$$

In matrix form

$$\begin{pmatrix} x & y & x^2+y^2 & 1 \\ x_1 & y_1 & x_1^2+y_1^2 & 1 \\ x_2 & y_2 & x_2^2+y_2^2 & 1 \\ x_3 & y_3 & x_3^2+y_3^2 & 1 \end{pmatrix} \begin{pmatrix} -2a \\ -2b \\ 1 \\ a^2+b^2-R^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

A nontrivial solution exists if the determinant of the coefficient matrix vanishes.

Example: Suppose $P_1:(2,6)$, $P_2:(6,4)$, $P_3:(7,1)$

Then $\boxed{(x-2)^2+(y-1)^2=25}$

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(cont.)

d) Equation of a sphere that passes through four points:

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$

is worked out similary.

Example: If $P_1:(0,0,5)$, $P_2:(4,0,1)$, $P_3:(0,4,1)$, $P_4:(0,0,-3)$, then $\boxed{x^2+y^2+(z-1)^2=16}$

6.8.30) Invert the coordinate transformation

$$y_1=x_1$$
$$y_2=x_2\cos\theta+x_3\sin\theta$$
$$y_3=-x_2\sin\theta+x_3\cos\theta$$

In matrix form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or write $Y=AX$

so that $X=A^{-1}Y$. We have

$$A^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and explicitly

$$x_1=y_1$$
$$x_2=\cos\theta\, y_2-\sin\theta\, y_3$$
$$x_3=\sin\theta\, y_2+\cos\theta\, y_3.$$

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7.1.10) Eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -10 & 10 & -15 \\ 10 & 5 & -30 \\ -5 & -10 & 0 \end{pmatrix}$$

1) $\lambda = -15$ with multiplicity 2

with corresponding eigen vectors

$$\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

2) $\lambda = 25$ with multiplicity 1

with eigenvector

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

7.2.5) Find the principal directions and corresponding factors of extension/contraction of the elastic deformation $Y = AX$ with

$$A = \begin{pmatrix} 3/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix}$$

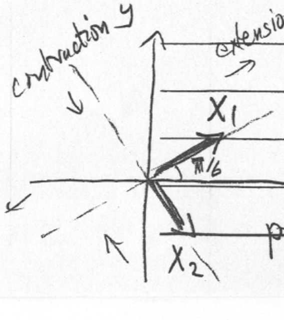
Eigenvalues are

$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$

"extension"

$\lambda_2 = 1/2$ with eigenvector $\begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$

"contraction"



principal directions

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7.2.15) Given $A = \begin{pmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{pmatrix}$ and $Y = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.1 \end{pmatrix}$,

find X such that $X - AX = Y$.

That is $(I - A)X = Y \Rightarrow X = (I-A)^{-1}Y$

We have $I - A = \begin{pmatrix} 0.9 & -0.4 & -0.2 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.4 & 0.6 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 9 & -4 & -2 \\ -5 & 10 & -1 \\ -1 & -4 & 6 \end{pmatrix}$

and

$$(I-A)^{-1} = \begin{pmatrix} 7/4 & 1 & 3/4 \\ 31/32 & 13/8 & 19/32 \\ 30/32 & 40/32 & 70/32 \end{pmatrix}$$

Therefore

$$Y = \begin{pmatrix} 176/320 \\ 89/320 \\ 220/320 \end{pmatrix}$$

7.5.4) Given $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}$

we have $\mathrm{Det}P = 1$ and $P^{-1} = \begin{pmatrix} -7 & 5 \\ -10 & 7 \end{pmatrix}$.

Then $P^{-1}AP = \begin{pmatrix} -29 & 20 \\ -42 & 29 \end{pmatrix} = \tilde{A}$.

Eigenvalues of A : $\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

Eigenvectors of A : $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eigenvalues of $\tilde{A}$: $\begin{vmatrix} -29-\lambda & 20 \\ -42 & 29-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda - 29)(\lambda + 29) - 840 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

(cont.)

Let $\tilde{X}_1 = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

and calculate

$$\tilde{A}\tilde{X}_1 = \begin{pmatrix} -29 & 20 \\ -42 & 29 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Similarly let $\tilde{X}_2 = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \end{pmatrix}$

and calculate

$$\tilde{A}\tilde{X}_2 = \begin{pmatrix} -29 & 20 \\ -42 & 29 \end{pmatrix}\begin{pmatrix} -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} = -\begin{pmatrix} -5 \\ -7 \end{pmatrix}$$

7.5.8

$$A = \begin{pmatrix} 4 & 1+i \\ 1-i & 4 \end{pmatrix}$$

Eigenvalues of A : $\begin{vmatrix} 4-\lambda & 1+i \\ 1-i & 4-\lambda \end{vmatrix} = 0$

Thus $(4-\lambda)^2 - 2 = 0$ or $\lambda_1 = 4+\sqrt{2}$, $\lambda_2 = 4-\sqrt{2}$.

Eigenvectors of A :

$$\begin{pmatrix} 4 & 1+i \\ 1-i & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (4+\sqrt{2})\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \Rightarrow y_1 = \frac{1-i}{\sqrt{2}}x_1$$

$$\therefore X_1 = \begin{pmatrix} 1/\sqrt{2} \\ \frac{1-i}{2} \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1+i \\ 1-i & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (4-\sqrt{2})\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow -\frac{1+i}{\sqrt{2}}y_2 = x_2$$

$$\therefore X_2 = \begin{pmatrix} -\frac{1+i}{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\{X_1, X_2\}$ form a unitary system.

That is, $X_1^+X_1 = 1 = X_2^+X_2$ and $X_1^+X_2 = X_2^+X_1 = 0$

Finally,

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1+i}{2} \\ \frac{1-i}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ is a unitary matrix.

7.5.12 Diagonalize the matrix $A = \begin{pmatrix} -43 & 77 \\ 13 & 93 \end{pmatrix}$

Eigenvalues of A are $\lambda_1 = 100$, $\lambda_2 = -50$ with normalized eigenvectors

$$X_1 = \frac{1}{\sqrt{218}}\begin{pmatrix} 7 \\ 13 \end{pmatrix}, \quad X_2 = \frac{1}{\sqrt{122}}\begin{pmatrix} 11 \\ -1 \end{pmatrix}$$

$\therefore$ Similarity matrix $P = \begin{pmatrix} 7/\sqrt{218} & 11/\sqrt{122} \\ 13/\sqrt{218} & -1/\sqrt{122} \end{pmatrix}$

$$\Rightarrow P^{-1} = \frac{1}{150}\begin{pmatrix} \sqrt{218} & 11\sqrt{218} \\ 13\sqrt{122} & -7\sqrt{122} \end{pmatrix}$$

One can then check that

$$P^{-1}AP = \begin{pmatrix} 100 & 0 \\ 0 & -50 \end{pmatrix}$$