Global Solution of Coupled Kuramoto–Sivashinsky and Ginzburg–Landau Equations

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To Olga Aleksandrovna Ladyzhenskaya
on occasion of her 80th birthday

Introduction

We study the initial boundary-value problems for the coupled Kuramoto–Sivashinsky and Ginzburg–Landau (KSGL) equations arising in modelling of the Marangoni convection. We begin with the problem

$$\partial_t A - \mu A - \partial_x^2 A + k|A|^2 A = Ah, \quad x \in (0,l),\ t > 0, \tag{A}$$

$$\partial_t h + m\partial_x^2 h + \nu\partial_x^4 h = \alpha\partial_x^2(|A|^2), \quad x \in (0,l),\ t > 0, \tag{H}$$

$$A(x,0) = A_0(x), \quad h(x,0) = h_0(x), \quad x \in (0,l),$$

$$A(0,t) = A(l,t) = \vec{0}, \quad t > 0,$$

$$h(0,t) = h(l,t) = \partial_x^2 h(0,t) = \partial_x^2 h(l,t) = 0, \quad t > 0,$$

where $\vec{0}$ is the zero vector in R^N, μ, k, m, ν and α are given numbers, $A(x,t) = (A_1(x,t), \ldots, A_N(x,t))$ is the unknown vector-valued function, $h(x,t)$ is the unknown scalar function, $A_0(x)$ and $h_0(x)$ are given functions.

In the case where A is a complex-valued function, i.e., $N = 2$, the system (A), (H) was derived in [**1**] as a simplified model of the surface tension–driven

Marangoni convection. A numerical simulation of the KSGL-equations wiht periodic boundary conditions was given in [**2**]. The global unique solvability of the periodic initial boundary-value problem in the case $\mu = \nu = m = 1$ was studied in [**3**]–[**5**]. In particular, the global existence and uniqueness of a regular solution of the periodic initial boundary-value problem was established in [**3**] under the condition $0 < \alpha < 2$. As was shown in [**4**], the problem has a unique solution if $\alpha < 0$. For $\alpha > 0$ the global existence and uniqueness of a weak solution was proved in [**5**].

In this paper, we establish the global unique solvability of the Dirichlet and periodic initial boundary-value problem for the KSGL-equations without the above restrictions, i.e., for all real m, μ, α and for all $\nu > 0$, $k > 0$. In the case of Dirichlet boundary conditions, we show that if $\alpha \geqslant 0$, $\nu\lambda_1 > m$ or $\alpha < 0$, $\nu\lambda_1 - m + \alpha/k > 0$ (hereinafter, λ_1 is the first eigenvalue of the problem $-\psi'' = \lambda\psi$ for $x \in (0,l)$, $\psi(0) = \psi(l) = 0$), then the semigroup generated by this initial boundary-value problem has a minimal global attractor in the phase space $L_2(0,l) \times H^{-1}(0,l)$.

Introduce the notation. For

$$f(x) = (f_1(x), \dots, f_N(x)), \quad g(x) = (g_1(x), \dots, g_N(x))$$

we set

$$\|f\| \equiv \Big(\int_\Omega |f(x)|^2 dx \Big)^{1/2}, \quad (f,g) \equiv \int_\Omega \langle f(x), g(x) \rangle dx,$$

where $\langle f, g \rangle = f_1 g_1 + f_2 g_2 + \dots + f_N g_N$ and $|f|^2 = \langle f, f \rangle$. For real-valued functions $f(x)$ and $g(x)$ we set

$$\|f\| \equiv \Big(\int_0^l f^2(x) dx \Big)^{1/2}, \quad (f,g) \equiv \int_\Omega f(x) g(x) dx.$$

To simplify the notation, we also denote by $(\cdot,\cdot)$ the duality pairing between elements of $H_0^1(\Omega)$ and elements of $H^{-1}(\Omega)$.

We often use the Cauchy inequality

$$|ab| \leqslant \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \quad a, b \in R, \ \varepsilon > 0,$$

the Poincare–Friedrichs inequality

$$\|f\|^2_{L_2(0,l)} \leqslant \lambda_1 \|f'\|^2, \quad \lambda_1 = \frac{l^2}{\pi^2},$$

and the following well-known inequalities:

$$\|f\|^2_{L_\infty(0,l)} \leqslant l \|f'\|^2, \tag{I$_0$}$$

$$\|f\|^2_{L_\infty(0,l)} \leqslant 2\|f\| \, \|f'\| \tag{I$_1$}$$

for functions in $H_0^1(0,l)$.

1. Existence and Uniqueness

Let P^2 be the inverse of the operator $L = -\dfrac{d^2}{dx^2}$ with the domain $D(L) = H^2(0,l) \cap H_0^1(0,l)$. Applying the operator P^2 to both sides of Eq. (H), we arrive at the problem

$$\partial_t A - \mu A - \partial_x^2 A + k|A|^2 A = Ah, \quad x \in (0,l),\ t > 0, \tag{1.1}$$

$$P^2 \partial_t h - mh - \nu \partial_x^2 h = -\alpha |A|^2, \quad x \in (0,l),\ t > 0, \tag{1.2}$$

$$A(x,0) = A_0(x), \quad h(x,0) = h_0(x), \quad x \in (0,l), \tag{1.3}$$

$$A(0,t) = A(l,t) = \vec{0}, \quad h(0,t) = h(l,t) = 0, \quad t > 0, \tag{1.4}$$

Multiplying Eq. (1.1) by A and Eq. (1.2) by h, we obtain the equalities

$$\frac{1}{2}\frac{d}{dt}\|A(t)\|^2 - \mu\|A(t)\|^2 + \|\partial_x A(t)\|^2 + k\int_0^l |A(x,t)|^4 dx = \int_0^l |A(x,t)|^2 h(x,t)dx, \tag{1.5$_1$}$$

$$\frac{1}{2}\frac{d}{dt}\|Ph(t)\|^2 - m\|h(t)\|^2 + \nu\|\partial_x h(t)\|^2 = -\alpha\int_0^l |A(x,t)|^2 h(x,t)dx. \tag{1.5$_2$}$$

Adding (1.5$_1$) and (1.5$_2$), we find

$$\frac{1}{2}\frac{d}{dt}[\|A(t)\|^2 + \|Ph(t)\|^2] + \|\partial_x A(t)\|^2 + k\int_0^l |A(x,t)|^4 dx + \nu\|\partial_x h(t)\|^2 - m\|h(t)\|^2 = \mu\|A(t)\|^2 + (1-\alpha)\int_0^l |A(x,t)|^2 h(x,t)dx. \tag{1.6}$$

Using the inequalities

$$\mu\|A(t)\|^2 \leqslant \frac{k}{4}\int_0^l |A(x,t)|^4 dx + \frac{l\mu^2}{k},$$

$$|(1-\alpha)\int_0^l |A(x,t)|^2 h(x,t)dx| \leqslant \frac{k}{4}\int_0^l |A(x,t)|^4 dx + \frac{(1-\alpha)^2}{k}\int_0^l h(x,t)^2 dx$$

in (1.6), we get

$$\frac{1}{2}\frac{d}{dt}\left[\|A(t)\|^2+\|Ph(t)\|^2\right]+\|\partial_x A(t)\|^2+\frac{k}{2}\int\limits_0^l |A(x,t)|^4dx+\nu\|\partial_x h(t)\|^2$$

$$\leqslant [m+(1-\alpha)^2/k]\|h(t)\|^2+\frac{l\mu^2}{k}. \tag{1.7}$$

By the Schwarz inequality and Cauchy inequality, we have

$$\|h\|^2=(P^{-1}h,Ph)\leqslant\|P^{-1}h\|\,\|Ph\|\leqslant\varepsilon\|\partial_x h\|^2+C_1(\varepsilon)\|Ph\|^2. \tag{I$_2$}$$

Using the last inequality with $\varepsilon=\nu/[2m+2(1-\alpha)^2/k]$ in (1.7), we get

$$\frac{1}{2}\frac{d}{dt}\left[\|A(t)\|^2+\|Ph(t)\|^2\right]+\|\partial_x A(t)\|^2+\frac{k}{2}\int\limits_0^l |A(x,t)|^4dx$$

$$+\frac{\nu}{2}\|\partial_x h(t)\|^2\leqslant\frac{l\mu^2}{k}+\widetilde{C}_1(\nu)\|Ph(t)\|^2. \tag{1.8}$$

From (1.8) it follows that

$$\|A(t)\|^2+\|Ph(t)\|^2\leqslant D_1(t)\quad\forall t\in R^+, \tag{1.9}$$

where

$$D_1(t)=\left[\|A_0\|^2+\|Ph_0\|^2+t\frac{2}{k}l\mu^2\right]e^{2\widetilde{C}_1(\nu)t}$$

and

$$\int\limits_0^t\|\partial_x A(s)\|^2ds,\ \int\limits_0^t\|\partial_x h(s)\|^2ds,\ \int\limits_0^l\int\limits_0^t|A(x,s)|^4ds\leqslant C(t)\quad\forall t\in R^+. \tag{1.10}$$

Hereinafter, $C(t)$ is a positive continuous function on $[0,\infty)$.

On the basis of the estimates (1.9) and (1.10), we can use the standard Faedo–Galyorkin method to prove the existence of a global weak solution $[A,h]$ of the problem (1.1)–(1.4) such that

$$A\in C(0,T;L_2(0,l))\cap L_2(0,T;H_0^1(0,l)), \tag{1.11}$$

$$h\in C(0,T;H^{-1}(0,l))\cap L_2(0,T;H_0^1(0,l)). \tag{1.12}$$

Let us show that the problem (1.1)–(1.4) has at most one solution.

Let $[A,h]$ be a weak solution of the problem (1.1)–(1.4) with the initial data $[A_0,h_0]$, and let $[\widetilde{A},\widetilde{u}]$ be a weak solution of the same problem but with the initial data $[\widetilde{A}_0,\widetilde{h}_0]$. Then $[a,H]=[A-\widetilde{A},h-\widetilde{h}]$ is a solution of the problem

$$\partial_t a-\mu a-\partial_x^2 a+k|A|^2A-k|\widetilde{A}|^2\widetilde{A}=Ah-\widetilde{A}\widetilde{h},\quad x\in(0,l),\ t>0, \tag{1.13}$$

$$P^2\partial_t H-mH-\nu\partial_x^2 H=-\alpha|A|^2+\alpha|\widetilde{A}|^2,\quad x\in(0,l),\ t>0, \tag{1.14}$$

where

$$a(x,0) = A_0(x) - \widetilde{A}_0(x), \quad H(x,0) = h_0(x) - \widetilde{h}_0(x), \quad x \in (0,l),$$
$$a(0,t) = a(l,t) = \vec{0}, \quad H(0,t) = H(l,t) = 0, \quad x \in (0,l),\ t > 0.$$

Since

$$Ah - \widetilde{A}\widetilde{h} = Ah - \widetilde{A}h + \widetilde{A}h - \widetilde{A}\widetilde{h} = ah + \widetilde{A}H$$

and

$$\begin{aligned} -\alpha|A|^2 + \alpha|\widetilde{A}|^2 &= -\alpha(\langle A, A\rangle - \langle \widetilde{A}, A\rangle + \langle \widetilde{A}, A\rangle - \langle \widetilde{A}, \widetilde{A}\rangle) \\ &= -\alpha\langle a, A\rangle - \alpha\langle \widetilde{A}, a\rangle, \end{aligned}$$

it follows from (1.13) and (1.14) that $[a, H]$ satisfies the system

$$\partial_t a - \mu a - \partial_x^2 a + k|A|^2 A - k|\widetilde{A}|^2 \widetilde{A} = ah + \widetilde{A}H, \tag{1.15}$$
$$P^2 \partial_t H - mH - \nu \partial_x^2 H = -\alpha\langle \widetilde{A}, a\rangle - \alpha\langle a, A\rangle. \tag{1.16}$$

Multiplying (1.15) by a, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|a(t)\|^2 - \mu\|a(t)\|^2 + \|\partial_x a(t)\|^2 \\ &+ k\int_0^l \langle |A(x,t)|^2 A(x,t) - |\widetilde{A}(x,t)|^2 \widetilde{A}(x,t), A(x,t) - \widetilde{A}(x,t)\rangle dx \\ &= \int_0^l |a(x,t)|^2 h(x,t)dx + \int_0^l \langle \widetilde{A}(x,t), a(x,t)\rangle H(x,t)dx. \end{aligned} \tag{1.17}$$

Using the inequality (I_1), we can estimate the first integral on the right-hand side of (1.17) as follows:

$$\begin{aligned} &\left|\int_0^l |a(x,t)|^2 h(x,t)dx\right| \leqslant \max_{x\in[0,l]} |a(x,t)|^2 \int_0^l |h(x,t)|dx \\ &\leqslant \sqrt{l} \max_{x\in[0,l]} |a(x,t)|^2 \|h(t)\| \leqslant 2\sqrt{l}\|a(t)\|\,\|\partial_x a(t)\|\,\|h(t)\| \\ &\leqslant \varepsilon_1 \|\partial_x a(t)\|^2 + \frac{l}{\varepsilon_1}\|a(t)\|^2 \|h(t)\|^2. \end{aligned} \tag{1.18}$$

Using (I_0) and (I_2), we can estimate the second term on the right-hand side of (1.17) as follows:

$$\left|\int_0^l \langle\widetilde{A}(x,t), a(x,t)\rangle H(x,t))dx\right| \leqslant \int_0^l |\widetilde{A}(x,t)||a(x,t)||H(x,t)|dx$$
$$\leqslant \max_{x\in[0,l]} |\widetilde{A}(x,t)|\|a(t)\|\,\|H(t)\| \leqslant \frac{1}{2}\max_{x\in[0,l]} |\widetilde{A}(x,t)|^2\|a(t)\|^2 + \frac{1}{2}\|H(t)\|^2$$
$$\leqslant \frac{l}{2}\|\partial_x\widetilde{A}(t)\|^2\|a(t)\|^2 + \varepsilon_2\|\partial_x H(t)\|^2 + C_1(\varepsilon_2)\|PH(t)\|^2. \tag{1.19}$$

Taking into account that the expression $(|A(t)|^2A(t) - |\widetilde{A}(t)|^2\widetilde{A}(t), A(t) - \widetilde{A}(t))$ is nonnegative and using (1.18) and (1.19), from (1.17) we deduce the inequality

$$\frac{1}{2}\frac{d}{dt}\|a(t)\|^2 + (1-\varepsilon_1)\|\partial_x a(t)\|^2 \leqslant \mu\|a(t)\|^2 + \frac{l}{\varepsilon_1}\|h(t)\|^2\|a(t)\|^2$$
$$+ \frac{l}{2}\|\partial_x\widetilde{A}(t)\|^2\|a(t)\|^2 + \varepsilon_2\|\partial_x H(t)\|^2 + C_1(\varepsilon_2)\|PH(t)\|^2. \tag{1.20}$$

Multiplying (1.16) by H, we get

$$\frac{1}{2}\frac{d}{dt}\|PH(t)\|^2 - m\|H(t)\|^2 + \nu\|\partial_x H(t)\|^2$$
$$= -\alpha(\langle\widetilde{A}(t), a(t)\rangle, H(t)) - \alpha(\langle a(t), A(t)\rangle, H(t)). \tag{1.21}$$

By (1.19), the right-hand side of (1.21) admits the following estimate:

$$|\alpha(\langle\widetilde{A}(t), a(t)\rangle, H(t)) + (\langle a(t), A(t)\rangle, H(t))| \leqslant \frac{|\alpha|l}{2}(\|\partial_x\widetilde{A}(t)\|^2$$
$$+ \|\partial_x A(t)\|^2)\|a(t)\|^2 + 2|\alpha|\varepsilon_2\|\partial_x H(t)\|^2 + 2|\alpha|C_1(\varepsilon_2)\|PH(t)\|^2.$$

Using the last inequality and (I_2), from (1.21) we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|PH(t)\|^2 + (\nu - 2|\alpha|\varepsilon_2 - |m|\varepsilon_2)\|\partial_x H(t)\|^2 \leqslant \frac{|\alpha|l}{2}(\|\partial_x\widetilde{A}(t)\|^2$$
$$+ \|\partial_x A(t)\|^2)\|a(t)\|^2 + (2|\alpha| + |m|)C_1(\varepsilon_2)\|PH(t)\|^2. \tag{1.22}$$

Adding (1.20) with (1.22) and taking $\varepsilon_1 = 1$, $\varepsilon_2 = \nu(2|\alpha| + |m|)^{-1}$, we get

$$\frac{d}{dt}\left[\|a(t)\|^2 + \|PH(t)\|^2\right] \leqslant N(t)\left[\|a(t)\|^2 + \|PH(t)\|^2\right], \tag{1.23}$$

where

$$N(t) = 2\mu + 2l\|h(t)\|^2 + l(|\alpha|+1)\|\partial_x\widetilde{A}(t)\|^2 + |\alpha|l\|\partial_x A(t)\|^2$$
$$+ (4|\alpha| + 4|m| + 2)C_1(\varepsilon_2).$$

By (1.9) and (1.10), we have $N \in L_1(0,T)$ for each $T > 0$. From (1.23) it follows that

$$\|A(t) - \widetilde{A}(t)\|^2 + \|Ph(t) - P\widetilde{h}(t)\|^2$$
$$\leqslant \exp\Big(\int_0^t N(s)ds)(\|A_0 - \widetilde{A}_0\|^2 + \|Ph_0 - P\widetilde{h}_0\|^2\Big).$$

Thus, we have proved the following assertion.

Theorem 1. *If $A_0 \in L_2(0,l)$ and $h_0 \in H^{-1}(0,l)$, then the problem* (1.1)–(1.4) *has a unique weak solution*

$$A \in C(0,T;L_2(0,l)) \cap L_2(0,T;H_0^1(0,l)),$$
$$h \in C(0,T;H^{-1}(0,l)) \cap L_2(0,T;H^1(0,l)).$$

The solution continuously depends on the initial data.

Thus, we have proved that the problem (1.1)–(1.4) generates a continuous semigroup V_t, $t \in R^+$, in the phase space $X_0 = L_2(0,l) \times H^{-1}(0,l)$.

2. Minimal Global Attractor

Let us recall the following well-known theorem

Theorem 2 ([**6**]). *If $V_t : X \to X$, $t \in R^+$, is a continuous, compact and bounded dissipative semigroup, then it has a minimal global attractor M which is connected, compact, and invariant.*

We show that the semigroup V_t generated by the problem (1)–(4) is bounded and dissipative in the following two cases.

Case 1. $\alpha > 0$ and $\gamma_0 = \nu\lambda_1 - m > 0$.

Case 2. $\alpha < 0$ and $\gamma_1 = \nu\lambda_1 - m + \alpha/k > 0$.

We begin with the case $\alpha > 0$. Multiplying (1.5_1) by α and adding to (1.5_2), we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\left[\alpha\|A(t)\|^2 + \|Ph(t)\|^2\right] + \alpha\|\partial_x A(t)\|^2 +$$
$$+ \alpha k \int_0^l |A(x,t)|^4 dx - m\|h(t)\|^2 + \nu\|\partial_x h(t)\|^2 = \alpha\mu\|A(t)\|^2.$$

By the Cauchy inequality and the Poincare–Friedrichs inequality, the last equality implies

$$\frac{1}{2}\frac{d}{dt}\left[\alpha\|A(t)\|^2 + \|Ph(t)\|^2\right] + \lambda_1\alpha\|A(t)\|^2 + \lambda_1\gamma_0\|Ph(t)\|^2 \leqslant \frac{\alpha l\mu^2}{4k}$$

or

$$\frac{d}{dt}y_\alpha(t) + d_1 y_\alpha(t) \leqslant \frac{\alpha l \mu^2}{2k},$$

where

$$y_\alpha(t) = \alpha\|A(t)\|^2 + \|Ph(t)\|^2, \quad d_1 = 2\lambda_1 \min\{1, \gamma_0\}.$$

Integrating, we obtain the inequality

$$\alpha\|A(t)\|^2 + \|Ph(t)\|^2 \leqslant \frac{\alpha l \mu^2}{2kd_1} + [\alpha\|A_0\|^2 + \|Ph_0\|^2]e^{-d_1 t}$$

which implies that the ball

$$B_0 = \{A, h\} \in X_0 : \|A\|^2 + \|Ph\|^2 \leqslant \frac{\alpha l \mu^2}{kd_1 \min\{1, \alpha\}}$$

is an absorbing ball of the semigroup V_t. Hence the semigroup $V_t : X_0 \to X_0$ is bounded dissipative.

Consider the case where

$$\alpha = -\alpha_1 < 0, \quad \gamma_1 = \nu\lambda_1 - m - \frac{\alpha_1}{k} > 0.$$

We choose a positive number k_1 such that

$$k_1 < k, \quad \gamma_2 = \nu\lambda_1 - m - \frac{\alpha_1}{k_1} > 0.$$

Multiplying (1.5_1) by α_1 and adding to (1.5_2), we find

$$\frac{1}{2}\frac{d}{dt}[\alpha_1\|A(t)\|^2 + \|Ph(t)\|^2] + \alpha_1\|\partial_x A(t)\|^2 - \alpha_1\mu\|A(t)\|^2$$
$$+ \alpha_1 k \int_0^l |A(x,t)|^4 dx + \nu\|\partial_x h(t)\|^2 - m\|h(t)\|^2$$
$$= 2\alpha_1 \int_0^l |A(x,t)|^2 h(x,t) dx \leqslant \alpha_1 k_1 \int_0^l |A(x,t)|^4 dx + \frac{\alpha_1}{k_1}\|h(t)\|^2. \tag{2.1}$$

By the Poincare–Friedrichs inequality and the inequality

$$\alpha_1\mu\|A(t)\|^2 \leqslant \alpha_1 \frac{k-k_1}{2} \int_0^l |A(x,t)|^4 dx + \frac{\alpha_1\mu^2 l}{2(k-k_1)},$$

from (2.1) it follows that

$$\frac{1}{2}\frac{d}{dt}\left[\alpha_1\|A(t)\|^2 + \|Ph(t)\|^2\right] + \alpha_1\|\partial_x A(t)\|^2 + \gamma_2\|h(t)\|^2$$
$$+ \alpha_1 \frac{k-k_1}{2} \int_0^l |A(x,t)|^4 dx \leqslant \frac{\alpha_1\mu^2 l}{2(k-k_1)}. \tag{2.2}$$

Thus, V_t is a bounded dissipative semigroup in X_0 for $\alpha < 0$ and $\nu\lambda_1 - m > \dfrac{|\alpha|}{k}$.

We show that the semigroup V_t is a compact semigroup for each real α without additional restrictions on the parameters of the system.

Multiplying Eq. (1.1) by $-t\partial_x^2 A$ and Eq. (1.2) by $-t\partial_x^2 h$ and integrating over $(0, l)$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(t\|\partial_x A(t)\|^2\right) + \frac{1}{2}\|\partial_x A(t)\|^2 - \mu t\|\partial_x A(t)\|^2 + t\|\partial_x^2 A(t)\|^2\\
&+ tk\int_0^l |A(x,t)|^2|\partial_x A(x,t)|^2 dx + 2tk\int_0^l \langle A(x,t), \partial_x A(x,t)\rangle^2 dx\\
&= -t\int_0^l h(x,t)\langle A(x,t), \partial_x^2 A(x,t)\rangle dx,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t\|h(t)\|^2) - \frac{1}{2}\|h(t)\|^2 - mt\|\partial_x h(t)\|^2 + \nu t\|\partial_x^2 h(t)\|^2\\
&= \alpha t\int_0^l |A(x,t)|^2\partial_x^2 h(x,t)dx
\end{aligned} \tag{2.4}$$

Using the Cauchy–Schwarz inequality and the inequality (I_0), we get

$$\begin{aligned}
t\left|\int_0^l h(x,t)\langle A(x,t), \partial_x^2 A(x,t)\rangle dx\right| &\leqslant t\max_{x\in[0,l]}|h(x,t)|\|A(t)\|\,\|\partial_x^2 A(t)\|\\
&\leqslant t\frac{1}{2}\|\partial_x^2 A(t)\|^2 + t\frac{l}{2}\|A(t)\|^2\|\partial_x h(t)\|^2.
\end{aligned} \tag{2.5}$$

Using (I_1), we can estimate the right-hand side of (2.4) as follows:

$$\begin{aligned}
&\left|t\alpha\int_0^l |A(x,t)|^2\partial_x^2 h(x,t)dx\right| \leqslant t|\alpha|\max_{x\in[0,l]}|A(x,t)|^2\int_0^l |\partial_x^2 h(x,t)|dx\\
&\leqslant t\frac{2\alpha^2 l}{\nu}\|A(t)\|^2\|\partial_x A(t)\|^2 + t\frac{\nu}{2}\|\partial_x^2 h(t)\|^2.
\end{aligned} \tag{2.6}$$

Using the estimates (2.5) and (2.6) in (2.3) and (2.4) respectively, we find

$$\begin{aligned}
&\frac{d}{dt}\left(t\|\partial_x A(t)\|^2\right) \leqslant 2\mu t\|\partial_x A(t)\|^2 + tl\|A(t)\|^2\|\partial_x h(t)\|^2,\\
&\frac{d}{dt}\left(t\|h(t)\|^2\right) \leqslant \|h(t)\|^2 + mt\|\partial_x h(t)\|^2 + t\frac{4\alpha^2 l}{\nu}\|A(t)\|^2\|\partial_x A(t)\|^2.
\end{aligned}$$

Integrating the last two inequalities and taking into account the estimates (1.9) and (1.10), we obtain the following regularity inequalities:

$$\|\partial_x A(t)\|^2 + \|h(t)\|^2 \leqslant M_1(t, \|A_0\|^2 + \|Ph_0\|^2) \quad \forall t > 0, \tag{2.7}$$

where $M_1(t,s)$ is a continuous function on $(0,+\infty)\times[0,\infty)$. Since $H_0^1(0,l) \hookrightarrow L_2(0,l)$ and $L_2(0,l) \hookrightarrow H^{-1}(0,l)$ are compact in view of (2.7), the semigroup $V_t : X_0 \to X_0$, $t \in R^+$, generated by the problem (1.1)–(1.4) is a compact semigroup. Thus, by Theorem 2, we obtain the following assertion.

Theorem 3. *Suppose that* $\nu\lambda_1 > m$ *for* $\alpha > 0$ *and* $\nu\lambda_1 - m > \dfrac{|\alpha|}{k}$ *for* $\alpha < 0$. *Then the semigroup* $V_t : X_0 \to X_0$, $t > 0$, *generated by the problem* (1)–(4) *has a minimal global attractor* M_0 *which is connected, compact, and invariant.*

3. Periodic Boundary Conditions

In this section, we consider the periodic initial boundary-value problem for Eqs. (A), (H), i.e., the system (A), (H) under the conditions

$$\begin{aligned} &A(x,0) = A_0(x), \quad h(x,0) = h_0(x), \\ &A(x,t) = A(x+l,t), \quad h(0,t) = h(l,t), \quad x \in R,\ t > 0. \end{aligned}$$

Integrating Eq. (H) over $(0,l)$ with respect to x and taking into account the periodic boundary conditions, we find

$$\frac{d}{dt}\int_0^l h(x,t)dx = 0.$$

Hence

$$\int_0^l h(x,t)dx = \int_0^l h_0(x)dx.$$

Let us consider the function

$$u(x,t) = h(x,t) - \frac{1}{l}\int_0^l h_0(x)dx.$$

We have

$$\int_0^l u(x,t)dx = 0,$$

and the pair $[A,u]$ is a solution of the problem

$$A_t - \lambda A - A_{xx} + k|A|^2 A = Au, \quad x \in R,\ t>0, \tag{3.1}$$

$$\partial_t u + m\partial_x^2 u + \nu\partial_x^4 u = \alpha\partial_x^2(|A|^2), \quad x\in R,\ t>0, \tag{3.2}$$

$$A(x,0) = A_0(x), \quad u(x,0) = h_0(x) - h_l, \quad x \in R, \tag{3.3}$$

$$A(x,t) = A(x+l,t), \quad u(x,t) = u(x+l,t), \quad x\in R,\ t>0, \tag{3.4}$$

where $\lambda = \mu + h_l, h_l = \frac{1}{l}\int\limits_0^l h_0(x)dx$.

Multiplying (3.1) by A, we find

$$\frac{1}{2}\frac{d}{dt}\|A(t)\|^2 - \lambda\|A(t)\|^2 + \|A_x(t)\|^2 + k\int\limits_0^l |A(x,t)|^4 dx$$

$$= \int\limits_0^l |A(x,t)|^2 u(x,t)dx \leqslant \frac{k}{2}\int\limits_0^l |A(x,t)|^4 dx + \frac{1}{2k}\|u(t)\|^2$$

or

$$\frac{d}{dt}\|A(t)\|^2 + 2\|\partial_x A(t)\|^2 + k\int\limits_0^l |A(x,t)|^4 dx \leqslant 2|\lambda|\|A(t)\|^2 + \frac{1}{k}\|u(t)\|^2. \tag{3.5}$$

Multiplying (3.2) by u, we find

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 - m\|\partial_x u(t)\|^2 + \nu\|\partial_x^2 u(t)\|^2 = \alpha\int\limits_0^l |A(x,t)|^2\partial_x^2 u(x,t)dx$$

$$\leqslant \frac{\nu}{4}\|\partial_x^2 u(t)\|^2 + \frac{\alpha^2}{\nu}\int\limits_0^l |A(x,t)|^4 dx.$$

By the Cauchy inequality and the inequality $\|\partial_x u\|^2 \leqslant \|u\|\|\partial_x^2 u\|$, from the last relation we obtain the estimate

$$\frac{d}{dt}\|u(t)\|^2 + \nu\|\partial_x^2 u(t)\|^2 \leqslant \frac{2m^2}{\nu}\|u(t)\|^2 + \frac{2\alpha^2}{\nu}\int\limits_0^l |A(x,t)|^4 dx. \tag{3.6}$$

Multiplying (3.6) by $b = \dfrac{\nu k}{4\alpha^2}$ and adding to (3.5), we find

$$\frac{d}{dt}\left[b\|u(t)\|^2 + \|A(t)\|^2\right] + b\nu\|\partial_x^2 u(t)\|^2 + 2\|\partial_x A(t)\|^2 + \frac{k}{2}\int\limits_0^2 |A(x,t)|^4 dx$$

$$\leqslant \left(\frac{2bm^2}{\nu} + \frac{1}{k}\right)\|u(t)\|^2 + 2|\lambda|\|A(t)\|^2. \tag{3.7}$$

Integrating (3.7) and using the Gronwall inequality, we get

$$b\|u(t)\|^2 + \|A(t)\|^2 \leqslant \left[b\|u_0\|^2 + \|A_0\|^2\right] e^{d_2 t}, \tag{3.8}$$

where

$$d_2 = \max\left\{\frac{2m^2}{\nu} + \frac{1}{kb}, 2|\lambda|\right\}.$$

From (3.7) and (3.8) it follows that

$$\int\limits_0^t \|\partial_x A(s)\|^2 ds \leqslant C(t), \quad \int\limits_0^t \|\partial_x^2 u(s)\|^2 ds \leqslant C(t). \tag{3.9}$$

Using the estimates (3.8) and (3.9), as in the case of the Dirichlet boundary conditions, we obtain the following assertion.

Theorem 4. *If $h_0 \in L_2(0,l)$ and $A \in L_2(0,l)$, then the problem* (3.1)–(3.4) *has a unique solution $[A,u]$ such that*

$$A \in C(0,T; L_2(0,l)) \cap L_2(0,T; H^1_{\text{per}}(0,l)),$$
$$u \in C(0,T; L_2(0,l)) \cap L_2(0,T; \dot{H}^2_{\text{per}}(0,l)).$$

We show that for $A_0 \in H^1_{\text{per}}(0,l)$ and $h_0 \in \dot{H}^1_{\text{per}}(0,l)$ we have

$$A \in L_\infty(0,T; H^1_{\text{per}}(0,l)), \quad u \in L_\infty(0,T; \dot{H}^1_{\text{per}}(0,l))$$

(we refer to [**7**] for the definition of $H^s_{per}(0,l)$ and $\dot{H}^s_{per}(0,l)$).

Indeed, multiplying (3.1) by $\partial_t A$, we get

$$\|\partial_t A(t)\|^2 + \frac{d}{dt}\left[-\frac{\lambda}{2}\|A(t)\|^2 + \frac{1}{2}\|\partial_x A(t)\|^2 + \frac{k}{4}\int\limits_0^l |A(x,t)|^4 dx\right]$$

$$= \int\limits_0^l u(x,t)\langle A(x,t), \partial_t A(x,t)\rangle dx \tag{3.10}$$

By the interpolation inequality

$$\|u\|_{L_\infty(0,l)} \leqslant \beta_0 \|u\|^{1/2}\|u\|^{1/2}_{H^1(0,l)}, \tag{A$_0$}$$

we have

$$\left| \int_0^l u(x,t)\langle A(x,t), \partial_t A(x,t)\rangle dx \right|$$

$$\leqslant \frac{1}{2}\|u(t)\|^2_{L_\infty(0,l)}\|A(t)\|^2 + \frac{1}{2}\|\partial_t A(t)\|^2$$

$$\leqslant \frac{\beta_0}{2}\|u(t)\|\,\|u(t)\|_{H^1(0,l)}\|A(t)\|^2 + \frac{1}{2}\|\partial_t A(t)\|^2$$

$$\leqslant \varepsilon_3\|u(t)\|^2_{H^1(0,l)} + \frac{\beta_0^2}{16\varepsilon_3}\|u(t)\|^2\|A(t)\|^4 + \frac{1}{2}\|\partial_t A(t)\|^2.$$

Using the last inequality, the estimate (3.8), and the inequality

$$\|\partial_x u\|^2 \leqslant \beta_1\|\partial_x^2 u\|^2,$$

from (3.10) we find

$$\frac{d}{dt}\left[-\frac{\lambda}{2}\|A(t)\|^2 + \frac{1}{2}\|\partial_x A(t)\|^2 + \frac{k}{4}\int_0^l |A(x,t)|^4 dx \right]$$

$$\leqslant -\frac{1}{2}\|\partial_t A(t)\|^2 + \varepsilon_3\|\partial_x^2 u(t)\|^2 + C(t). \tag{3.11}$$

Taking $\varepsilon_3 = \nu/2$ in (3.11) and adding to (3.6), we find

$$\frac{d}{dt}\left[-\frac{\lambda}{2}\|A(t)\|^2 + \frac{1}{2}\|\partial_x A(t)\|^2 + \frac{k}{4}\int_0^l |A(x,t)|^4 dx + \|u(t)\|^2 \right]$$

$$\leqslant -\frac{1}{2}\|\partial_t A(t)\|^2 - \frac{\nu}{2}\|\partial_x^2 u(t)\|^2 + \frac{2\alpha^2}{\nu}\int_0^l |A(x,t)|^4 + C(t). \tag{3.12}$$

It is easy to see that (3.12) implies the estimates

$$\int_0^t \|\partial_s A(s)\|^2 ds \leqslant C(t), \|\partial_x A(t)\|^2 \leqslant C(t). \tag{3.13}$$

Multiplying (3.2) by $-\partial_x^2 u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u(t)\|^2 - m\|\partial_x^2 u(t)\|^2 + \nu\|\partial_x^3 u(t)\|^2$$

$$= 2\alpha\int_0^l \langle A(x,t), \partial_x A(x,t)\rangle \partial_x^3 u(x,t) dx. \tag{3.14}$$

By the inequality (A_0) and (3.13), we have

$$2|\alpha| \int_0^l |\langle A(x,t), \partial_x A(x,t)\rangle||\partial_x^3 u(x,t)|dx$$

$$\leqslant 2|\alpha| \|A(t)\|_{L_\infty(0,l)} \|\partial_x A(t)\| \, \|\partial_x^3 u(t)\| \leqslant \frac{\nu}{4}\|\partial_x^3 u(t)\|^2 + C(t).$$

Using the last inequality and the inequality

$$m\|\partial_x^2 u\|^2 \leqslant \frac{\nu}{4}\|\partial_x^3 u(t)\|^2 + \frac{m^2}{\nu}\|\partial_x u\|^2,$$

from (3.14) we find

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u(t)\|^2 + \frac{\nu}{2}\|\partial_x^3 u(t)\|^2 \leqslant \frac{m^2}{\nu}\|\partial_x u(t)\|^2 + C(t). \tag{3.15}$$

From (3.15) we obtain the estimates

$$\|\partial_x u(t)\| \leqslant C(t), \quad \int_0^t \|\partial_x^3 u(s)\|^2 ds \leqslant C(t). \tag{3.16}$$

It is easy to see that if the initial functions are sufficiently smooth, then the corresponding solution is also sufficiently smooth (cf. [**8**]). Thus, the solution of the problem (3.1)–(3.4) cannot blow-up in a finite time for any real m, μ, α and positive ν, k in spite of the fact that the numerical studies indicate the opposite (cf. [**2**]). But for $k \leqslant 0$ and $\alpha \neq 0$, using the concavity method (cf. [**9**]) or the generalized concavity method (cf. [**10**]), it is possible to show that there is a large class of initial data such that the corresponding solutions blow-up in a finite time.

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