A NOTE ON FUNDAMENTAL GROUPS OF SYMPLECTIC TORUS COMPLEMENTS IN 4-MANIFOLDS

TOLGA ETGÜ AND B. DOUG PARK

Abstract. Previously, we constructed an infinite family of knotted symplectic tori representing a fixed homology class in the symplectic four-manifold $E(n)_K$, which is obtained by Fintushel-Stern knot surgery using a nontrivial fibred knot K in S^3, and distinguished the (smooth) isotopy classes of these tori by indirectly computing the Seiberg-Witten invariants of their complements. In this note, we compute the fundamental groups of the complements of these knotted tori and show that for each nontrivial fibred knot K these groups constitute an infinite collection of nonisomorphic groups. We also review some other constructions of symplectic tori in 4-manifolds and show that the fundamental groups of the complements do not distinguish homologous tori in those cases.

1. Introduction

In [5]–[8], we constructed *infinite* families of homologous symplectic tori in many symplectic 4-manifolds, in particular, in simply-connected elliptic surfaces and homotopy elliptic surfaces, and distinguished the smooth isotopy classes of these tori by using Seiberg-Witten invariants. In this note, we compute the fundamental groups of the complements of these knotted symplectic tori. After we announced our results in [5]–[8], many colleagues have asked us whether the isotopy classes of our tori can be distinguished by these fundamental groups alone, without resorting to the Seiberg-Witten invariants. It turns out that these groups are *not* sufficient to distinguish the knotted tori that represent the same homology class in $E(n)$, the simply-connected complex elliptic surface with Euler characteristic $12n$ $(n \geq 1)$ and without any multiple fiber, or $E(n)_p$, the result of a logarithmic transformation of multiplicity p on $E(n)$. On the other hand, we have the following theorem regarding the knotted symplectic tori in the homotopy elliptic surface $E(n)_K$, i.e. the symplectic four-manifold which is the result of Fintushel-Stern knot surgery on $E(n)$ using a fibred knot $K \subset S^3$.

Theorem 1. *If K is a nontrivial fibred knot in S^3, then there exist infinitely many homologous symplectic tori in $E(n)_K$ whose complements have mutually nonisomorphic fundamental groups.*

In [9], infinitely many nullhomologous *Lagrangian* tori are distinguished by the fundamental groups of their complements in $E(n)_K$ when the Alexander polynomial $\Delta_K(t)$ of the fibred knot K has a root which is not a root of unity. For example, many hyperbolic knots satisfy this condition on $\Delta_K(t)$, but no torus knot satisfies

Date: October 31, 2003. Revised on June 2, 2004.
2000 *Mathematics Subject Classification.* Primary 57R17, 57M05; Secondary 53D35, 57R95.
The second author was partially supported by NSERC and CFI/OIT grants.

it. The techniques used to distinguish the fundamental groups in [9] are quite different from the ones we use here. On the other hand, using and extending the techniques and certain results in this note, infinitely many homologous non-isotopic symplectic surfaces of higher genera are obtained in [18].

In the next section, we review the relevant constructions, and in the last section we compute the fundamental groups of the complements of the symplectic tori and prove Theorem 1 (see Proposition 10 and Lemma 12).

2. Constructions of symplectic tori in 4-manifolds

In this section, we first review the generalization of the link surgery construction of Fintushel and Stern [10] by Vidussi [22], and then review the constructions of symplectic tori in elliptic surfaces $E(n)$, $E(n)_p$ and symplectic 4-manifolds $E(n)_K$ which are homotopy equivalent (hence homeomorphic) to $E(n)$ as given in [5]–[8].

2.1. **Generalized link surgery.** For an ordered n-component link $L \subset S^3$, choose an ordered homology basis of simple closed curves $\{(\alpha_i, \beta_i)\}_{i=1}^n$ for the boundary of the tubular neighborhood of L such that the pair (α_i, β_i) lie in the i-th boundary component and the intersection number of α_i and β_i is 1. Let X_i $(i = 1, \dots, n)$ be a 4-manifold containing a 2-dimensional torus submanifold F_i of self-intersection 0. Choose a cartesian product decomposition $F_i = C_1^i \times C_2^i$, where each C_j^i $(j = 1, 2)$ is an embedded circle in X_i.

Definition 2. The ordered collection

$$\mathfrak{D} \;=\; \big(\{(\alpha_i, \beta_i)\}_{i=1}^n \,,\; \{(X_i,\, F_i = C_1^i \times C_2^i)\}_{i=1}^n\big)$$

is called a *link surgery gluing data* for an n-component link L. We define the *link surgery manifold corresponding to* $\mathfrak{D}$ to be the closed 4-manifold

$$L(\mathfrak{D}) \;=\; [\coprod_{i=1}^n X_i \setminus \nu F_i] \bigcup_{F_i \times \partial D^2 = (S^1 \times \alpha_i) \times \bar{\beta}_i} [S^1 \times (S^3 \setminus \nu L)]\,,$$

where ν denotes the tubular neighborhoods. Here, the gluing diffeomorphisms between the boundary 3-tori identify the torus $F_i = C_1^i \times C_2^i$ of X_i with $S^1 \times \alpha_i$ factor-wise, and act as the complex conjugation on the last S^1 factors, ∂D^2 and β_i.

Remark. Strictly speaking, the diffeomorphism type of the link surgery manifold $L(\mathfrak{D})$ may possibly depend on the chosen trivialization of $\nu F_i \cong F_i \times D^2$ (the framing of F_i). However, we will suppress this dependence in our notation. It is well known (see e.g. [11]) that the diffeomorphism type of $L(\mathfrak{D})$ is independent of the framing of F_i when $(X_i, F_i) = (E(1), F)$.

Notation. For a knot $K \subset S^3$, $\mu(K)$ and $\lambda(K)$ denote the meridian and the longitude, respectively, in the complement $S^3 \setminus \nu K$.

For an embedding ι of a closed, connected, oriented surface Σ in a manifold X, $[\Sigma]$ denotes the image of the fundamental class of Σ under the homomorphism $\iota_* : H_2(\Sigma; \mathbb{Z}) \to H_2(X; \mathbb{Z})$.

F denotes a regular fiber of the elliptic fibration of $E(n)$. From now on, we shall always work with a fixed cartesian product decomposition $F = C_1 \times C_2$ in all link surgery gluing data containing the pair $(E(n), F)$. Let R denote either one of the rim tori of F, i.e. $R = R_i = C_i \times \partial D^2 \subset F \times D^2 \cong \nu F \subset E(n)$.

2.2. **Construction 1.** We review the construction in [5] which gives an infinite family of symplectic tori $\{T_{p,q}\}_{p\geq 1}$ for each $q \geq 2$, representing $q[F]$ in $E(n)$ for any $n \geq 1$.

For any pair of integers $p \geq 1$ and $q \geq 2$, consider the q-strand braid $\beta_{p,q}$ in Figure 1. Let A denote the closure of the axis of the braid $\beta_{p,q}$ and $\mathfrak{D}_1 =$

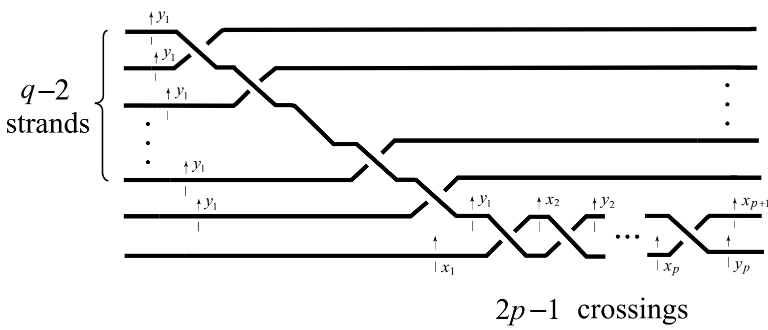

FIGURE 1. Braid $\beta_{p,q}$

$\big(\{(\mu(A), \lambda(A))\}, \{(E(n), F)\}\big)$. Note that the link surgery manifold $A(\mathfrak{D}_1)$ is diffeomorphic to $E(n)$ since A is the unknot. Now consider the closed braid $\hat{\beta}_{p,q}$ as a knot inside the solid torus $S^3 \setminus \nu A$. Let $T_{p,q}$ denote the torus $S^1 \times \hat{\beta}_{p,q}$ embedded in $[S^1 \times (S^3 \setminus \nu A)] \subset A(\mathfrak{D}_1) = E(n)$.

Proposition 3. (*See* [5]). *The torus $T_{p,q}$ is a symplectic submanifold of $E(n)$ and represents the homology class $q[F]$.*

2.3. **Construction 2.** We review two closely related constructions in [6] which give an infinite family of symplectic tori $\{T_{p,q,m}\}_{p\geq 1}$ representing $q[F] + m[R]$ in $E(n)$ for each pair of positive integers $(q, m) \neq (1, 1)$ and $n \geq 2$.

Let $L \subset S^3$ be the Hopf link in Figure 2. For the link surgery gluing data

$$\mathfrak{D}_2 = \big(\{(\mu(A), \lambda(A)), (\lambda(B), -\mu(B))\}, \{(E(n-1), F), (E(1), F)\}\big),$$

we obtain $L(\mathfrak{D}_2) \cong E(n)$.

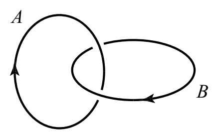

FIGURE 2. Hopf link $L = A \cup B$

Let $\beta = \beta_{p,q}$ be the family of q-strand braids given by Figure 1. A generic member of this family is shown in Figure 3 as the upper left part (inside the dotted rectangle) of the larger braid $\beta'_{p,q,m}$, for which the 3-component link $L \cup \hat{\beta}$ is $A \cup \hat{\beta}'_{p,q,m}$, where A is the closure of the axis of the braid $\beta'_{p,q,m}$ as well as one

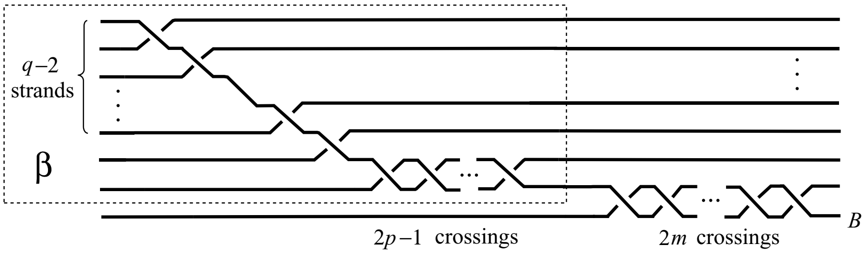

FIGURE 3. $(q+1)$-strand braid $\beta'_{p,q,m}$ with $p, m \geq 1$, $q \geq 2$

of the components of the Hopf link $L = A \cup B$. There is a symplectic form ω on $E(n)$ such that the torus $T_{p,q,m} = S^1 \times \hat{\beta} \subset [S^1 \times (S^3 \setminus \nu L)] \subset E(n)$ is symplectic with respect to ω. In fact, we could switch the roles of A and B (see Figure 4 and compare it with Figure 3) and obtain another torus which is also symplectic (with respect to a different symplectic form) in $E(n)$.

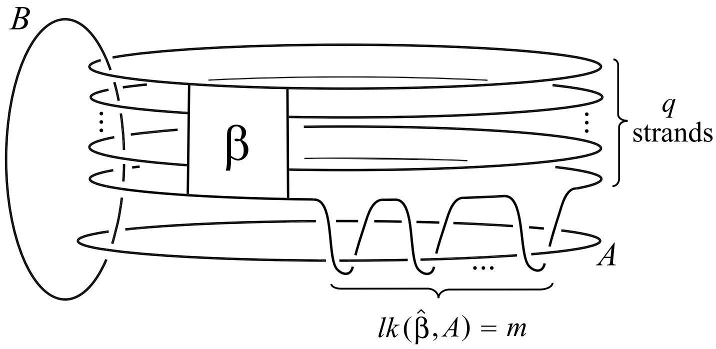

FIGURE 4. An alternative embedding of $\hat{\beta}$ into $S^3 \setminus \nu L$

Proposition 4. (*See* [6]). *Fix a pair of integers $q \geq 2$ and $m \geq 1$. Let $\beta = \beta_{p,q}$ be the q-strand braid in Figure* 1.
(i) *The torus $S^1 \times \hat{\beta} \subset E(n)$ embedded according to Figure* 3 *is symplectic with respect to a symplectic form on $E(n)$, and represents the homology class $q[F]+m[R]$.*
(ii) *The torus $S^1 \times \hat{\beta} \subset E(n)$ embedded according to Figure* 4 *represents $m[F]+q[R]$, and there is a symplectic form on $E(n)$ with respect to which this torus is symplectic.*

2.4. **Construction 3.** We review the construction in [8] which gives an infinite family of symplectic tori $\{T'_m\}_{m\geq 1}$ in $E(n)_p$ representing the homology class of the multiple fiber $[F_p]$ for each $n \geq 1$, $p \geq 2$; in particular an infinite family of symplectic tori in $E(1) \cong E(1)_p$ representing $[F]$.

As in previous constructions, we fix a cartesian product decomposition of a regular torus fiber $F = C_1 \times C_2$ in the elliptic surface $E(n)$. Let $L \subset S^3$ be the

Hopf link in Figure 2. For the link surgery gluing data

$$\begin{aligned}\mathfrak{D}_3 &= \big(\{(\mu(A),\lambda(A)),(\mu(B),\lambda(B)-p\mu(B))\},\\ &\quad \{(E(n),F),(S^1\times S^1\times S^2,F_2=S^1\times S^1\times\{\text{pt}\})\}\big),\end{aligned}$$

we shall denote $L(\mathfrak{D}_3)$ by $E(n)_p$. This notation is consistent with the existing literature as there is a diffeomorphism between our link surgery manifold $L(\mathfrak{D}_3)$ and the logarithmic transform $[E(n)\setminus\nu F]\cup_\varphi[T^2\times D^2]$ of multiplicity p, where the gluing diffeomorphism $\varphi: T^2\times\partial D^2\to\partial(\nu F)$ induces the linear map

$$\varphi_* = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -p\end{pmatrix}$$

between the first homology groups with respect to the obvious choice of bases.

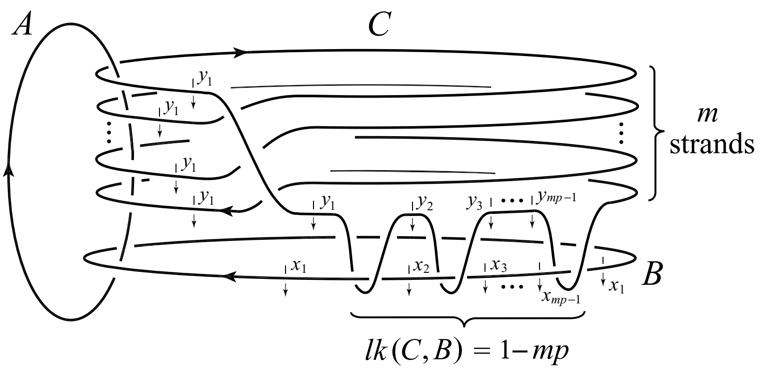

FIGURE 5. 3-component link $L_{m,p}=A\cup B\cup C$ in S^3

Proposition 5. (*See* [8] *and* [11]). *If $p\ge 1$, then $E(n)_p$ is a symplectic 4-manifold and $E(1)_p$ is diffeomorphic to $E(1)$. The diffeomorphism type of the logarithmic transform $E(n)_p$ does not depend on the choice of the gluing map φ. The homology class $[F_p]\in H_2(E(n)_p;\mathbb{Z})$ is primitive.*

Let T'_m be the torus $S^1\times C\subset[S^1\times(S^3\setminus\nu L)]\subset E(n)_p$, where the closed curve $C=C'_{m,p}\subset(S^3\setminus\nu L)$ is given by Figure 5. For every pair of integers $p\ge 2$ and $m\ge 1$, T'_m is a symplectic submanifold of $E(n)_p$ and we have $[T'_m]=[F_p]$ in $H_2(E(n)_p;\mathbb{Z})$.

2.5. **Construction 4.** We review the construction in [7] which gives an infinite family of symplectic tori $\{T'_{p,q}\}_{q\ge 1}$ with $[T'_{p,q}]=p[F]$ in $E(n)_K$ for any fibred knot K and any pair of integers $n,p\ge 1$.

Let K be a tame knot in S^3, and let M_K denote the 3-manifold that is the result of the 0-surgery on K. Choose a meridian circle $\mu=\mu(K)$ in M_K. Let $L\subset S^3$ be the Hopf link in Figure 2 as before. For the link surgery gluing data

$$\mathfrak{D}_4=\big(\{(\mu(A),\lambda(A)),(\lambda(B),-\mu(B))\},\{(E(n),F),(S^1\times M_K,T_\mu=S^1\times\mu)\}\big),$$

we shall denote $L(\mathfrak{D}_4)$ by $E(n)_K$. This notation is consistent with that of Fintushel and Stern in [10] as there is a diffeomorphism between our $L(\mathfrak{D}_4)$ and their fiber sum $E(n)_K=E(n)\#_{F=T_\mu}(S^1\times M_K)$.

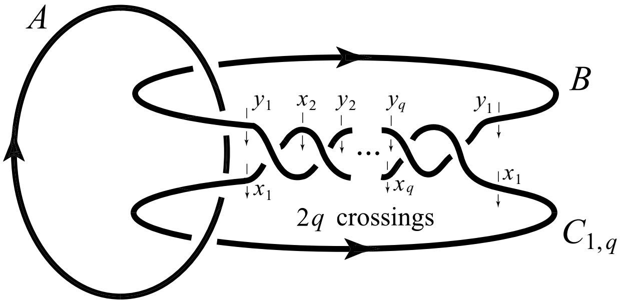

FIGURE 6. 3-component link $A \cup B \cup C_{1,q}$ in S^3

Proposition 6. (*See* [7]). *If $K \subset S^3$ is a fibred knot, then $E(n)_K$ is a symplectic 4-manifold. The homology class $[F] = [T_\mu] \in H_2(E(n)_K;\mathbb{Z})$ is primitive. Let $T'_{p,q}$ be the torus $S^1 \times C_{p,q} \subset [S^1 \times (S^3 \setminus \nu L)] \subset E(n)_K$, where the closed curve $C_{p,q} \subset (S^3 \setminus \nu L)$ is the (p,q)-cable of B, representing $\lambda(B)^p \mu(B)^q$ in $\pi_1(\partial(\nu B))$. See Figure 6 for the special case when $p=1$. If K is a fibred knot, then $T'_{p,q}$ is a symplectic submanifold of $E(n)_K$, and we have $[T'_{p,q}] = p[F]$ in $H_2(E(n)_K;\mathbb{Z})$.*

3. Fundamental groups of torus complements

In this section, we determine the fundamental groups of the complements of the symplectic tori constructed in the previous section. It turns out that these groups do not distinguish homologous tori we get in $E(n)$ or $E(n)_p$. On the other hand, the fundamental groups can be used to show that Construction 4 gives infinitely many homologous non-isotopic tori in $E(n)_K$, when K is a nontrivial fibred knot.

In the fundamental group computations that follow, we repeatedly use the well-known fact that the complement of a regular fiber in the elliptic surface $E(n)$ is simply-connected (see [11]). The main tools are Van Kampen's Theorem and the Wirtinger presentation of knot and link complements in S^3.

Proposition 7. *Let $T_{p,q}$ be a symplectic torus with $[T_{p,q}] = q[F]$ in $E(n)$ obtained by Construction 1. Then*

$$\pi_1(E(n) \setminus T_{p,q}) \;\cong\; \mathbb{Z}/q\mathbb{Z}$$

for all integers $n, p \geq 1$ and $q \geq 2$.

Proof. Using Van Kampen's Theorem, we compute that

$$\begin{aligned}
& \pi_1(E(n) \setminus \nu T_{p,q}) \\
\cong\; & \pi_1\Big(E(n) \setminus \nu F \bigcup_{F \times \partial D^2 = (S^1 \times \mu(A)) \times \overline{\lambda(A)}} S^1 \times \big(S^3 \setminus \nu(A \cup \hat{\beta}_{p,q})\big)\Big) \\
\cong\; & \pi_1(S^3 \setminus \hat{\beta}_{p,q}) / \langle \lambda(A) \rangle \\
\cong\; & \langle x_1, \ldots, x_{p+1}, y_1, \ldots, y_p \,|\, x_1 y_1 = y_1 x_2 = x_2 y_2 = \cdots = x_p y_p = y_p x_{p+1}, \\
& \qquad x_{p+1} = y_1,\; y_p = x_1,\; x_1 y_1^{q-1} = 1 \rangle \\
\cong\; & \langle x_1, y_1 \,|\, x_1 = y_1,\; x_1 y_1^{q-1} = 1 \rangle \;\cong\; \langle x_1 \,|\, x_1^q = 1 \rangle \;\cong\; \mathbb{Z}/q\mathbb{Z}\,.
\end{aligned}$$

Here, the second to last line is obtained using the Wirtinger presentation with generators given in Figure 1. Note that $x_{p+1} = y_1$ and $y_p = x_1$ when we close up the braid, and the generators around the upper strands are all the same because of the relations obtained from the crossings between those strands. Also note that $x_1 y_1^{q-1} = 1$ implies that x_1 and y_1 commute. □

Proposition 8. *Let $T_{p,q,m}$ be a symplectic torus with $[T_{p,q,m}] = q[F] + m[R]$ or $[T_{p,q,m}] = m[F] + q[R]$ in $E(n)$ obtained by Construction 2. If g is the greatest common divisor of m and q, then*

$$\pi_1(E(n) \setminus T_{p,q,m}) \;\cong\; \mathbb{Z}/g\mathbb{Z}$$

for all integers $m, p \geq 1$ and $n, q \geq 2$.

Proof. We will do the computation only for the first type of tori and the symmetry will give the same result for the second type of tori.

$$\begin{aligned}
& \pi_1(E(n) \setminus \nu T_{p,q,m}) \\
\cong\; & \pi_1\Bigg(E(n-1) \setminus \nu F \bigcup_{F \times \partial D^2 = (S^1 \times \mu(A)) \times \overline{\lambda(A)}} S^1 \times \big(S^3 \setminus \nu(A \cup B \cup \hat{\beta})\big) \\
& \qquad\qquad \bigcup_{(S^1 \times \lambda(B)) \times \mu(B) = F \times \partial D^2} E(1) \setminus \nu F\Bigg) \\
\cong\; & \pi_1(S^3 \setminus \hat{\beta}) / \langle \lambda(A), \lambda(B) \rangle\,.
\end{aligned}$$

Recall that the braid β is exactly the same as $\beta_{p,q}$ in Figure 1. So we can use the same Wirtinger presentation as we did in the proof of Proposition 7. Hence

$$\begin{aligned}
& \pi_1(E(n) \setminus T_{p,q,m}) \\
\cong\; & \langle x_1, \ldots, x_{p+1}, y_1, \ldots, y_p \,|\, x_1 y_1 = y_1 x_2 = x_2 y_2 = \cdots = x_p y_p = y_p x_{p+1}, \\
& \; x_{p+1} = y_1,\; y_p = x_1,\; x_1 y_1^{q-1} = 1,\; y_p^m = 1 \rangle \\
\cong\; & \langle x_1 \,|\, x_1^q = 1,\; x_1^m = 1 \rangle \;\cong\; \langle x_1 \,|\, x_1^g = 1 \rangle \;\cong\; \mathbb{Z}/g\mathbb{Z}\,,
\end{aligned}$$

where $g = \gcd(m, q)$. □

Proposition 9. *Let T'_m be a symplectic torus with $[T'_m] = [F_p]$ in $E(n)_p$ obtained by Construction 3. Then $\pi_1(E(n)_p \setminus T'_m)$ is trivial for all integers $m, n \geq 1$ and $p \geq 2$.*

Proof. Recall that $E(n)_p \setminus \nu T'_m = (E(n) \setminus \nu F) \cup (S^1 \times (S^3 \setminus \nu L_{m,p})) \cup (T^2 \times D^2)$, where the first boundary identification is $F \times \partial D^2 = (S^1 \times \mu(A)) \times \overline{\lambda(A)}$, and the second one is $S^1 \times \mu(B) \times (\lambda(B) - p\mu(B)) = S^1 \times S^1 \times \overline{\partial D^2}$.

$$\begin{aligned}
& \pi_1\big((E(n) \setminus \nu F) \cup (S^1 \times (S^3 \setminus \nu L_{m,p}))\big) \;\cong\; \pi_1\big(S^3 \setminus \nu(B \cup C)\big) / \langle \lambda(A) \rangle \\
\cong\; & \langle x_1, \ldots, x_{mp-1}, y_1, \ldots, y_{mp-1} \,| \\
& \quad y_1 x_1 = x_1 y_2 = y_2 x_2 = x_2 y_3 = \cdots = y_{mp-1} x_{mp-1} = x_{mp-1} y_1 \rangle / \langle \lambda(A) \rangle \\
\cong\; & \langle x_1, \ldots, x_{mp-1}, y_1, \ldots, y_{mp-1} \,| \\
& \quad y_1 x_1 = x_1 y_2 = y_2 x_2 = x_2 y_3 = \cdots = y_{mp-1} x_{mp-1} = x_{mp-1} y_1,\; y_1^m x_1 = 1 \rangle \\
\cong\; & \langle y_1 \,|\; \rangle\,,
\end{aligned}$$

where x_i and y_i are the generators of the Wirtinger presentation as given in Figure 5. Hence we have

$$\begin{aligned} \pi_1(E(n)_p \setminus \nu T'_m) &\cong \langle y_1 \mid \ \rangle *_{\left\{\substack{1=a \\ \mu(B)=b \\ \lambda(B)\mu(B)^{-p}=1}\right\}} \pi_1(T^2 \times D^2) \\ &\cong \langle y_1, b \mid y_1^{-m} = b,\ y_1^{1-mp}(y_1^{-m})^{-p} = 1\rangle \\ &\cong \langle y_1 \mid y_1 = 1\rangle \ \cong\ 1\,. \end{aligned}$$

Here, a and b denote the generators of $\pi_1(T^2 \times D^2)$. □

Proposition 10. *For any fibred knot K and any triple of integers $n, p, q \geq 1$, let $T'_{p,q}$ be a symplectic torus with $[T'_{p,q}] = p[F]$ in $E(n)_K$ obtained by Construction 4. Then*

$$\pi_1(E(n)_K \setminus T'_{p,q}) \ \cong\ \frac{\pi_1(S^3 \setminus K)}{\mu^p \lambda^q = 1} \ \cong\ \pi_1(M_K(p/q))\,,$$

where λ and μ represent a longitude and a meridian of K in the knot complement, respectively, and $M_K(p/q)$ is the three-manifold obtained by Dehn (p/q)-surgery on K.

Proof. Let us first work out the $p = 1$ case in detail. Using Van Kampen's Theorem, we compute that

$$\begin{aligned} &\pi_1(E(n)_K \setminus \nu T'_{1,q}) \\ \cong\ &\pi_1\Big(E(n) \setminus \nu F \bigcup_{F \times \partial D^2 = (S^1 \times \mu(A)) \times \overline{\lambda(A)}} S^1 \times \big(S^3 \setminus \nu(A \cup B \cup C_{1,q})\big) \\ &\qquad \bigcup_{S^1 \times \lambda(B) \times \mu(B) = S^1 \times \mu(K) \times \lambda(K)} S^1 \times (S^3 \setminus \nu K)\Big) \\ \cong\ &\Big(\pi_1\big(S^3 \setminus \nu(B \cup C_{1,q})\big)/\langle \lambda(A)\rangle\Big) *_{\left\{\substack{\lambda(B)=\mu(K) \\ \mu(B)=\lambda(K)}\right\}} \pi_1(S^3 \setminus \nu K) \\ \cong\ &\langle x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_q \mid \\ &\quad y_1 x_1 = x_2 y_1 = y_2 x_2 = x_3 y_2 = \cdots = y_q x_q = x_1 y_q = y_1 x_1\rangle/\langle y_1 x_1\rangle \\ &\quad *_{\left\{\substack{x_1 x_q x_{q-1} \cdots x_2 = \mu(K) \\ y_1 = \lambda(K)}\right\}} \pi_1(S^3 \setminus \nu K) \\ \cong\ &\langle x \rangle *_{\left\{\substack{x^q = \mu(K) \\ x^{-1} = \lambda(K)}\right\}} \pi_1(S^3 \setminus \nu K) \\ \cong\ &\pi_1(S^3 \setminus \nu K)/\langle \lambda(K)^q \mu(K)\rangle\,, \end{aligned}$$

where x_i and y_i denote the generators of the Wirtinger presentation as given in Figure 6.

More generally, it is not too difficult to show (again using Van Kampen's Theorem) that

$$\pi_1\big(S^3 \setminus \nu(B \cup C_{p,q})\big) \ \cong\ \frac{\langle \lambda(A)\rangle * \pi_1(\partial(\nu B))}{\lambda(A)^q = \lambda(B)^p \mu(B)^q}\,. \tag{3.1}$$

We refer to [18] for a proof of (3.1). The general formula can now be derived in the same way as before. □

Proving Theorem 1 now boils down to showing that the fundamental groups of Dehn surgery manifolds $\{M_K(p/q)\}_{q\geq 1}$ comprise an infinite collection for a fixed positive integer p and a fixed knot K. Recall from [21] that a tame knot K in S^3 is either hyperbolic, torus or satellite. For the first two cases, we can resort to the following lemma, the proof of which does not involve any gauge theory.

Lemma 11. *Let $\mathfrak{G}_p(K) = \{\pi_1(M_K(p/q)) \mid q \geq 1,\ \gcd(p,q) = 1\}$, where $M_K(p/q)$ is the three-manifold obtained by Dehn (p/q)-surgery on K. If K is a hyperbolic or a nontrivial torus knot in S^3, then $\mathfrak{G}_p(K)$ contains infinitely many mutually nonisomorphic groups for each integer $p \geq 1$.*

Proof. Throughout this proof, let p be a fixed positive integer. By a theorem of Thurston (see [21]), when K is a hyperbolic knot, for infinitely many q, $M_K(p/q)$ is hyperbolic. Moreover, infinitely many of these hyperbolic manifolds have different volumes since these volumes asymptotically approach (strictly from below) the volume of the complement of K as q goes to infinity (see [17] or [20]). Hence infinitely many of these hyperbolic manifolds are not mutually isometric. On the other hand, hyperbolic three-manifolds with finite volume and isomorphic fundamental groups are isometric by Mostow Rigidity Theorem [16].

Another proof for the hyperbolic K case goes as follows: For large enough q, the core of the surgery solid torus is the unique shortest closed geodesic in the hyperbolic manifold $M_K(p/q)$ (see p. 610 of [4]). Moreover, its length tends to zero as q goes to infinity (see [17]). Thus there are infinitely many nonisometric hyperbolic manifolds among $M_K(p/q)$'s, hence by Mostow Rigidity, infinitely many nonisomorphic groups among $\pi_1(M_K(p/q))$'s.

A third proof can be given by combining Mostow Rigidity with Theorem 1 and Lemma 2 in [1], which say that if K is a hyperbolic knot in S^3 and r, r' are two surgery coefficients yielding homeomorphic hyperbolic manifolds, then K is amphicheiral and $r' = -r$, except for finitely many surgery coefficients. Since we are only considering positive surgery coefficients, this implies that we have mutually nonhomeomorphic $M_K(p/q)$'s when q is larger than some constant that depends only on the knot K.

Finally, let K be an (m,n) torus knot with $|m|, |n| \geq 2$. Recall from [15] that if $p/q \neq mn$, then $M_K(p/q)$ is a Seifert fibred space with base orbifold of the form $S^2(|m|, |n|, |qmn - p|)$. Using this fact, we can show that the quotient of the fundamental group of $M_K(p/q)$ by its center is the $(|m|, |n|, |qmn - p|)$ triangle group (see p. 94 of [13]). Thus there are at most two q's which can give the same fundamental group. □

Now to prove Theorem 1 for *all* nontrivial fibred knots, we must use gauge theory. The following lemma applies to all nontrivial knots, but its proof is far less constructive than the proof of Lemma 11.

Lemma 12. *Let K be a nontrivial knot in S^3 and p be a fixed positive integer. The number of nonisomorphic groups in $\mathfrak{G}_p(K) = \{\pi_1(M_K(p/q)) \mid q \geq 1,\ \gcd(p,q) = 1\}$ is infinite.*

Proof. First let us recall the "pillowcase" picture of the $SU(2)$ representations of $\pi_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$. We choose a "meridian" m and a "longitude" l of this torus so that, up to conjugation, any representation $h : \pi_1(T^2) \to SU(2)$ sends $[m]$ to a diagonal matrix $Diag(e^{i\phi}, e^{-i\phi})$ and $[l]$ to $Diag(e^{i\theta}, e^{-i\theta})$. This is obvious if $h([m])$ and $h([l])$

are both $\pm I \in SU(2)$. Otherwise pick one of these commuting matrices that is not $\pm I$ and diagonalize it using a matrix A. Then change h by conjugation with A. This new representation h' sends $[m]$ and $[l]$ to commuting matrices, one of which is diagonal and not $\pm I$, hence the other one has to be diagonal too. Here, h uniquely determines $e^{i\phi}$ and $e^{i\theta}$ up to simultaneous conjugation. In other words, there is a one-to-one map from the space of conjugacy classes of $SU(2)$ representations of $\pi_1(T^2)$ to the "pillowcase" obtained by dividing $\mathbb{R}^2$ by the equivalence relation $(\phi, \theta) \sim (\phi + 2\pi j, \theta + 2\pi k)$ for any $(j,k) \in \mathbb{Z}^2$, and $(\phi,\theta) \sim (-\phi, -\theta)$. We will denote this quotient space by $\mathcal{R}(T^2)$.

From [14] we know that for a nontrivial knot K, the instanton Floer homology $HF_*(M_K(0))$ of the 0-surgery $M_K(0)$ is nontrivial, implying that the moduli space of flat connections on the $SO(3)$ bundle $P \to M_K(0)$ with nonzero w_2, denoted by $\mathfrak{M}^{SO(3)}_{\text{flat}}(M_K(0))$, is nonempty. On the other hand, this moduli space can be embedded inside the moduli space $\mathfrak{M}^{SU(2)}_{\text{flat}}(S^3 \setminus \nu K)$, which is thought of as the variety $\mathcal{R}(S^3 \setminus \nu K)$ of irreducible $SU(2)$ representations of $\pi_1(S^3 \setminus \nu K)$, as those representations that send the homotopy class of a longitude λ to $-I$ (see p. 149 of [2]). Therefore, we conclude that there exists a non-abelian representation $\rho : \pi_1(S^3 \setminus \nu K) \to SU(2)$ sending $[\lambda]$ to $-I$. Moreover, by using standard perturbation arguments (see [3], [12] and [19]), it is possible to show that not all such representations are isolated and in fact there is at least one such ρ which is a point on a path γ of $SU(2)$ representations of $\pi_1(S^3 \setminus \nu K)$ not all of which map $[\lambda]$ to $-I$.

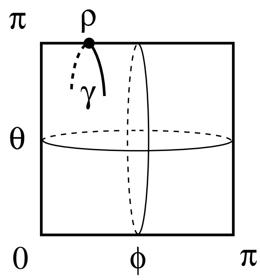

FIGURE 7. The image of path γ in $\mathcal{R}(T^2)$

From the inclusion $\partial \nu K \to S^3 \setminus \nu K$, we obtain an induced map of representation varieties $\mathcal{R}(S^3 \setminus \nu K) \to \mathcal{R}(T^2)$, as any $SU(2)$ representation of $\pi_1(S^3 \setminus \nu K)$ gives rise to an $SU(2)$ representation of the fundamental group of the boundary $\partial \nu K \cong T^2$. On the other hand, a representation of $\pi_1(S^3 \setminus \nu K)$ extends to a representation of $\pi_1(M_K(p/q))$ if and only if its induced representation h of $\pi_1(\partial \nu K)$, when considered inside the pillowcase $\mathcal{R}(T^2)$, is on the (quotient of the) line $\mathfrak{l}$ passing through the origin with slope $-p/q$. For a fixed p and as q goes to infinity, the intersection number of the path of representations γ in Figure 7 with this line $\mathfrak{l}$ goes to infinity, implying that the number of $SU(2)$ representations of $\pi_1(M_K(p/q))$ goes to infinity as q goes to infinity, and this finishes the proof. □

Acknowledgments

We would like to thank Hans U. Boden, Steven Boyer, Olivier Collin, Andrew Nicas and Stefano Vidussi for their indispensable help in writing the proofs of Lemmas 11 and 12. The figures were produced by the second author using Adobe® Illustrator® Version 10.

References

[1] S. A. Bleiler, C. D. Hodgson and J. R. Weeks: Cosmetic surgery on knots, in *Proceedings of the Kirbyfest* (*Berkeley, CA,* 1998), pp. 23–34, Geom. Topol. Monogr., 2, Geom. Topol. Publ., Coventry, 1999. math.GT/9911247.

[2] O. Collin: Non-trivial involutions on Floer Homology, *Math. Ann.* **326** (2003), 147–153.

[3] ———: Private communication.

[4] D. Cooper and M. Lackenby: Dehn surgery and negatively curved 3-manifolds, *J. Differential Geom.* **50** (1998), 591–624.

[5] T. Etgü and B. D. Park: Non-isotopic symplectic tori in the same homology class, *Trans. Amer. Math. Soc.* **356** (2004), 3739–3750.

[6] ———: Homologous non-isotopic symplectic tori in a $K3$-surface, *Commun. Contemp. Math.* (to appear), math.GT/0305201.

[7] ———: Homologous non-isotopic symplectic tori in homotopy rational elliptic surfaces, *Math. Proc. Cambridge Philos. Soc.* (to appear), math.GT/0307029.

[8] ———: Symplectic tori in rational elliptic surfaces, math.GT/0308276.

[9] T. Etgü, D. McKinnon and B. D. Park: Lagrangian tori in homotopy elliptic surfaces, *Trans. Amer. Math. Soc.* (to appear).

[10] R. Fintushel and R. J. Stern: Knots, links and 4-manifolds, *Invent. Math.* **134** (1998), 363–400.

[11] R. E. Gompf and A. I. Stipsicz: *4-Manifolds and Kirby Calculus.* Graduate Studies in Mathematics, 20. Amer. Math. Soc., Providence, RI, 1999.

[12] C. M. Herald: Legendrian cobordism and Chern-Simons theory on 3-manifolds with boundary, *Comm. Anal. Geom.* **2** (1994), 337–413.

[13] W. Jaco: *Lectures on Three-Manifold Topology.* CBMS Regional Conference Series in Mathematics, 43. Amer. Math. Soc., Providence, RI, 1980.

[14] P. B. Kronheimer and T. S. Mrowka: Witten's conjecture and Property P, *Geom. Topol.* **8** (2004), 295–310.

[15] L. Moser: Elementary surgery along a torus knot, *Pacific J. Math.* **38** (1971), 737–745.

[16] G. D. Mostow: *Strong Rigidity of Locally Symmetric Spaces.* Ann. of Math. Stud., 78. Princeton Univ. Press, Princeton, NJ, 1973.

[17] W. D. Neumann and D. Zagier: Volumes of hyperbolic three-manifolds, *Topology* **24** (1985), 307–332.

[18] B. D. Park and S. Vidussi: Homologous non-isotopic symplectic surfaces of higher genus, preprint.

[19] N. Saveliev: *Invariants for Homology 3-Spheres.* Encyclopaedia of Mathematical Sciences, 140. Low-Dimensional Topology, 1. Springer-Verlag, Berlin, 2002.

[20] W. P. Thurston: *The Geometry and Topology of Three-Manifolds.* Lecture notes, Princeton University, 1980. `http://www.msri.org/publications/books/gt3m/`

[21] ———: Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* **6** (1982), 357–381.

[22] S. Vidussi: Smooth structure of some symplectic surfaces, *Michigan Math. J.* **49** (2001), 325–330.

Department of Mathematics, Koç University, Rumelifeneri Yolu, 34450 Sariyer, Istanbul, Turkey

E-mail address: etgut@math.mcmaster.ca

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: bdpark@math.uwaterloo.ca