Symplectic structures on product four-manifolds

by

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Abstract

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This thesis concerns existence of symplectic structures on product four–manifolds. If a three–manifold M fibers over the circle S^1, then the product $S^1 \times M$ admits a symplectic structure. Conversely, we have the following conjecture.

Taubes Conjecture. *Let M be a closed, oriented three–manifold. If $S^1 \times M$ admits a symplectic structure, then M fibers over the circle.*

In this thesis, this conjecture is proved in three special cases, namely when $S^1 \times M$ is a four–manifold which admits a Lefschetz fibration, when it is a complex surface and when it is a Seifert fibered four–manifold (assuming the three-manifold in question has no fake 3-cells). As a byproduct, some interesting results on the relationship between the geometrization of three–manifolds and the symplectic topology of four–manifolds are also obtained.

W. Chen and R. Matveyev classified all symplectic Lefschetz fibrations on $S^1 \times M$ and proved that M fibers over the circle if $S^1 \times M$ admits a symplectic Lefschetz fibration and there is no fake 3–cell in M. First of all, this result is generalized to include the manifolds which admit Lefschetz fibrations that are not compatible with any symplectic structure. Then by using the Enrique-Kodaira classification of compact complex surfaces and the Seiberg-Witten invariants of non-Kähler complex surfaces the Taubes conjecture is proved when $S^1 \times M$ admits a complex structure (not necessarily compatible with any symplectic structure). M. Ue generalized the concept of Seifert fibration to define Seifert fibered 4–manifolds which have nice geometric properties similar to those of their 3–dimensional counterparts. By using these geometric properties, the conjecture is proved for product four–manifolds that admit Seifert fibrations.

Most of the three-manifolds which fiber over the circle admit geometric structures (in the sense of W. Thurston). It is also known that the existence of a symplectic structure on $S^1 \times M$ implies that M is the connected sum of a prime manifold and a homology 3–sphere (which is S^3 assuming the geometrization conjecture of Thurston). In the last part of the thesis, the relationship between the geometry of a compact three–manifold M and the symplectic topology of $S^1 \times M$ is discussed. Most of the geometric nonhyperbolic three-manifolds are Seifert fibered and as a consequence Taubes' conjecture holds for all geometric, nonhyperbolic 3-manifolds. An essential part of this discussion is the fact that the product of a Seifert fibered 3–space with the circle admits a complex structure. A proof to a partial converse of this well-known fact is also obtained in the thesis.

Professor Robion C. Kirby
Dissertation Committee Chair

To my dearest wife,

Pelin Özaltan Etgü

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Chapter 1

Introduction

This thesis concerns existence of symplectic structures on product four–manifolds. A closed, oriented, smooth 4–manifold X which fibers over a Riemann surface admits a symplectic structure unless the fiber class is torsion in $H_2(X;\mathbb{Z})$. In particular, a fibration of a closed, oriented 3–manifold M over S^1 induces a symplectic form on $S^1 \times M$.

Conjecture T. *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. Then M fibers over S^1.*

This conjecture was first stated by C. Taubes [33] and is still open. Recent work of W. Chen and R. Matveyev [4] proves that it holds when M has no fake 3–cells, $S^1 \times M$ admits a symplectic structure and a Lefschetz fibration with symplectic fibers.

In the first part of this thesis, we generalize Chen and Matveyev's result proving that the conjecture holds when $S^1 \times M$ admits an arbitrary Lefschetz fibration (possibly with nonsymplectic fibers). More generally, we prove the following

Theorem 1.0.1. *Suppose M is a closed 3–manifold without a fake 3–cell.*

(L) If $S^1 \times M$ admits a Lefschetz fibration, then Conjecture T holds.

(S) If $S^1 \times M$ admits a Seifert fibration, then Conjecture T holds.

(K) If $S^1 \times M$ admits a Kähler structure, then Conjecture T holds.

(C) If $S^1 \times M$ admits a complex structure, then Conjecture T holds.

Here, a fake 3–cell means a compact, contractible 3–manifold which is not homeomorphic to D^3. Note that the Poincaré conjecture implies that there is no fake 3–cell.

Remark 1.0.2. We'll see that a nonsymplectic Lefschetz fibration on a product 4–manifold has no singular fibers and its fibers are tori. Since a Seifert fibration can be thought of as a T^2–fibration with multiple fibers, (S) is a further generalization of (L). Statement (C) is clearly a generalization of (K). Note that all (symplectic) product 4–manifolds which admit complex structures turn out to be Seifert fibered. This means that all other statements follow from (S) using the result of Chen and Matveyev on symplectic Lefschetz fibrations.

In the remark above and the rest of the thesis, by a product 4–manifold we mean the product of S^1 with a (compact, oriented, connected) 3–manifold.

In order to prove (L), we first generalize another result of Chen and Matveyev and show that any Lefschetz fibration on a product 4–manifold is a locally trivial fiber bundle. we also use the Seiberg-Witten theory of circle bundles over surfaces to analyze different cases.

The Enriques-Kodaira classification of compact complex surfaces is used to handle symplectic 4-manifolds that admit complex structures which may or may not be compatible with any symplectic form on them. Seiberg-Witten invariants of non-Kähler complex surfaces give an interesting obstruction which is also helpful in the proof of (K) and (C).

Remark 1.0.3. In their paper [11] on taut contact circles on 3–manifolds, Geiges and Gonzalo classified product 4–manifolds carrying complex structures with respect to which the obvious circle action is holomorphic. Since we don't require this action to be holomorphic and we are mainly interested in the symplectic structure on product manifolds, we prove different type of results even though we use similar methods.

Research on Seifert fibered spaces has been an important part of 3–dimensional topology and geometry. In [41], M. Ue generalized the concept of Seifert fibration to define Seifert fibered 4–manifolds which have nice geometric properties similar to those of their 3–dimensional counterparts. These manifolds admit torus fibrations over surfaces with finitely many multiply covered fibers. Smooth elliptic surfaces with vanishing Euler characteristics are all Seifert fibered. We prove (S) by using geometric properties of Seifert fibered 4–manifolds.

In the second part, we discuss the relationship between the geometry of a compact 3–manifold M and the symplectic topology of $S^1 \times M$. A manifold is called geometric if it admits a locally homogeneous metric, i.e. a metric for which one can find a local isometry that carries any point of the underlying manifold to a preassigned point. Thurston showed that there are only eight 3–dimensional geometries, i.e. simply-connected and geometric 3–manifolds which have compact quotients, and six of these geometries correspond to Seifert fibered spaces. He also conjectured that one can cut any compact 3–manifold along finitely many spheres and tori so that all the resulting pieces are geometric. As a result of his work towards proving this conjecture, we now know that most 3–manifolds which fiber over the circle are in fact geometric [40].

The following theorem shows the relevance of geometric 3–manifolds to the study of Taubes conjecture.

Theorem 1.0.4. *Suppose that M is a closed, geometric nonhyperbolic 3–manifold. If $S^1 \times M$ admits a symplectic structure, then M fibers over the circle.*

An essential part of the proof of this theorem is the fact that the product of a Seifert fibered space with the circle admits a complex structure. A proof to the following partial converse of this well-known result is also provided.

Theorem 1.0.5. *If $S^1 \times M$ admits a complex structure and M is sufficiently large, then M is a Seifert fibered space.*

Remark 1.0.6. Theorem 1.0.4 says that when M is a nonhyperbolic geometric 3–manifold Conjecture T holds. On the other hand, if $S^1 \times M$ admits a symplectic structure, then M is prime (assuming Thurston's conjecture on the geometrization of 3–manifolds) [21] , [43]. So it might be interesting to try to prove Conjecture T (at least up to the geometrization conjecture) by first proving it when M is hyperbolic, then considering geometric 3–manifolds with boundary (disjoint union of tori) and finally using Seiberg–Witten theory of 4–manifolds glued along T^3.

The organization of the thesis is as follows. In the next chapter we recall definitions and some basic theorems on Lefschetz fibrations, complex surfaces, Seiberg–Witten invariants, Seifert fibrations and geometric structures on 3 and 4–manifolds. In Chapter 3, we discuss nonsymplectic Lefschetz fibrations on $S^1 \times M$. By using the Seiberg–Witten theory of symplectic 4–manifolds and S^1–bundles over surfaces, we prove (L) of Theorem 1.0.1 . In Chapter 4, product 4–manifolds which admit complex structures are considered and (K) is proved first. As a result of a slightly more careful investigation we prove (C). Finally, we consider Seifert fibered 4–manifolds and prove (S). In the last chapter, we discuss the relation between the geometry of M and the symplectic structures on $S^1 \times M$.

In this thesis, by a fiber bundle we mean a locally trivial one and an F–bundle means a (locally trivial) fiber bundle with fiber F. All fibrations (of any kind) are oriented and all manifolds are compact, smooth, oriented and connected, unless clearly stated otherwise.

Chapter 2

Background

2.1 Symplectic manifolds

A symplectic form on a 2n–dimensional manifold X is a closed 2–form ω which is also non-degenerate, i.e. the n-fold wedge product $\omega \wedge \omega \cdots \wedge \omega$ never vanishes. In particular, the existence of a symplectic form implies that X is orientable. In case X is oriented, we insist that the given orientation coincides with the one that comes from the symplectic form. If there exists a symplectic form on a manifold X we say that X *admits a symplectic structure*, or simply X *is symplectic*.

There is a natural symplectic form on $\mathbb{R}^{2n}$ given by $\omega_0 = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$. In fact, any symplectic manifold is locally given by this form, i.e. there is a local coordinate chart where the symplectic form is the same as ω_0.

The volume form of a closed, connected, oriented surface Σ is obviously a symplectic form, and moreover, the sum of the pull-backs of the volume forms of Σ and Σ' is a symplectic form on $\Sigma \times \Sigma'$. On the other hand, product four–manifolds $S^1 \times M^3$ are not all symplectic. For example, $S^1 \times S^3$ has no symplectic form on it since $b_+(S^1 \times S^3) = 0$ and a symplectic form would give a positive definite subspace in $H^2(S^1 \times S^3; \mathbb{R})$.

[22] is an excellent reference for the symplectic topology of manifolds.

2.2 Topology of product four–manifolds

The following lemma summarizes the topological properties of $S^1 \times M$.

Lemma 2.2.1. *Let M be a closed, oriented and connected 3–manifold. Then $X = S^1 \times M$ is a spin 4-manifold with $\sigma(X) = \chi(X) = 0$, $b_\pm(X) = b_1(M)$ (in particular, $b_2(X)$ is even), where σ, χ and b_* denote the signature, Euler characteristic and the corresponding Betti number, respectively.*

Proof. Both S^1 and M are spin, so X is spin. Since $\chi(S^1) = 0$, the Euler characteristic of X vanishes. The boundary of $D^2 \times M$ is X, so $\sigma(X) = 0$. The facts about the Betti numbers follow easily from the definitions of σ and χ in terms of Betti numbers. ☐

2.3 Lefschetz fibrations and pencils

Definition 2.3.1. Let X be a compact, connected, oriented and smooth 4–manifold. A *Lefschetz fibration* on X is a smooth map $\pi : X \longrightarrow \Sigma$, where Σ is a compact, connected, oriented surface and $\pi^{-1}(\partial\Sigma) = \partial X$, such that each critical point of π lies in the interior of X and has an orientation-preserving (complex-valued) coordinate chart on which $\pi(z_1, z_2) = z_1^2 + z_2^2$ relative to a suitable smooth (complex-valued) chart on Σ.

Definition 2.3.2. A *Lefschetz pencil* on a closed, connected, oriented, smooth 4–manifold X is a non-empty finite subset B of X called the base locus, together with a smooth map $\pi : X - B \longrightarrow \mathbb{C}P^1$ such that each point $b \in B$ has an orientation-preserving coordinate chart in which π is given by the projectivization $\mathbb{C}^2 - \{0\} \longrightarrow \mathbb{C}P^1$, and each critical point has a local coordinate chart as in the definition of a Lefschetz fibration above.

Definition 2.3.3. A Lefschetz fibration is called *relatively minimal* if no fiber contains an exceptional sphere, in other words it cannot be obtained by blowing up another Lefschetz fibration.

Remark 2.3.4. Here by blowing up we mean the smooth generalization of the complex blow-up process. For a smooth, oriented 4–manifold X, the connected sum $X\#\overline{\mathbb{C}P^2}$ is called the *blow-up of* X. The sphere $\overline{\mathbb{C}P^1}$ in the $\overline{\mathbb{C}P^2}$ summand is called *the exceptional sphere.* The blow-up process can also be defined in the symplectic category, because for a symplectic 4–manifold X, $X\#\overline{\mathbb{C}P^2}$ is also symplectic.

Definition 2.3.5. A Lefschetz fibration is called a *symplectic Lefschetz fibration* if the total space admits a symplectic structure such that generic fibers are symplectic submanifolds, otherwise it is called a *nonsymplectic Lefschetz fibration.*

Theorem 2.3.6 (Gompf). *A Lefschetz fibration on a 4–manifold X is symplectic if and only if the homology class of the fiber is not torsion in $H_2(X;\mathbb{Z})$.*

The close relation between Lefschetz fibrations and symplectic structures is stated in the following theorems.

Theorem 2.3.7 (Donaldson [5]). *Every symplectic 4–manifold admits a Lefschetz pencil by symplectic surfaces.*

Theorem 2.3.8 (Gompf [12]). *If a 4–manifold admits a Lefschetz pencil (with non-empty base locus), then it admits a symplectic structure.*

It is necessary that the base locus is non-empty as we have examples of 4–manifolds, e.g. $S^1 \times S^3$, which admit Lefschetz fibrations over S^2 but no symplectic structure.

If a manifold admits a Lefschetz pencil, then one can blow-up the points of the base locus and construct a Lefschetz fibration (over S^2). So Donaldson's theorem implies that every symplectic 4–manifold has a blow-up which admits a Lefschetz fibration. Even though it is always possible to put a Lefschetz pencil on a symplectic $S^1 \times M$ it may not be possible to find a Lefschetz fibration on it. Note that a blow-up of $S^1 \times M$ can never be a product for topological reasons.

For more details on Lefschetz pencils and fibrations see [12].

2.4 Seiberg–Witten invariants

Let X be a closed, oriented, connected, homology oriented Riemannian 4–manifold with $b_+(X) > 0$. The Seiberg–Witten invariant SW of a $Spin_c$ structure on X was first extracted from monopole equations by Witten in [46]. If $b_+(X) > 1$, then SW is an integer-valued diffeomorphism invariant of X. When $b_+(X) = 1$ it may depend on the chosen metric. The Seiberg–Witten invariant of a $Spin_c$ structure ξ on X is denoted by $SW_X(\xi)$. We call $\alpha \in H^2(X;\mathbb{Z})$ a basic class if there exists a $Spin_c$ structure ξ such that $SW_X(\xi) \neq 0$ with $c_1(\det(\xi)) = \alpha$, where $\det(\xi)$ denotes the determinant (complex) line bundle of ξ. If there is no 2–torsion in $H^2(X;\mathbb{Z})$, then there is a unique $Spin_c$ structure ξ with $c_1(\det(\xi)) = \alpha$ for any characteristic class $\alpha \in H^2(X;\mathbb{Z})$. In general, the set of isomorphism classes of $Spin_c$ structures on X is an affine space modelled on $H^2(X;\mathbb{Z})$.

Seiberg–Witten invariants of 3–dimensional manifolds are defined similarly. As we state in Theorem 3.2.4, Seiberg–Witten invariants of a 3–manifold M carry exactly the same information as those of $S^1 \times M$ at least when $b_1(M) > 1$. The reader is referred to [18] and [28] for the theory of Seiberg–Witten invariants in dimension 3.

We have the following important theorem on the Seiberg–Witten invariants of symplectic manifolds.

Theorem 2.4.1 (Taubes [31], [32]). *Let X be a closed 4–manifold with $b_+(X) > 1$ and a symplectic form ω. Then there is a canonical $Spin_c$ structure ξ on X such that $SW_X(\xi) = \pm 1$ and $\det(\xi)$ is the canonical line bundle K of (X, ω).*

Moreover,

$$0 \leq |\alpha \cdot [\omega]| \leq |c_1(K) \cdot [\omega]| \quad ,$$

where α is any basic class; $0 = \alpha \cdot [\omega]$ if and only if $\alpha = 0$; $|\alpha \cdot [\omega]| = |c_1(K) \cdot [\omega]|$ if and only if $\alpha = \pm c_1(K)$.

See [12], [24] and [18] for more details on Seiberg-Witten invariants of 4-manifolds.

2.5 Geometric structures on manifolds

Definition 2.5.1. A metric on a manifold is called *locally homogeneous* if any pair of points can be mapped to each other by isometries of open neighborhoods.

Definition 2.5.2. A manifold is called *geometric* if it admits a complete, locally homogeneous metric.

Definition 2.5.3. A simply connected geometric manifold together with the isometry group corresponding to a complete (locally) homogeneous metric is called a *geometry.*

Up to isometry, there are eight 3–dimensional and nineteen 4–dimensional geometries with compact quotients. These are classified by Thurston and Filipkiewicz [7] respectively. See [30] and [44] for detailed discussions on 3 and 4–dimensional geometries.

A manifold is called *prime* if it cannot be written as the connected sum of two manifolds none of which is a sphere. In [23] Milnor showed that, up to homeomorphism and the permutation of the summands, there is a unique way to write a compact, oriented 3–manifold as the connected sum of prime manifolds. There is also a reasonably canonical way to cut compact, prime 3–manifolds along tori into pieces which no longer have embedded tori in them other than their boundary components (up to homology). Thurston's geometrization conjecture asserts that these pieces should all be geometric.

2.6 Seifert fibered manifolds

A *trivial fibered solid torus* is $S^1 \times D^2$ with the product foliation by circles. A *fibered solid torus* is a solid torus with a foliation by circles that is finitely covered by a trivial fibered solid torus. It can be constructed by gluing two ends $D^2 \times \{0\}$ and $D^2 \times \{1\}$ of $D^2 \times I$ after a q/p rotation.

A *Seifert fibered space* is a 3–manifold with a decomposition into disjoint circles such that each circle has a neighborhood isomorphic to a fibered solid torus. A circle bundle over a surface is naturally a Seifert fibered space. By identifying each of these circles with a point, we can consider a Seifert fibered space as a fibration over a 2–orbifold base. Such a fibration is called a *Seifert fibration.* Fibers of a Seifert fibration are obviously circles and singularities of the base orbifold correspond to the fibers without trivial fibered solid torus neighborhoods. A fiber is called *regular* if it projects to a nonsingular point of the base, otherwise it is called a *multiple fiber.*

Lemma 2.6.1 (cf. Lemma 3.2 in [30]). *Suppose M admits a Seifert fibration over a 2–orbifold X. Then there is a short exact sequence*

$$1 \longrightarrow G \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(X) \longrightarrow 1 \quad ,$$

where G denotes the cyclic subgroup of $\pi_1(M)$ generated by a regular fiber and $\pi_1^{orb}(X)$ denotes the fundamental group of X as an orbifold. The subgroup G is infinite except in cases where M is covered by S^3.

Note that a presentation for $\pi_1^{orb}(X)$ is

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_n \middle| x_i^{p_i} = 1, \prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{i=1}^{n} x_i = 1 \right\rangle \quad ,$$

where g is the genus of the underlying surface of X, assuming X is closed and orientable with n singular points of multiplicities $p_1, \ldots p_n$. The Euler characteristic $\chi(X)$ of such a 2–orbifold X is defined by

$$\chi(X) = 2 - 2g - \sum_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) \quad .$$

An orbifold is called *spherical* (*hyperbolic*) if its Euler characteristic is positive (negative). It is called *Euclidean* if the Euler characteristic vanishes.

For more details on Seifert fibered spaces see [27] and [26]. For geometric structures on Seifert fibered spaces see [30].

A Seifert fibration on a 4–manifold is analogous to a Seifert fibration on a 3–manifold.

Definition 2.6.2. A smooth map $\pi : X \longrightarrow \Sigma$ from a smooth 4–manifold X to a surface Σ is called a *Seifert fibration* if there exists a finite set of isolated points B in

Σ such that the restriction of π to $\pi^{-1}(\Sigma - B)$ is a torus bundle and for each element $b \in B$, $\pi^{-1}(b)$ has a tubular neighborhood diffeomorphic to the product of a fibered solid torus with a circle.

A Seifert fibration can be thought of as a torus fibration over a 2–orbifold. In the complex category it corresponds to an elliptic fibration without singular fibers (possibly with multiple ones). If a 4–manifold admits a Seifert fibration it is called a *Seifert 4–manifold*. We have analogous statements for Seifert fibered 4–manifolds to most of the properties of Seifert fibered spaces, e.g. Lemma 2.6.1. See [44] and [45] for geometric structures on elliptic surfaces without singular fibers, [41] and [42] for a general picture of Seifert 4–manifolds in terms of geometric structures.

Chapter 3

Lefschetz fibrations on product four–manifolds

In this chapter we generalize the results of Chen and Matveyev to include product 4–manifolds which admit Lefschetz fibrations that aren't compatible with a symplectic structure.

3.1 Nonsymplectic Lefschetz fibrations on product four–manifolds

In this section we first show that nonsymplectic Lefschetz fibrations on $S^1 \times M$ cannot have singular fibers hence they are in fact locally trivial torus bundles. We then investigate which of these fibrations have symplectic total spaces and which of them give rise to fibrations of M over S^1.

Theorem 3.1.1 (Chen-Matveyev [4]). *Let π be a symplectic Lefschetz fibration on $S^1 \times M$, where M is a closed, connected, oriented 3–manifold without any fake 3–cells. Then there exists a fibration p on M over S^1. Moreover, the symplectic structure with which π is compatible is deformation equivalent (up to self-diffeomorphisms of $S^1 \times M$) to the canonical symplectic structure associated to the fibration $Id \times p : S^1 \times M \to S^1 \times S^1$.*

The symplectic form (canonical up to deformation equivalence) on the total space of a surface bundle over a compact, oriented surface is obtained by extending a symplectic form on a fiber and adding a (sufficiently large) multiple of the pullback of a symplectic form on the base to it (see [38] and [22] for details and more general cases). The following lemma plays a crucial role in the proof of the theorem above.

Lemma 3.1.2 ([4]). *Let π be a symplectic Lefschetz fibration on $S^1 \times M$, where M is a closed, connected, oriented 3–manifold. Then π doesn't have any critical points.*

First of all, we give the following generalization of this lemma.

Lemma 3.1.3. *Let π be a Lefschetz fibration on $S^1 \times M$, where M is a closed, connected, oriented 3–manifold. Then π is a fiber bundle. If π is not symplectic, then it is a torus bundle.*

Proof. We only need to consider the case where π is not symplectic, i.e. fibers are not symplectic submanifolds of $X = S^1 \times M$. By Theorem 2.3.6 the fiber class $[F]$ is torsion in $H_2(X;\mathbb{Z})$. This is possible only if F is a torus since otherwise

$$0 \neq \chi(F) = \langle e(TF), [F] \rangle \quad .$$

Note that $e(TF)$ extends to $H^2(X;\mathbb{Z})$ since TF is the pull-back (by the inclusion $F \hookrightarrow X$) of the vertical (with respect to π) subbundle of TX. On the other hand, the Euler characteristic of the total space of a Lefschetz fibration is equal to the product of the those of the base and the fiber plus the number of vanishing cycles. In our case this leads to

$$0 = \chi(S^1 \times M) = \chi(T^2) \cdot \chi(B) + \#\{\text{vanishing cycles}\} \quad .$$

Hence there are no vanishing cycles and π is a torus bundle. $\square$

This lemma shows that nonsymplectic Lefschetz fibrations on $S^1 \times M$ are all torus bundles over Riemann surfaces. We investigate these bundles in three groups according to base genus.

Lemma 3.1.4. *Let $S^1 \times M$ be the total space of a nontrivial T^2–bundle over S^2. Then $S^1 \times M$ carries no symplectic form.*

Proof. Since the torus bundle is nontrivial, $b_1(S^1 \times M) < 2$ and therefore $b_2(S^1 \times M) = 2 \cdot b_1(M) = 0$. Hence all closed 2–forms on $S^1 \times M$ are degenerate. $\square$

Remark 3.1.5. As we mentioned before, a fibration of M over S^1 induces a symplectic form on $S^1 \times M$. Therefore, when $S^1 \times M$ is as in the lemma M doesn't fiber over the circle.

We have a totally different picture for T^2–bundles over T^2.

Theorem 3.1.6 (Geiges [10]). *Let X be the total space of an oriented T^2–bundle over T^2. Then X admits a symplectic structure. Moreover, there exists a symplectic T^2–bundle over T^2 with total space X unless X is the total space of a nontrivial S^1–bundle over the total space of a nontrivial S^1–bundle over T^2.*

Let X be an exception, i.e. a twisted circle bundle over a twisted circle bundle over the torus. Then $b_1(X) = b_2(X) = 2$. Moreover, $H^1_{DR}(X;\mathbb{R})$ is generated by $[\alpha]$ and $[\beta]$, where α and β are closed 1–forms on X such that $n \cdot \alpha \wedge \beta = d\gamma$, where n is the Euler number of the (nontrivial) S^1–bundle over T^2 and γ is a 1–form on X (see [6] for details). In particular, $(H^1(X;\mathbb{R}))^{\cup 2} = 0$, where $(H^1(X;\mathbb{R}))^{\cup 2}$ denotes the image of the cup product of $H^1(X;\mathbb{R})$ with itself. On the other hand, $H^1(S^1 \times M;\mathbb{R}) \cong H^1(S^1;\mathbb{R}) \oplus H^1(M;\mathbb{R})$ and obviously $(H^1(S^1 \times M;\mathbb{R}))^{\cup 2} \neq 0$. Therefore we have the following corollary.

Corollary 3.1.7. *If $S^1 \times M$ is the total space of a T^2–bundle over T^2, then $S^1 \times M$ admits a symplectic Lefschetz fibration.*

For T^2–bundles over higher genus surfaces we have the following lemma.

Lemma 3.1.8. *Let $S^1 \times M$ be the total space of a T^2–bundle over B, where B is a closed, oriented surface of genus ≥ 2. Also assume that M has no fake 3–cells. Then M fibers over the circle if and only if the torus bundle is trivial.*

We are going to use the following lemma to prove the one above.

Lemma 3.1.9 (cf. [27] Theorem 7.2.4). *Let M be a closed,oriented 3–manifold which is the total space of a circle bundle over a closed, oriented surface B of genus ≥ 2. Then M fibers over the circle if and only if $M = S^1 \times B$.*

Proof. Recall that $\pi_1(M)$ has the presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, \alpha \middle| [a_i, \alpha] = [b_i, \alpha] = 1, [a_1, b_1] \cdots [a_g, b_g] = \alpha^k \right\rangle \ ,$$

where $g = genus(B)$ and k is the Euler number of the S^1–bundle. In particular, $H_1(M) \cong \mathbb{Z}^{2g+1}$ if $k = 0$ and $H_1(M) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_{|k|}$ otherwise.

We also have the following commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(S^1) \arrow[r, "j_\#"] \arrow[d, "\cong"] & \pi_1(M) \arrow[r] \arrow[d] & \pi_1(B) \arrow[r] \arrow[d] & 1 \\
 & H_1(S^1) \arrow[r, "j_*"] & H_1(M) \arrow[r] & H_1(B) \arrow[r] & 0
\end{tikzcd}$$

where vertical maps are Hurewicz epimorphisms. Note that the homomorphism j_* is injective if and only if $Im(j_\#) \cap [\pi_1(M) : \pi_1(M)] = \{1\}$ (here $[G : G]$ stands for the commutator subgroup of G). Now suppose that $F \longrightarrow M \longrightarrow S^1$ is a fibration. There exists a normal subgroup $N \cong \pi_1(F)$ in $\pi_1(M)$ such that $\pi_1(M)/N \cong \mathbb{Z}$. Assume that there exists an element $u \in N \backslash \{1\}$ such that $u = j_\#(v)$. Then there is a normal infinite cyclic subgroup (generated by u) in N and this implies that F is a torus, but M cannot be the total space of a torus bundle over the circle since $b_1(M) \geq 2g \geq 4 > 3$. Therefore $Im(j_\#) \cap N = \{1\}$. On the other hand, $[\pi_1(M) : \pi_1(M)] \subset N$ because $\pi_1(M)/N \cong \mathbb{Z}$. So $Im(j_\#) \cap [\pi_1(M) : \pi_1(M)] = \{1\}$, j_* is injective and we have the short exact sequence

$$0 \longrightarrow H_1(S^1) \longrightarrow H_1(M) \longrightarrow H_1(B) \longrightarrow 0$$

which clearly splits. Hence $b_1(M) = 2g + 1$ and M is the product $S^1 \times B$. $\square$

Proof of Lemma 3.1.8. We have the homotopy sequence of the T^2–bundle

$$0 \longrightarrow \pi_1(T^2) \xrightarrow{j_\#} \pi_1(S^1 \times M) \xrightarrow{\pi_\#} \pi_1(B) \longrightarrow 1 \ . \tag{3.1}$$

Let u be a generator of $\pi_1(S^1 \times pt)$. Assume that $\pi_\#(u) = v \neq 1 \in \pi_1(B)$. Then v generates a normal cyclic subgroup in $\pi_1(B)$ and this contradicts the fact that $genus(B) \geq 2$. Therefore $u \in ker(\pi_\#) = im(j_\#)$, where j is the inclusion map. Let a be $j_\#^{-1}(u)$. We can find another element $b \in \pi_1(T^2)$ such that the restriction of $j_\#$ to the subgroup $\langle b \rangle$ generated by b gives the short exact sequence

$$1 \longrightarrow \langle b \rangle \longrightarrow \pi_1(M) \longrightarrow \pi_1(B) \longrightarrow 1 \quad . \tag{3.2}$$

By Theorem 11.10 in [14] M admits an S^1–bundle over B (we use the assumption that M has no fake 3–cells). Lemma 3.1.9 finishes the proof. □

We should note that the idea of extracting (3.2) from (3.1) was used by Chen and Matveyev in [4].

Proposition 3.1.10. *Suppose $S^1 \times M$ admits a nonsymplectic Lefschetz fibration, where M is a closed, oriented 3–manifold. If the base space of the fibration is a torus, then $S^1 \times M$ admits a symplectic form and a symplectic Lefschetz fibration. Otherwise M doesn't fiber over S^1 or it has a fake 3–cell.*

Proof. Let π be a nonsymplectic Lefschetz fibration on $X = S^1 \times M$. By Lemma 3.1.3, π is relatively minimal, has no critical points and the fibers are tori. It is a nontrivial bundle since otherwise it would be symplectic. If the base space B is a torus, then X admits a symplectic Lefschetz fibration by Corollary 3.1.7. If $B = S^2$, then X doesn't admit a symplectic structure by Lemma 3.1.4 and in particular, M doesn't fiber over S^1 since such a fibration would induce a symplectic form on X. Finally, if the genus of B is at least 2 and M has no fake 3–cells, then Lemma 3.1.8 implies that M doesn't fiber over S^1. □

3.2 Seiberg–Witten invariants of symplectic manifolds and circle bundles over surfaces

In this section we use Seiberg–Witten theory of symplectic manifolds and S^1–bundles over closed, oriented surfaces to prove the following theorem which in turn implies that the existence of a symplectic form and a Lefschetz fibration on $S^1 \times M$ is possible only if there is a symplectic Lefschetz fibration on $S^1 \times M$ (Theorem 3.2.5). Statement (L) of Theorem 1.0.1 is a consequence of this result.

Theorem 3.2.1. *Let M be the total space of an oriented S^1–bundle over a Riemann surface B. Then $X = S^1 \times M$ admits a symplectic structure if and only if the bundle is trivial or B is a torus.*

The following theorem follows from the work of Mrowka, Ozsváth and Yu on the SW invariants of Seifert fibered spaces [25]. See [1] for a different (and more elementary) approach.

Theorem 3.2.2. *Let M be the S^1–bundle over a Riemann surface B of genus $g \geq 1$ with Euler class $n\lambda$, where λ is the (positive) generator of $H^2(B;\mathbb{Z})$. If $n \neq 0$, then all basic classes of M are in $\{k \cdot \pi^*(\lambda) \mid 0 \leq k \leq |n| - 1\}$, where π is the bundle projection. Moreover, we have*

$$SW_M(k \cdot \pi^*(\lambda)) = \sum_{s \equiv k \pmod{n}} SW_{S^1 \times B}(s \cdot pr_2^*(\lambda)) \quad , \tag{3.3}$$

where pr_2 is the projection $S^1 \times B \to B$.

It is well-known that the Seiberg–Witten invariants of $S^1 \times B$ are given by

$$SW_{S^1 \times B}(t) = (t - t^{-1})^{2g-2} \quad ,$$

where g is the genus of B and the coefficient of t^p on the right hand side corresponds to the Seiberg–Witten invariant of the $Spin_c$ structure with determinant line bundle L with $c_1(L) = p \cdot pr_2^*(\lambda)$. Therefore the sum of all Seiberg–Witten invariants of $S^1 \times B$ is 0 if $g > 1$. This sum is preserved under twisting of the S^1–bundle as can be seen from (3.3).

Corollary 3.2.3. *Let M be as in the previous theorem and $g > 1$. Then*

$$\sum_{\alpha} SW_M(\alpha) = 0 \quad ,$$

where the sum is over all $Spin_c$ structures on M.

The following is also well-known and relates the Seiberg–Witten invariants of $S^1 \times M$ with those of M. For a proof see [28].

Theorem 3.2.4. *If M is a closed, oriented 3–manifold, then*

$$SW_M(\alpha) = SW_{S^1\times M}(pr_2^*(\alpha))$$

for any $\alpha \in H^2(M;\mathbb{Z})$, where pr_2 is the projection $S^1 \times M \to M$. Moreover, if $b_+(S^1 \times M) = b_1(M) > 1$, then all basic classes of $S^1 \times M$ are pull-backs of basic classes of M .

Proof of Theorem 3.2.1. If the bundle is trivial then $X = T^2 \times B$ and there is a symplectic form on X which is simply the sum of symplectic forms on T^2 and B.

If B is a torus, then X is a torus bundle over a torus and by Theorem 3.1.6 it admits a symplectic structure.

If the bundle is nontrivial and B is a sphere, then X is a nontrivial T^2–bundle over S^2 and cannot be symplectic as we proved in Lemma 3.1.4.

From now on we will assume that the bundle is nontrivial and the genus of B is at least 2.

By Corollary 3.2.3 and Theorem 3.2.4 (as $b_1(M) \geq 2b_1(B) \geq 4$)

$$\sum_\alpha SW_M(\alpha) = \sum_\beta SW_X(\beta) = 0 \quad , \tag{3.4}$$

where sums are over all $Spin_c$ structures on M and X respectively.

Assume that X admits a symplectic form ω. First of all, by the conditions on equality in Theorem 2.4.1, the canonical class $K = c_1(X,\omega)$ cannot be a nonzero torsion class. On the other hand, since M is an S^1-bundle over a surface of genus ≥ 2 we can apply Theorem 3.2.4, and then the first part of Theorem 3.2.2 implies that all basic classes of X are torsion. Therefore the only basic class of X is $K = 0$ and $SW_X(0) = \pm 1$, in particular,

$$\sum_\beta SW_X(\beta) = \pm 1 \quad ,$$

where the sum is over all $Spin_c$ structures on X. This contradicts (3.4) hence X does not admit a symplectic structure. $\square$

Theorem 3.2.5. *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a Lefschetz fibration and a symplectic form. Then $S^1 \times M$ admits a symplectic Lefschetz fibration or M has a fake 3–cell.*

Proof. Let $X = S^1 \times M$ admit a Lefschetz fibration and a symplectic form. Assume that there is no symplectic Lefschetz fibration on it. Then by Lemma 3.1.3 it admits a torus bundle over a Riemann surface B. Any such bundle should be nontrivial since otherwise it would be symplectic. By Theorem 3.1.10, B is not a torus, and it cannot be a sphere by Lemma 3.1.4. So $genus(B) \geq 2$. If M has no fake 3–cells, then as we have seen in the proof of Lemma 3.1.8, M is the total space of an S^1–bundle over B and this contradicts Theorem 3.2.1. $\square$

This theorem (together with Theorem 3.1.1) finishes the proof of statement (L) of Theorem 1.0.1.

Remark 3.2.6. Symplectic Lefschetz fibrations on product 4–manifolds were classified in [4]. As a result of our discussion, we see that nonsymplectic Lefschetz fibrations on nonsymplectic $S^1 \times M$ are nontrivial torus bundles over spherical or hyperbolic surfaces. On the other hand, nonsymplectic Lefschetz fibrations on a symplectic $S^1 \times M$ are torus bundles over tori and by Proposition 3.1.10 any such manifold admits a symplectic Lefschetz fibration.

Chapter 4

Complex structures and Seifert fibrations on product 4–manifolds

In this chapter, we use the classification of complex surfaces to prove statements (K) and (C) of Theorem 1.0.1. To prove the latter, we also use an interesting result in Seiberg–Witten theory of complex surfaces due to O. Biquard. Then we consider Seifert fibered product 4–manifolds and prove that those which admit symplectic structures also admit either Kähler structures or torus bundles over tori. This observation finishes the proof of Theorem 1.0.1.

At this point we know exactly when the existence of a Lefschetz fibration on $S^1 \times M$ is sufficient for M to fiber over the circle. Our motivation is to determine whether the existence of a symplectic structure on $S^1 \times M$ is sufficient for M to fiber over the circle, so it is quite natural to ask which symplectic (product) 4–manifolds admit Lefschetz fibrations. This question doesn't seem to be any easier than Conjecture T itself even though Donaldson proved that every symplectic 4–manifold admits a Lefschetz pencil. In fact, statement (L) of Theorem 1.0.1 implies that they are equivalent when M has no fake 3–cells. On the other hand, allowing multiple fibers and considering Seifert fibrations, one can still get interesting results on Conjecture T. Seifert fibered product 4–manifolds turn out to be closely related to complex surfaces and this is the main reason of our discussion on complex structures on product 4–manifolds.

4.1 Complex structures on product 4–manifolds

Suppose that $S^1 \times M$ is a closed complex surface. Since it is a spin 4–manifold its intersection form is even, so there is no exceptional sphere to blow-down, thus it is a minimal complex surface. We are going to use the Enriques–Kodaira classification of compact complex surfaces (see [12] or [3]) to prove the following theorem.

Theorem 4.1.1 (cf. Theorem 4.1 in [11]). *Let $S^1 \times M$ be a closed 4–manifold.*

If $S^1 \times M$ admits a complex structure, then it is either an elliptic surface or of Class VII_0.

If $S^1 \times M$ is also symplectic, then the only possibilities are the following:

(i) $S^1 \times M \cong S^2 \times T^2$.

(ii) $S^1 \times M$ admits a T^2–bundle over T^2.

(iii) $S^1 \times M$ admits a Seifert fibration over a hyperbolic orbifold.

Proof. Let $\kappa(X)$ be the Kodaira dimension of $X = S^1 \times M$ as a complex surface.

Case 1: $\kappa(X) = -\infty$. In this case X is either $\mathbb{C}P^2$ or geometrically ruled or of *Class* VII_0. The complex projective plane $\mathbb{C}P^2$ is simply-connected, but X is not. If X is a complex surface of *Class* VII_0, then $0 = b_1(X) - 1 = b_+(X)$ hence it cannot be symplectic. If it is geometrically ruled, then it is the total space of a $\mathbb{C}P^1$–bundle over a Riemann surface B and $0 = \chi(X) = \chi(\mathbb{C}P^1) \cdot \chi(B)$, hence B is a torus. Moreover, X is diffeomorphic to $S^2 \times T^2$ since the total space of the nontrivial S^2–bundle over T^2 is not spin.

Case 2: $\kappa(X) = 0$. Any minimal complex surface of Kodaira dimension 0 is a $K3$ surface, an Enriques surface, a primary Kodaira surface, a secondary Kodaira surface, a hyperelliptic surface or a complex torus. Since $b_1(X) \geq 1$ X cannot be a $K3$ or an Enriques surface. In three of the other four cases, X is diffeomorphic to the total space of a T^2–bundle over T^2. When X is a secondary Kodaira surface it admits an elliptic fibration over $\mathbb{C}P^1$ (without singular fibers) and $b_1(X) = 1$. So in this case, X cannot be symplectic because $b_+(X) = b_1(X) - 1 = 0$.

Case 3: $\kappa(X) = 1$. In this case X is a (properly) elliptic surface. An elliptic fibration on X cannot have singular fibers but only multiple fibers since the Euler characteristic of X vanishes. In particular, X is a Seifert fibered 4–manifold. While investigating geometric structures on elliptic surfaces Wall (see [44] or [45]) proves that the base orbifold of such a fibration must be hyperbolic when $\kappa(X) = 1$.

These are the only possibilities since every minimal surface of general type has positive Euler characteristic and $\chi(X) = 0$. ☐

Remark 4.1.2. By a well-known result of Bogomolov [37] a complex surface of *Class* VII_0 with vanishing second Betti number is either a Hopf surface or an Inoue surface. Since the center of the fundamental group of an Inoue surface is trivial (cf. Proposition 4.2 in [11]) no Inoue surface is a product. On the other hand, Kato's work on Hopf surfaces [16] implies that if a Hopf surface is diffeomorphic to a product, then it must be elliptic. In particular, it is Seifert fibered since vanishing of the Euler characteristic implies that an elliptic fibration on a product can have no singular fibers (but only multiple ones).

Recall that a closed complex surface is Kähler if and only if its first Betti number is even. Therefore statement (K) of Theorem 1.0.1 is a consequence of the following theorem.

Theorem 4.1.3. *Let* $S^1 \times M$ *be a closed, connected complex surface. If* $b_1(M)$ *is odd and* M *has no fake 3–cells, then* M *is a Seifert fibered space which fibers over* S^1*.*

Proof. Since $b_1(X) = b_1(M) + 1$ is even, $X = S^1 \times M$ admits a Kähler structure. By Theorem 4.1.1, X is diffeomorphic to $S^2 \times T^2$ or admits a T^2–bundle over T^2 or a properly elliptic fibration without any singular (possibly with multiple) fibers.

If X is diffeomorphic to $S^2 \times T^2$, then M fibers over S^1 by Theorem 3.1.1. Moreover, the diffeomorphism between $S^1 \times M$ and $S^1 \times (S^2 \times S^1)$ gives a homotopy equivalence between M and $S^2 \times S^1$ and as they both fiber over S^1 this homotopy equivalence must be a homeomorphism, in particular, M is a Seifert fibered space.

If X admits a T^2–bundle over T^2, then by Corollary 3.1.7 and Theorem 3.1.1 M fibers over S^1 with fiber a torus and in particular it is geometric. On the other hand, by Theorem 3 in [10] the geometric type of M is $\mathbb{E}^3$, where $\mathbb{E}^n$ is $\mathbb{R}^n$ with its standard metric. This implies that $M = T^3$ (see p.446 in [30]). In particular, M is Seifert fibered.

If X admits a Seifert fibration over a hyperbolic orbifold B, then it is geometric and the geometric type of it must be $\mathbb{E}^2 \times \mathbb{H}^2$ by Theorem 4.5 in [45] as X admits a Kähler structure, where $\mathbb{H}^2$ is the hyperbolic plane. It should be noted that there is a mistake in [45] which was later corrected by Kotschick in [17]; since it concerns manifolds with nonvanishing Euler characteristic, it doesn't effect our discussion on product 4–manifolds. On the other hand, we get the following exact sequence from the Seifert fibration

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(S^1 \times M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1 \quad ,$$

where F is a regular fiber and $\pi_1^{orb}(B)$ denotes the fundamental group of B as an orbifold. This exact sequence leads to another one

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1 \quad ,$$

just as in the proof of Lemma 3.1.8, since B is hyperbolic and its orbifold fundamental group doesn't contain an infinite cyclic normal subgroup. So there exists an infinite cyclic normal subgroup in $\pi_1(M)$ and M is a Seifert 3–manifold by Corollary 12.8 in [14]. (Note that as $b_1(M)$ is odd it is nonzero and M is sufficiently large.) In particular, M is geometric. Since $S^1 \times M$ is type $\mathbb{E}^2 \times \mathbb{H}^2$, M must be type $\mathbb{E}^1 \times \mathbb{H}^2$, in other words the rational Euler class of a Seifert fibration on M is 0. A generalization of Lemma 3.1.9 (e.g. Theorem 8.1 in [26]) implies that M fibers over S^1. □

In order to prove statement (C) of Theorem 1.0.1 we use the following result of Biquard (cf. Théorème 8.2 in [2]):

Theorem 4.1.4. *A properly elliptic non–Kähler surface admits no symplectic structure.*

Proof of Statement (C) in Theorem 1.0.1. We have seen in Theorem 4.1.1 that if $X = S^1 \times M$ admits a complex and a symplectic structure, then there are three possibilities. The product $S^2 \times T^2$ admits a Kähler structure hence if $X = S^2 \times T^2$, then M fibers over S^1 by Theorem 4.1.3. If X admits a T^2–bundle over T^2, then M fibers over S^1 by Corollary 3.1.7 and Theorem 3.1.1. If X is a properly elliptic surface, then it has to be Kähler by Theorem 4.1.4 hence M fibers over S^1 by Theorem 4.1.3. □

4.2 Seifert fibrations of product four–manifolds

The following is a well-known theorem. For a nice proof see [47].

Theorem 4.2.1. *If M is a closed, oriented Seifert fibered space, then $S^1 \times M$ admits a complex structure.*

Proposition 4.2.2. *Let M be a closed, oriented 3–manifold with no fake 3–cells. Suppose $S^1 \times M$ admits a symplectic structure and a Seifert fibration. Then $S^1 \times M$ admits a Kähler structure or a T^2–bundle over T^2.*

Proof. We have the following short exact sequence coming from the Seifert fibration

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(S^1 \times M) \xrightarrow{\pi_\#} \pi_1^{orb}(B) \longrightarrow 1 \quad ,$$

where F is a generic fiber, $\pi_1^{orb}(B)$ denotes the fundamental group of B as an orbifold and π is the projection map of the fibration. Let u be a generator of $\pi_1(S^1 \times \{pt\})$ in $\pi_1(S^1 \times M)$ as in the proof of Lemma 3.1.8.

First assume that $\pi_\#(u)$ is nontrivial in $\pi_1^{orb}(B)$. Then it generates an infinite, cyclic, normal subgroup (cf. proof of Lemma 3.1.8). Existence of such a subgroup in $\pi_1^{orb}(B)$ is possible only if B is a nonsingular orbifold of genus 1, i.e. a torus. So the Seifert fibration we have is in fact a T^2–bundle over T^2.

Now assume $u \in ker(\pi_\#)$. Then as in the proof of Theorem 4.1.3 we have

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1 \quad .$$

In particular, there is an infinite cyclic normal subgroup of $\pi_1(M)$. Since X admits a symplectic structure $b_+(X) \geq 1$ and so is $b_1(M)$. This implies that M is sufficiently large. Therefore we can use Corollary 12.8 in [14] to conclude that M is a Seifert fibered space. So $S^1 \times M$ admits a complex structure by Theorem 4.2.1, hence it admits a Kähler structure or a T^2–bundle over T^2 as in the proof of statement (C). □

This proposition (together with Theorem 4.1.3 and Corollary 3.1.7) finishes the proof of Theorem 1.0.1.

Chapter 5

The geometrization of 3-manifolds and symplectic 4-manifolds

During the course of our proof of Theorem 1.0.1 in the previous chapters we made observations on the interaction between various structures and fibrations on M and $S^1 \times M$. In this chapter, we recall some of these observations and use them to prove some theorems on the relation between the geometry of M and $S^1 \times M$.

Throughout this chapter we will assume that M is a closed, connected and oriented 3–manifold with no fake 3–cells.

5.1 Geometric structures on three–manifolds that fiber over the circle

We consider closed, oriented 3–manifolds that fiber over the circle in three different classes according to the genus of the fiber.

If a closed, oriented 3–manifold M is the total space of a sphere bundle over the circle, then M is homeomorphic to $S^1 \times S^2$, and in particular, geometric of type $\mathbb{E}^1 \times S^2$.

Torus bundles over the circle are three types according to the monodromy of the fibration. If the monodromy is periodic, i.e. the monodromy is an automorphism of the torus which has a positive power that is homotopic to the identity, then the total space is geometric of type $\mathbb{E}^3$. If the monodromy is not periodic but reducible, in other words, if there is a simple closed curve preserved by the monodromy, then the total space is in fact the total space of a circle bundle over the torus and it is geometric of type Nil^3. If the monodromy is neither periodic nor reducible, then it is Anosov and the total space is geometric of type Sol^3.

In case the fiber of a fibration over the circle is a closed, oriented hyperbolic surface, then we again have three cases according to the type of the monodromy. If the monodromy is periodic, then the total space is geometric of type $\mathbb{E}^1 \times \mathbb{H}^2$. A remarkable theorem of Thurston [39] says that if the monodromy is pseudo-Anosov, i.e. neither periodic nor reducible, then the total space is hyperbolic.

The above discussion can be summarized as

Proposition 5.1.1. *Let M be a closed, oriented 3–manifold which is the total space of a surface bundle over the circle. Unless the fiber is hyperbolic and the monodromy is reducible, M has a geometric structure. If M is not geometric, then there is an incompressible torus embedded in M.*

5.2 Seifert fibered and complex $S^1 \times M$ and the geometry of M

In the proof of Proposition 4.2.2 we used the existence of a symplectic structure on $S^1 \times M$ to conclude that $b_+(S^1 \times M) = b_1(M) > 0$ and as a consequence M is sufficiently large.

Theorem 5.2.1. *If $S^1 \times M$ is Seifert fibered and M is sufficiently large, then M admits a nonhyperbolic geometric structure.*

Proof. As in the proof of Proposition 4.2.2 we look at the homotopy sequence of the Seifert fibration. There are two different cases depending on the image of a generator u of $\pi_1(S^1 \times \{pt\}) \subset \pi_1(S^1 \times M)$:

If u is in the kernel, then we have an infinite cyclic normal subgroup in $\pi_1(M)$. Since M is sufficiently large, Corollary 12.8 in [14] implies that M is a Seifert fibered space.

If u is not in the kernel, then $S^1 \times M$ admits a T^2–bundle over T^2, in particular it is symplectic. Hence (e.g. by (L) of Theorem 1.0.1) M fibers over the circle with fiber a torus. As discussed in the previous section, M is geometric of type $\mathbb{E}^3$, Nil^3 or Sol^3.

It is now clear that in any case M is geometric but not hyperbolic. □

Theorem 4.2.1 says that if M is Seifert fibered, then $S^1 \times M$ admits a complex structure. If M is geometric of type Sol^3, then $S^1 \times M$ is obviously geometric of type $\mathbb{E}^1 \times Sol^3$ and $S^1 \times M$ doesn't admit any complex structure as a consequence of [44].

On the other hand, Theorem 4.1.1 says that if $S^1 \times M$ admits a complex structure, then it is either of *Class* VII_0 or an elliptic surface and in any case, by the remark following Theorem 4.1.1, $S^1 \times M$ is Seifert fibered.

This discussion leads us to the following conclusion which is a partial converse of the well-known Theorem 4.2.1.

Theorem 5.2.2. *If $S^1 \times M$ admits a complex structure and M is sufficiently large, then M is a Seifert fibered space.*

5.3 Symplectic $S^1 \times M$ and the geometry of M

J. McCarthy and S. Vidussi independently proved that if M is a closed, oriented, 3–manifold such that $S^1 \times M$ admits a symplectic structure, then M can be uniquely decomposed as the connected sum of a prime manifold and a homology 3–sphere ([21] and [43]). In fact, they proved that M itself is prime (assuming the geometrization conjecture). Note that all prime 3–manifolds are irreducible except $S^1 \times S^2$. Throughout this section, we will assume that M is a closed, oriented, irreducible 3–manifold such that $S^1 \times M$ admits a symplectic structure.

First of all, as we saw in Lemma 2.2.1, $b_1(M) = b_+(S^1 \times M)$. On the other hand, a symplectic form on a 4–manifold X induces a cohomology class which generates a positive definite subspace in $H^2(X;\mathbb{R})$, hence $b_+(X) \geq 1$ for such a manifold X. Therefore $b_1(M) > 0$ and this implies that M is sufficiently large and Haken.

Thurston proved the geometrization conjecture for Haken manifolds. So there is a canonical way to cut M into geometric pieces along embedded incompressible tori. In fact, if M is atoroidal, then there is no incompressible torus embedded in M and hence M itself is geometric, and in fact, it is hyperbolic. Even if M is not atoroidal, there may still not be an incompressible torus embedded in M. If this is the case, i.e. if M is not atoroidal and there is no incompressible torus in M, then, by the Torus Theorem, M is Seifert fibered. So the only remaining case is the one where M has an incompressible torus. It should be kept in mind that even in this case M might be geometric. For example, the total space of any torus bundle over the circle with Anosov monodromy is geometric of type Sol^3, but still admits incompressible tori which are fibers of the torus bundle.

Here are two statements that wrap up this discussion:

Proposition 5.3.1. *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. Then $M = N \# \Sigma$ for a prime manifold N and a homology sphere Σ. Moreover, if there is no incompressible torus in N, then N is either Seifert fibered or hyperbolic (in particular, geometric). If there is an incompressible torus in N, then N could be canonically cut into geometric pieces along tori.*

Proposition 5.3.2. *Let M be a prime, closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. If there is no incompressible torus embedded in M, then M is either $S^1 \times S^2$ or type $\mathbb{H}^2 \times \mathbb{E}^1$ or hyperbolic.*

5.4 Taubes conjecture and hyperbolic 3–manifolds

As it is mentioned in the first section of this chapter, most of the 3–manifolds that fiber over the circle are hyperbolic, but it is not known whether Conjecture T holds for hyperbolic 3–manifolds or not. In fact, it might be possible to construct counterexamples to Conjecture T by using hyperbolic 3–manifolds which don't fiber over S^1 but have finite covers that fiber over S^1.

Question Is there a hyperbolic 3–manifold M which satisfies the following conditions?

1. M doesn't fiber over the circle.
2. There exists a finite cover of M that fibers over the circle.
3. $S^1 \times M$ is symplectic.

Such a three–manifold would obviously be a counterexample to Conjecture T. There are examples in [9] and [29] which satisfy 1 and 2. In fact, the virtual bundle conjecture of Thurston claims that any hyperbolic 3–manifold has a finite cover which fibers over the circle. On the other hand, since the geometrization conjecture implies that M is Haken when $S^1 \times M$ is symplectic, it is not expected that the example in [29] satisfies 3.

Bibliography

[1] **S Baldridge**, *Seiberg–Witten invariants of 4–manifolds with free circle actions*, Commun. Contemp. Math. 3 (2001), 341–353.

[2] **O Biquard**, *Les équations de Seiberg–Witten sur une surface complexe non Kählérienne*, Comm. Anal. Geom. 6 (1998), 173–197.

[3] **W Barth**, **C Peters**, **A van de Ven**, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol.4, Springer–Verlag, 1984.

[4] **W Chen**, **R Matveyev**, *Symplectic Lefschetz fibrations on $S^1 \times M$*, Geometry and Topology, 4 (2000), 517–535.

[5] **S Donaldson**, *Lefschetz fibrations in symplectic geometry*, Proc. ICM, Vol. II (Berlin, 1998), Doc. Math., Extra Vol. II, 1998, 309–314.

[6] **M Fernández**, **M Gotay**, **A Gray**, *Compact parallelizable four dimensional symplectic and complex manifolds*, Proc. Amer. Math. Soc. 103 (1988), no.4, 1209–1212.

[7] **R Filipkiewicz**, *Four–dimensional geometries*, Ph.D. Thesis, University of Warwick, 1984.

[8] **R Fintushel**, **R Stern**, *Knots, links and 4–manifolds*, Invent. Math. 134 (1998), 363–400.

[9] **D Gabai**, *On three-manifolds finitely covered by surface bundles*, Low–dimensional topology and Kleinian groups (Coventry/Durham, 1984), London

Math. Soc. Lecture Note Ser., 112, Cambridge Univ. Press, Cambridge, 1986, 145–155.

[10] **H Geiges**, *Symplectic structures on* T^2*–bundles over* T^2, Duke Math. J. 67 (1992), no.3, 539–555.

[11] **H Geiges**, **J Gonzalo**, *Contact geometry and complex surfaces*, Invent. Math. 121 (1995), 147–209.

[12] **R Gompf**, **A Stipsicz**, *4–manifolds and Kirby calculus*, Graduate Studies in Math., vol.20, American Mathematical Society, 1999.

[13] **M Gromov**, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307–347.

[14] **J Hempel**, *3–manifolds*, Ann. of Math. Studies, Princeton University Press, 1976.

[15] **E Ionel**, **T Parker**, *The Gromov invariants of Ruan–Tian and Taubes*, Math. Res. Lett. 4 (1997), 521–532.

[16] **M Kato**, *Topology of Hopf surfaces*, J. Math. Soc. Japan 27 (1975), 222–238 and 41 (1989), 173–174.

[17] **D Kotschick**, *Remarks on geometric structures on compact complex surfaces*, Topology 31 (1992), no.2, 317–321.

[18] **P Kronheimer**, *Embedded surfaces and gauge theory in three and four dimensions*, Surveys in Differential Geometry, (Cambridge, MA, 1996), vol. III, International Press, 1998, 243–298.

[19] **P Kronheimer**, *Minimal genus in* $S^1 \times M^3$, Invent. Math. 135 (1999), 45–61.

[20] **Y Matsumoto**, *Diffeomorphism types of elliptic surfaces*, Topology 25 (1986), 549–563.

[21] **J McCarthy**, *On the asphericity of a symplectic* $M^3 \times S^1$, Proc. Amer. Math. Soc. 129 (2001), 257–264.

[22] **D McDuff**, **D Salamon**, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford University Press, 1995.

[23] **J Milnor**, *A unique factorisation theorem for 3–manifolds*, Amer. J. Math. 79 (1957), 623–630.

[24] **J Morgan**, *The Seiberg–Witten invariants and applications to the topology of smooth four–manifolds*, Mathematical Notes, vol.44, Princeton University Press, 1996.

[25] **T Mrowka**, **P Ozsváth**, **B Yu**, *Seiberg–Witten monopoles on Seifert fibered spaces*, Comm. Anal. Geom. 5 (1997), 685–791.

[26] **W Neumann**, **F Raymond**, *Seifert manifolds, plumbing,* μ*–invariant and orientation reversing maps*, Algebraic and Geometric Topology (Santa Barbara, 1977), Lecture Notes in Mathematics, vol.664, Springer–Verlag, 1978, pp.163–196.

[27] **P Orlik**, *Seifert manifolds*, Lecture Notes in Mathematics, vol.291, Springer–Verlag, 1972.

[28] **C Okonek**, **A Teleman**, *3–dimensional Seiberg–Witten invariants and non–Kählerian geometry*, Math. Ann. 312 (1998), 261–288.

[29] **A W Reid**, *A non-Haken hyperbolic* 3*-manifold covered by a surface bundle* Pacific J. Math. 167 (1995), 163–182.

[30] **P Scott**, *The geometries of 3–manifolds*, Bull. London Math. Soc. 15 (1983), 401–487.

[31] **C Taubes**, *The Seiberg–Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994), 809–822.

[32] **C Taubes**, *More constraints on symplectic forms from the Seiberg–Witten invariants*, Math. Res. Lett. 2 (1995), 9–13.

[33] **C Taubes**, *The geometry of the Seiberg–Witten invariants*, Proc. ICM, Vol. II (Berlin, 1998), Doc. Math., Extra Vol. II, 1998, 493–504.

[34] **C Taubes**, SW $\Rightarrow$ Gr*: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996), 845–918.

[35] **C Taubes**, Gr $\Rightarrow$ SW*: from pseudo-holomorphic curves to Seiberg-Witten solutions*, J. Differential Geom. 51 (1999), 203–334.

[36] **C Taubes**, Gr = SW*: counting curves and connections*, J. Differential Geom. 52 (1999), 453–609.

[37] **A Teleman**, *Projectively flat surfaces and Bogomolov's theorem on Class* VII_0 *surfaces*, Internat. J. Math. 5 (1994), 253–264.

[38] **W Thurston**, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (1976), no.2, 467–468.

[39] **W Thurston**, *Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle*, preprint, ArXiv:GT/9801045.

[40] **W Thurston**, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982), 357–381.

[41] **M Ue**, *Geometric 4–manifolds in the sense of Thurston and Seifert 4–manifolds I*, J. Math. Soc. Japan 42 (1990), no.3, 511–540.

[42] **M Ue**, *Geometric 4–manifolds in the sense of Thurston and Seifert 4–manifolds II*, J. Math. Soc. Japan 43 (1991), no.1, 149–183.

[43] **S Vidussi**, *The Alexander norm is smaller than the Thurston norm: a Seiberg–Witten proof*, Prépublication École Polytechnique 99–6.

[44] **C T C Wall**, *Geometries and geometric structures in real dimension 4 and complex dimension 2*, Geometry and Topology (College Park, Maryland, 1983/84), Lecture Notes in Mathematics, vol.1167, Springer–Verlag, 1985, 268–292.

[45] **C T C Wall**, *Geometric structures on compact complex analytic surfaces*, Topology 25 (1986), no.2, 119–153.

[46] **E Witten**, *Monopoles and four–manifolds*, Math. Res. Lett. 1 (1994), 769–796.

[47] **J Wood**, *Harmonic morphisms, conformal foliations and Seifert fibre spaces*, Geometry of low–dimensional manifolds:1, (Durham, 1989), London Math. Soc. Lecture Note Series, vol.150, Cambridge University Press, 1990, 247–259.