A NOTE ON THE GEOMETRY OF M^3 AND SYMPLECTIC STRUCTURES ON $S^1 \times M^3$

TOLGA ETGÜ

Abstract. We investigate the relationship between the geometry of a closed, oriented 3–manifold M and the symplectic structures on $S^1 \times M$. In most cases the existence of a symplectic structure on $S^1 \times M$ and Thurston's geometrization conjecture imply the existence of a geometric structure on M. This observation together with the existence of geometric structures on most 3-manifolds which fiber over the circle suggests a different approach to the problem of finding a fibration of a 3–manifold over the circle in case its product with the circle admits a symplectic structure.

1. Introduction

One of the most natural bridges between closed 3–manifolds and 4–manifolds is constructed by taking the product of 3–manifolds with the circle S^1. By construction, one direction on this bridge is more accessible than the other one, i.e. it is easier to understand the 4–manifold $S^1 \times M$ once some information is given about the 3–manifold M, but it often requires more work to recover M from $S^1 \times M$. For example, if M fibers over the circle, then it is not difficult to construct a symplectic form on $S^1 \times M$. The converse of this statement is still a conjecture. Starting from a Seifert fibration on M one can construct an elliptic fibration (in particular a complex structure) on $S^1 \times M$. In [1], it was proved (not without the help of Seiberg-Witten theory) that the converse of this statement is also true under additional assumptions.

In this note, we summarize the results achieved after our initial attemp to relate the symplectic geometry of a product manifold $S^1 \times M$ to the geometry of M. It is proved that if $S^1 \times M$ admits a symplectic structure, then the closed, oriented 3–manifold M admits a geometric structure under some mild conditions and assuming Thurston's geometrization conjecture.

The reason of our interest in this subject is mainly because of the following conjecture which was first stated by Taubes.

Conjecture. *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. Then M fibers over S^1.*

In [1], as a result of our discussion on this conjecture, especially for Lefschetz fibered, Seifert fibered and complex manifolds, we proved Conjecture T for nonhyperbolic geometric 3–manifolds. On the other hand, McCarthy and Vidussi independently proved that if M is a closed, oriented, 3–manifold such that $S^1 \times M$ admits a symplectic structure, then M can be uniquely decomposed as the connected sum of a prime manifold and a homology 3–sphere ([3] and [9]). Moreover, assuming the geometrization conjecture, they proved that M itself is prime.

2. From symplectic structures on $S^1 \times M$ to the geometry of M

As we mentioned in the introduction, if M is a closed, oriented, 3–manifold such that $S^1 \times M$ admits a symplectic structure, then M can be uniquely decomposed as the connected sum of a prime manifold and a homology 3–sphere which is S^3 assuming that the geometrization conjecture. Note that all prime 3–manifolds are irreducible except $S^1 \times S^2$ which is obviously geometric. Throughout this section, we will assume that M is a closed, oriented, irreducible 3–manifold such that $S^1 \times M$ admits a symplectic structure.

First of all, it can easily be seen that $b_1(M) = b_+(S^1 \times M)$ (cf. Lemma 2.1 in [1]). On the other hand, a symplectic form on a 4–manifold X induces a cohomology class which generates a positive definite subspace in $H^2(X;\mathbb{R})$, hence $b_+(X) \geq 1$ for such a manifold X. Therefore $b_1(M) > 0$ and this implies that M is sufficiently large and Haken.

Thurston proved the geometrization conjecture for Haken manifolds. So there is a canonical way to cut M into geometric pieces along embedded incompressible tori. In fact, if M is atoroidal, then there is no incompressible torus embedded in M and hence M itself is geometric. Actually M is hyperbolic. Even if M is not atoroidal, there may still not be an incompressible torus embedded in M, in which case M is Seifert fibered by the Torus Theorem. So the only remaining case is the one where M has an incompressible torus. It should be kept in mind that even in this case M may be geometric. For example, the total space of any torus bundle over the circle with Anosov monodromy is geometric of type Sol^3, but still admit incompressible tori which are fibers of the torus bundle.

Here are two statements that wrap up this discussion:

Proposition 1. *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. Then $M = N \# \Sigma$ for a prime manifold N and a homology sphere Σ. Moreover, if there is no incompressible torus in N, then N is either Seifert fibered or hyperbolic (in particular geometric) and otherwise (if there is an incompressible torus in N, then) N could be canonically cut into geometric pieces along tori.*

Remark 2. Thurston's geometrization conjecture implies that for any such decomposition Σ is in fact homeomorphic to S^3.

Proposition 3. *Let M be a prime, closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. If there is no incompressible torus embedded in M, then M is either $S^1 \times S^2$ or type $\mathbb{H}^2 \times \mathbb{E}^1$ or hyperbolic.*

3. Geometric structures on 3–manifolds that fiber over the circle

We consider closed, oriented 3–manifolds that fiber over the circle in three different classes according to the genus of the fiber.

If a closed, oriented 3–manifold M is the total space of a sphere bundle over the circle, then M is homeomorphic to $S^1 \times S^2$ hence geometric of type $\mathbb{E}^1 \times S^2$.

Torus bundles over the circle come in three types according to their monodromy. If the monodromy is periodic, i.e. the monodromy is an automorphism of the torus which has a positive power that is homotopic to the identity, then the total space is geometric of type $\mathbb{E}^3$. If the monodromy is reducible, in other words, if there is a simple closed curve preserved by the monodromy, then the total space is in fact the total space of a circle bundle over the torus, too and in particular it is geometric of type Nil^3. If the monodromy is neither periodic nor reducible, then it is Anosov and the total space is geometric of type Sol^3.

In case the fiber of a fibration over the circle is a closed, oriented hyperbolic surface, then we again have three cases according to the type of the monodromy. If the monodromy is periodic, then the total space is geometric of type $\mathbb{E}^1 \times \mathbb{H}^2$. A remarkable theorem of Thurston in [7] says that if the monodromy is pseudo-Anosov, i.e. neither periodic nor reducible, then the total space is hyperbolic.

The above discussion can be summarized as

Proposition 4. *Let M be a closed, oriented 3–manifold which is the total space of a surface bundle over the circle. Unless the fiber is hyperbolic and the monodromy is reducible, M has a geometric structure. If M is not geometric, then there is an incompressible torus embedded in M.*

4. Further remarks

Under the light of the discussion in this paper and [1], it can be seen that to prove Conjecture T one strategy could be to try to understand incompressible T^3 in (symplectic) 4–manifolds. It might also be interesting to prove that M itself is prime without assuming the geometrization conjecture. There is also another conjecture of Thurston, known as the virtual bundle conjecture, which seems to be related to our discussion. It claims that any hyperbolic 3–manifold admits a finite covering which fibers over the circle [8].

There are well-known results on the Seiberg-Witten invariants of 4–manifolds in terms of those of the 4–manifold(s) you get when you cut it along embedded tori. Note that the Seiberg-Witten theory of a 3–manifold M is parallel to that of the 4–manifold $S^1 \times M$. Another information which could be helpful is to understand the Seiberg-Witten invariants of 3–manifolds that fiber over the circle. These have been studied extensively in [2] by Ionel and Parker, in a rather indirect way, under their discussion of Gromov invariants of symplectic manifolds. Since the work of Taubes [4], [5], [6] relate Gromov invariants of symplectic 4–manifolds to the Seiberg-Witten invariants of them, we already have an understanding of the Seiberg-Witten invariants of fibered 3–manifolds. We should also note that in [9], Vidussi proved that if $S^1 \times M$ admits a symplectic structure, then the Alexander and Thurston norms of M are related in the same way as those of 3–manifolds which fiber over the circle.

References

[1] T. Etgü, *Lefschetz fibrations, complex structures and Seifert fibrations on $S^1 \times M$*, Algebr. Geom. Topol. **1** (2001), 469–489.

[2] E. Ionel and T. Parker, *The Gromov invariants of Ruan–Tian and Taubes*, Math. Res. Lett., **4** (1997), 521–532.

[3] J. McCarthy, *On the asphericity of a symplectic $M^3 \times S^1$*, Proc. Amer. Math. Soc. **129** (2001), 257–264.

[4] C. Taubes, SW $\Rightarrow$ Gr*: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc., **9**, (1996), 845–918.

[5] C. Taubes, Gr $\Rightarrow$ SW*: from pseudo-holomorphic curves to Seiberg-Witten solutions*, J. Differential Geom., **51** (1999), 203–334.

[6] C. Taubes, Gr = SW*: counting curves and connections*, J. Differential Geom., **52**, (1999), 453–609.

[7] W. Thurston, *Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle*, preprint, ArXiv:GT/9801045.

[8] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

[9] S. Vidussi, *The Alexander norm is smaller than the Thurston norm: a Seiberg–Witten proof*, Prépublication École Polytechnique 99–6.

Department of Mathematics, Koç University, Sariyer, 34450 Istanbul, Turkey
E-mail address: `tetgu@ku.edu.tr`