SYMPLETIC AND LAGRANGIAN SURFACES IN 4–MANIFOLDS

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ABSTRACT. This is a brief summary of recent examples of isotopically different symplectic and Lagrangian surfaces representing a fixed homology class in a simply-connected symplectic 4–manifold.

1. INTRODUCTION

In recent years there has been considerable activity that resulted in the construction of connected symplectic (and more recently Lagrangian) surfaces which are different up to smooth isotopy, but nonetheless represent the same homology class in a simply-connected symplectic 4–manifold. The first of such examples were given by R. Fintushel and R. Stern [10] who utilized their link surgery construction [9] to obtain non-isotopic tori representing certain multiples of the homology class of a generic fiber in the rational elliptic surface $E(1) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ and n-fold fiber sum $E(n)$ of $E(1)$. They distinguished innitely many of these tori by using the Seiberg-Witten invariants of double covers of the ambient 4–manifold branched along these tori. Based on their techniques many more examples were constructed in [3], [4], [5], [6], [21], [23] . All these examples are tori and the rst homologous non-isotopic surfaces of higher genera are constructed in [14] (in certain 4–manifolds which are not simply-connected, I. Smith constructed homologous, isotopically different higher genus surfaces in [19]).

The interest in this subject mainly stems from the ever tempting comparison between the complex and symplectic categories. A simply-connected complex surface always carries a Kähler form hence it is, in particular, a symplectic 4–manifold. It is a classical fact that in a simply-connected complex surface, two complex curves that represent the same homology class are smoothly isotopic. In fact there is reason to conjecture that in $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ with $n < 9$ (viewed as symplectic 4–manifolds) a similar uniqueness result holds. First of all, the techniques used to produce infinitely many non-isotopic homologous tori mentioned above do not work in this case since they depend on the existence of a symplectic torus of self-intersection zero, and $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ has not even a smoothly embedded such torus in it. Moreover, in [17], using Gromovs compactness theorem and extending the results of V. Shevchishin [16] and J.-C. Sikorav [18], B. Siebert and G. Tian proved that in $\mathbb{CP}^2$, any symplectic surface representing $d[\mathbb{CP}^1]$ for $d \leq 17$ is smoothly isotopic to the unique complex curve in that homology class. They have similar partial results for $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$, i.e. $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^1\times\mathbb{CP}^1$.

On the other hand, one might expect to have uniqueness for Lagrangian representatives as the Lagrangian condition is a closed one like the complex one and unlike the symplectic one, but this was proven to be false first by S. Vidussi in [22] and more counterexamples followed [11], [8], [12], [15].

This note is an attempt to summarize the existence results in the subject. We try to emphasize only the main tools and ideas, and refer elsewhere for the details. In the rest of this note the term 4–manifold refers to a compact, smooth and oriented 4–manifold. All surfaces are assumed to be closed and connected.

2. Preliminaries

2.1. **Symplectic 4–manifolds, symplectic and Lagrangian submanifolds.** A differential 2–form ω on a 4–manifold X is called a symplectic form if it is closed, i.e. $d\omega = 0$, and non-degenerate, i.e. $\omega\wedge\omega > 0$. The manifold X which carries such a symplectic form is said to admit a symplectic structure and the pair (X,ω) is called a symplectic manifold. We sometimes drop ω from the notation and call X a symplectic manifold.

A 2–dimensional submanifold Σ of X is called a symplectic submanifold of X if the symplectic form ω restricts to a volume form on Σ. If ω vanishes on Σ, then Σ is called a Lagrangian submanifold of X. An interesting observation of R. Gompf is that a Lagrangian submanifold Σ of (X,ω) is a symplectic submanifold of (X,ω') for a perturbation ω' of ω if Σ is homologically essential in X , i.e. $[\Sigma]\neq 0\in H_2(X;\mathbb{Z})$.

2.2. **Seiberg-Witten invariants.** Seiberg-Witten invariants of a smooth 4–manifold X are integer-valued invariants defined on the set of $Spin_c$-structures of X . Since this set could be identified with $H_2(X;\mathbb{Z})$ in the absence of 2-torsion and since the Seiberg-Witten invariants of only finitely many $Spin_c$-structures are nonzero when $b_+(X) > 1$, we can view the Seiberg-Witten invariants of X as a polynomial (or more precisely, as an element of the group ring $\mathbb{Z}[H_2(X;\mathbb{Z})]$), for example, when X is simply-connected with $b_+(X) > 1$. In that case, we write $SW_X = \sum_g a_g\cdot g$, where the Seiberg-Witten invariant of the $Spin_c$-structure that corresponds to the homology class g is equal to a_g.

2.3. **Fintushel-Stern link surgery.** In their seminal work [9], Fintushel and Stern introduced a method to construct new 4–manifolds out of old ones by replacing regular neighborhoods of certain tori by circle times knot (or more generally link) complements. This construction has many interesting features: under mild conditions the new manifold, X_K, is homeomorphic to the old one, X; if we use a fibred knot K and a symplectic X , then X_K is also symplectic, etc. Moreover, the symplectic structure on the surgery manifold is an extension of the natural symplectic form on $N_K = S^1\times(S^3-\nu K)$ as the total space of a fiber bundle over T^2 when K is bred. This allows us to construct symplectic or Lagrangian surfaces in N_K and consider them in X_K . Genus-1 case can be made particularly efcient by constructing tori T_C as circle times a second knot C (seen as a loop) in the complement of K. There are several advantages in this: one can read off the homology class $[T_C]$ simply from the linking number of C with K, and T_C is symplectic (resp. Lagrangian) whenever C is transverse to (resp. lies in) a Seifert surface, i.e. a fiber of the fibration of $S^3 - \nu K$ over S^1.

Another useful feature of Fintushel-Stern link surgery is the description of the Seiberg-Witten polynomial of X_K in terms of SW_X and the Alexander polynomial of K [9].

3. Summary of results

3.1. **Symplectic tori.** In this subsection we list the results on the existence of homologous non-isotopic symplectic tori in symplectic 4–manifolds. In fact, in all of the following cases we get an innite family of such examples. In some of these cases it is possible to conclude that the homologous tori constructed are inequivalent under diffeomorphisms of the ambient 4manifold. Moreover, some of them have complements with nonisomorphic fundamental groups (see [7]).

One may think that there are different conditions that the tori T and T_i should satisfy in the following statements, but in fact the prototype which satises all these conditions is a regular fiber in the elliptic surface $E(n)$. We will emphasize the interpretation of each result in this prototypical case.

Theorem 1 (Fintushel-Stern [10], [12])**.** *Let T be a c-embedded symplectic torus in a simply-connected 4–manifold X . The for each $q \geq 2$ there exists an infinite family of mutually non-isotopic symplectic tori representing the homology class $2q[T]$.*

A symplectically embedded torus is called c-embedded if it has self-intersection 0 and has a pair of simple closed curves which generate its first homology and bound self-intersection -1 disks in the ambient 4–manifold. Again, a regular fiber F in $E(n)$ is the prototype we should keep in mind. The theorem above implies the existence of an infinite family of mutually non-isotopic symplectic tori in each even multiple of the fiber class except for $2[F]$. The following result generalizes the elliptic surface case to every positive multiple of the fiber class for most $E(n)$.

Theorem 2 (Vidussi [21])**.** *For every $q \geq 1$ and $n \geq 3$, there exists an infinite family of mutually non-isotopic symplectic tori representing the homology class $q[F] \in H_2(E(n); \mathbb{Z})$.*

Theorem 3 (Etgü-Park [3])**.** *Let T be an essentially embedded symplectic 2-torus in a symplectic 4–manifold X with $b_2^+(X) > 1$. If $[T] \in H_2(X; \mathbb{Z})$ is primitive, $[T]^2 = 0$, and $H^1(X - \nu T; \mathbb{Z}) = 0$. Then for any integer $q \geq 3$, there exists an infinite family of mutually non-isotopic symplectic tori representing the homology class $q[T]$.*

The theorem above together with the corollary below almost ends the story in the case of positive multiples of the ber class in $E(n)$. There is only the ber class in $E(2)$ left, and most probably there is an infinite family of non-isotopic symplectic tori representing it, too.

Theorem 4 (Etgü-Park [6], Park-Vidussi [15])**.** *Let T be a symplectic 2-torus in a symplectic 4–manifold X. Suppose that $[T] \in H_2(X;\mathbb{Z})$ is primitive, $[T]^2 = 0$, and that T lies in a fishtail neighborhood. If $b_2^+(X) = 1$, then we also assume that the Seiberg-Witten invariant of $X - \nu T$ is nontrivial and a finite sum. Then there exists an infinite family of mutually non-isotopic symplectic tori in X_p representing the homology class $q[T_p] \in H_2(X_p;\mathbb{Z})$ for every $p > 1$ and $q \geq 1$.*

Here X_p stands for the 4–manifold obtained by applying a generalized logarithmic transform of multiplicity p to X along T. Using the fact that no logarithmic transform on $E(1)$ along a regular fibre changes its diffeomorphism type, we get the following corollary.

Corollary 5 (Etgü-Park [6], Park-Vidussi [15])**.** *For a suitable choice of a symplectic form on $E(1)$, there exists an infinite family of mutually non-isotopic symplectic tori representing $q[F]$ for each $q \geq 1$.*

Theorem 6 (Etgü-Park [4])**.** *Let T_i be a symplectically embedded 2-torus in a closed symplectic 4–manifold X_i with $b_2^+(X_i) > 1$, $[T_i]^2 = 0$ and $H^1(X_i - \nu T_i;\mathbb{Z}) = 0$, for each $i \in \{1, 2\}$, and let $X = X_1 \#_{T_1=T_2} X_2$ be the symplectic fiber sum of X_1 and X_2 along T_1 and T_2. If $[T]$ and $[R]$ are the homology classes of $T_1 = T_2$ and a rim torus in X , respectively, then for each pair of positive integers $(q,m) \neq (1,1)$ there exists an infinite family of mutually non-isotopic symplectic tori representing the homology class $q[T] + m[R] \in H_2(X;\mathbb{Z})$.*

As usual we can apply the theorem above to the elliptic surface case: For any pair of positive integers $(q,m) \neq (1,1)$ there exists an innite family of mutually non-isotopic symplectic tori representing the homology class $q[F] + m[R]$ of an elliptic surface $E(2)$, where $[R]$ is the homology class of a rim torus.

Theorem 7 (Vidussi [23], Etgü-Park [5])**.** *Let T be a symplectic 2-torus in a symplectic 4–manifold X with primitive homology class, $[T]^2 = 0$, and $H^1(X - \nu T;\mathbb{Z}) = 0$. Also assume that the Seiberg-Witten polynomial of $X - \nu T$ is a nontrivial finite sum in case $b_2^+(X) = 1$. Then there exists an infinite family of mutually non-isotopic symplectic tori in X_K representing $[T] \in H_2(X_K;\mathbb{Z})$ for any nontrivial fibred knot K in S^3.*

The examples that lead to the theorem above are particularly interesting, especially when $X = E(n)$, as it can be seen in the following result. Moreover, these tori are used to construct the first examples of homologous non-isotopic symplectic surface of higher genus in simply-connected 4–manifolds as explained in Subsection 3.3.

Theorem 8 (Etgü-Park [7]). *If K is a nontrivial fibred knot in S^3, then there exists an infinite family of homologous symplectic tori in $E(n)_K$ whose complements have mutually nonisomorphic fundamental groups.*

In contrast, the complements of the homologous (non-isotopic) tori constructed in [3], [4] and [6] have isomorphic fundamental groups [7].

3.2. **Lagrangian tori.**

Theorem 9 (Vidussi [22]). *Let K be a knot in S^3 which has a trefoil summand. There exists a primitive homology class $[R] \in H_2(E(n)_K; \mathbb{Z})$ such that $q[R]$ is represented by infinitely many mutually non-isotopic Lagrangian tori for each $q \geq 1$ and $n \geq 2$.*

In [1], D. Auckly showed that in case K is the sum of the trefoil knot with its reection, the complements of the non-isotopic homologous Lagrangian tori constructed by Vidussi have nonisomorphic fundamental groups.

Fintushel and Stern generalized Vidussi's result to obtain the following results.

Theorem 10 (Fintushel-Stern [11]). *Let X be a symplectic 4–manifold with $b_2^+(X) > 1$ which contains a symplectic torus of self-intersection 0 in a fishtail neighborhood. For each nontrivial fibred knot K, X_K contains an infinite family of nullhomologous mutually non-isotopic Lagrangian tori.*

Theorem 11 (Fintushel-Stern [11]). *Let X_i be a symplectic 4–manifold which contains a symplectic torus T_i of self-intersection 0 for each $i = 1, 2$ and let T_1 be embedded in a fishtail neighborhood. For each nontrivial fibred knot K, X_K contains an infinite family of mutually non-isotopic Lagrangian tori representing a primitive homology class, where $X = X_1 \#_{T_1=T_2} X_2$.*

The Lagrangian tori constructed to prove the following theorem are distinguished by the fundamental groups of their complements.

Theorem 12 (Etgü-McKinnon-Park [8]). *Let K be a fibred knot in S^3 whose Alexander polynomial $\Delta_K(t)$ has an irreducible factor none of whose roots is a root of unity and let X be a symplectic 4–manifold with a symplectically embedded torus T of self-intersection 0. If $\pi_1(X - \nu T) = 1$, then there are infinitely many nullhomologous non-isotopic Lagrangian tori in X_K.*

The Lagrangian tori in [15] can be distinguished by using Seiberg-Witten theory but not by the fundamental groups of their complements.

Theorem 13 (Park-Vidussi [15]). *Let T be a symplectic 2-torus in a symplectic 4–manifold X. Suppose that $[T] \in H_2(X; \mathbb{Z})$ is primitive, $[T]^2 = 0$, and that T lies in a fishtail neighborhood. If $b_2^+(X) = 1$, then we also assume that the Seiberg-Witten invariant of*

$X - \nu T$ is nontrivial and a finite sum. Then there exists an infinite family of mutually non-isotopic Lagrangian tori in X_p representing the homology class $q[T_p] \in H_2(X_p;\mathbb{Z})$ for every $p > 1$ and $q \geq 1$.

In the multiples of the ber class in $E(1)$ one can find Lagrangian tori which are equivalent under the diffeomorphisms of $E(1)$, but smoothly non-isotopic.

Corollary 14 (Park-Vidussi [15])**.** *For a suitable choice of a symplectic form on $E(1)$, there exists an innite family of mutually non-isotopic Lagrangian tori representing $q[F] \in H_2(E(1);\mathbb{Z})$ for each $q \geq 1$.*

3.3. **Symplectic surfaces of higher genus.** The only examples of higher genus homologous mutually non-isotopic surfaces symplectically embedded in a simply-connected 4–manifold are constructed by Park, Poddar and Vidussi. In non-simply-connected 4–manifolds, we also have the examples of Smith [19].

Theorem 15 (Park-Poddar-Vidussi [14])**.** *For each integer $q \geq 2$, there exists a simply-connected symplectic 4–manifold which contains infinitely many homologous mutually non-isotopic symplectic surfaces of genus g.*

4. Constructions of sympectic surfaces

There are several different constructions of non-isotopic homologous symplectic surfaces in simply-connected 4–manifolds. Even though these have certain common features, each one of them has a different aspect. We start with the general framework and mention some of the differences along the way. Different techniques used to distinguish homologous tori are explained in Section 6.

4.1. **Constructions of symplectic tori.** We start with a closed symplectic 4–manifold X with a self-intersection 0 symplectic torus T embedded in it. We almost always assume that the homology class $[T]$ is primitive in $H_2(X;\mathbb{Z})$ and $H^1(X - \nu T;\mathbb{Z}) = 0$. For technical reasons in Seiberg-Witten theory, in case $b_2^+(X) = 1$ it may also be necessary to assume that $SW_{X-\nu T}$ is non-trivial and a finite sum. The elliptic surface $E(n)$ and a generic fiber of it are the ideal candidates frequently used for the roles of X and T, respectively.

At another part of this construction site we have a braid β . In the simplest case (e.g. as in [3]), the closure B of β is a knot and we consider its closure inside the complement of its axis A. Since A is an unknot, its complement is $S^1 \times D^2$ which trivially fibers over the circle. A fiber of this fibration is the obvious disk that is bounded by A. When we multiply this whole picture by another circle we obtain a disk bundle over a torus $N_A = S^1 \times (S^3 - \nu A)$ which admits a natural symplectic form which is essentially the sum of symplectic forms on the fiber and the base [20]. Hence the simple fact that B is transverse to the disk bounded by A implies that $T_B = S^1 \times B$ is a symplectic torus in N_A. To get more general results we may use a braid which closes to a multi-component link and

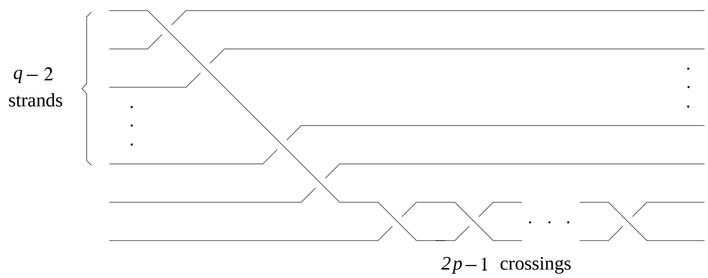

FIGURE 1. Braid $\beta_{p,q}$ used in [3]

consider one of these components in the complement of the rest of them union the axis A (see [4],[5],[6]) and we get a symplectic torus after crossing the whole picture by a circle.

Once we obtain a symplectic torus as above inside, for example, N_A which is homologically the same as a regular neighborhood of the self-intersection 0 torus T in X, we take such a regular neighborhood out of X and glue N_A instead of it to obtain a closed 4–manifold. In the more complicated cases, since the initial symplectic torus we construct is in a 4–manifold, say N_L, with more than one boundary component, gluing N_L to $X - \nu T$ produces a 4–manifold with boundary components diffeomorphic to T^3. Depending on the case, we may choose to close each of these boundary components by $T^2 \times D^2$, $E(1) - \nu F$ or $S^1 \times (S^3 \nu K)$ for a specific fibred knot K. No matter how we close the boundary, we always have to make sure that we respect the symplectic structures on all pieces. That way we obtain symplectic tori in the targeted closed 4–manifold.

The homology class of the resulting torus depends on the linking numbers of the components of the closure of the braid β. The way we glue our pieces makes some of this linking data irrelevant, hence one can obtain symplectic tori representing a fixed homology class in many different ways. Of course this is not enough to claim that these homologous tori are non-isotopic and the methods used to distinguish them is the subject of Section 6.

4.2. **Luttinger surgery and singular plane curves.** A small neighborhood of a Lagrangian torus in a symplectic 4–manifold can be removed and replaced by a standard $T^2 \times D^2$ symplectically. In some cases, i.e. when the gluing is made using certain diffeomorphisms, this operation can be considered as a 4–dimensional Dehn surgery, called Luttinger surgery. B. Moishezon's construction of singular curves in $\mathbb{CP}^2$ is discussed through the eyes of Luttinger surgery in [2]. Both this construction and the construction of symplectic tori summarized above can be seen as braiding of many copies of disjoint symplectic submanifolds and they are related based on the fact that any symplectic 4–manifold is a cover of $\mathbb{CP}^2$ branched along a singular curve.

4.3. **Construction of symplectic surfaces of higher genera.** The first examples of homologous non-isotopic higher genus symplectic surfaces in a simply-connected 4–manifold are given in [14]. This construction uses a certain class of homologous symplectic tori, constructed in [5] (also see [23]) in the knot surgery manifold $E(2)_K$, which were shown to have complements with non-isomorphic fundamental groups [7]. The diversity encoded in these fundamental groups survives (at least when K is a hyperbolic knot) if one uses Parks doubling construction [13] and two copies of a torus representing g times the fiber class in $E(2)_K$ to obtain a symplectic surface of genus $g + 1$ in the double of $E(2)_K$, i.e. the fiber sum of two copies of $E(2)_K$ along the surface Σ obtained from the union of a regular fiber away from the surgery region in $E(2)$ and a pseudo-section which is a minimal genus Seifert surface of K capped off by the punctured section of $E(2)$. Since the torus in $E(2)_K$ intersects with Σ at g points, after the fiber sum we have a g-punctured torus in each piece and the union of these two tori is the desired genus $g + 1$ surface.

5. Construction of Lagrangian tori

5.1. **Constructions of nullhomologous Lagrangian tori.** The rst examples of homologous non-isotopic Lagrangian tori in a 4–manifold were constructed by Vidussi in [22]. These examples are in knot surgery manifolds like $E(2)_K$. The way they are constructed is very similar to the construction of symplectic tori above. The main difference which makes the tori Lagrangian, as opposed to symplectic, is that the simple closed curves that give these tori after multiplication by the trivial S^1 factor in $S^1 \times (S^3 - \nu K)$ are not transverse to but embedded in a fiber of the fibration of the complement of the fibred knot K over the circle. It should be noted that these tori are necessarily nullhomologous.

After these first examples, and inspired by them, many other examples are given in more general classes of 4–manifolds (see [11],[8], [12], [15]).

5.2. **Circle sum construction and essential tori.** In [11], generalizing a technique used by Vidussi [22], Fintushel and Stern developed a tool called circle sum that enables one to construct homologically essential Lagrangian tori by using the nullhomologous Lagrangian tori above. This is done by xing an essential Lagrangian torus $T_\mu = S^1 \times \mu$ in the gluing region of the knot surgery and adding this loop μ to different simple closed curves in the Seifert surface in $S^3 - \nu K$ before multiplying by the trivial circle factor as above. These new tori represent the same homology class as T_μ.

6. Tools to distinguish homologous symplectic surfaces

6.1. **Seiberg-Witten invariants of branched covers.** One way to distinguish the homologous symplectic tori is to consider the double branched covers of the ambient manifold branched along the tori. This is exactly how it is done in [10]. These covers are distinguished by using their Seiberg-Witten invariants which can be calculated after observing that the covers are link surgery manifolds, and the diffeomorphism type of the cover depends only on the isotopy class of the branch tori. One minor disadvantage of this technique is the restriction to tori which represent even homology classes.

6.2. **Seiberg-Witten invariants of fiber sums.** To distinguish the homologous symplectic tori constructed in the general framework mentioned in this note one can also use the Seiberg-Witten invariants of the fiber sum of the ambient 4–manifold with $E(1)$ along the tori on one side and a generic fiber on the $E(1)$ side (see [21], [3], [4], [23], [5], [6]). The effectiveness of this method lies within the fact that such a fiber sum could also be interpreted as a link surgery manifold. Hence the calculation of the Seiberg-Witten invariants boils down to the calculation of the multivariable Alexander polynomial of certain links. Once the difference of these invariants is established, the non-isotopy of the tori could be claimed as the diffeomorphism type of the fiber sum depends only on the isotopy type of these tori.

6.3. **Fundamental group of the complement.** As another method of distinguishing homologous symplectic surfaces one can try to use the fundamental group of the complement (see [7] and [14]). In the examples on which this method is applied one needs only the Wirtinger presentation of link components and the Seifert-van Kampen theorem to calculate the fundamental groups. In general, complements of homologous non-isotopic symplectic tori do not necessarily have nonisomorphic fundamental groups (e.g. the examples in [3], [4], [6]). Even if they do, showing that two groups are not isomorphic is usually a challenging task. As it is demonstrated in [7], techniques from the theory of hyperbolic 3–manifolds or gauge theory (through its relationship with the $SU(2)$-representations of the fundamental groups of 3–manifolds) can be used to show the existence of innitely many homologous symplectic tori the complements of which have nonisomorphic fundamental groups. Moreover, the non-isotopy of higher genus surfaces obtained in [14] is established by extending the results in [7]. The fundamental group of the complement is especially signicant in the higher genus case since it is the only known way to distinguish homologous symplectic surfaces of genus > 1.

7. TOOLS TO DISTINGUISH HOMOLOGOUS LAGRANGIAN SURAFCES

7.1. **Seiberg-Witten invariants of fiber sums.** Seiberg-Witten invariants of the fiber sum of $E(2)_K$ and $E(1)$ along Lagrangian tori and a regular fiber is used to distinguish the tori in [22]. This technique is similar to the analogous computations for symplectic tori.

7.2. **Lagrangian framing defect.** Fintushel and Stern define the Lagrangian framing defect (see [11], [12]) of a nullhomologous Lagrangian torus and show that it is not only particularly easy to calculate for the tori constructed along the ideas in [22] but it is in fact a smooth isotopy invariant by demonstrating its relationship with Seiberg-Witten theory.

This defect is an integer invariant which is essentially the difference between the nullhomologous and Lagrangian framings of the torus. Another nice feature of this invariant is that it is compatible with the circle sum construction, i.e. it can be used to distinguished essential tori obtained by circle sum.

7.3. **Fundamental group of the complement.** As it is demonstrated in [1] and [8], certain classes of homologous Lagrangian tori have complements with nonisomorphic fundamental groups. Again, different techniques can be used to show the diversity of these groups. For example, in [8], the Alexander ideals of fundamental groups of certain homologous Lagrangian tori in specic knot surgery manifolds lead to an unexpected connection with algebraic number theory using which one can prove that the groups in question are different at least when the Alexander polynomial of the knot involved in the surgery has a root which is not a root of unity.

8. CONCLUSION

Although there has been extensive research on homologous and non-isotopic symplectic or Lagrangian surfaces in 4–manifolds there are still more questions waiting to be answered in this area. For example, there is the property of zero self-intersection that is shared by all the surfaces constructed (symplectic or Lagrangian, torus or higher genus). This may seem to be a technical point, yet there are no examples in non-zero self-intersection homology classes. Also the variety in genus-1 examples is yet to be seen in higher genus. Of course this could be explained by the ineffectiveness of Seiberg-Witten theory, as it is used in this subject, on distinguishing higher genus surfaces up to isotopy. Note that, since the normal bundle of a Lagrangian surface is isomorphic to its cotangent bundle, the self-intersection is determined by the genus, hence higher genus Lagrangian surfaces necessarily represent homology classes of positive square.

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