Multilateral Limit Pricing in Price-Setting Games

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Abstract

In this paper, we characterize the set of pure strategy equilibria in differentiated Bertrand oligopolies with linear demand and constant unit costs when firms may prefer not to produce. When there are two firms or all firms are active, there is a unique equilibrium. However, there is a continuum of Bertrand equilibria on a wide range of parameter values when the number of firms (n) is *more than two* and $n^* \in [2, n-1]$ firms are active. In each such equilibrium, the relatively more cost or quality efficient firms limit their prices to induce the exit of their rival(s). When $n \geq 3$, this game do not need to satisfy supermodularity, the single-crossing property (SCP), or log-supermodularity (LS). Moreover, best responses might have negative slopes. These results are very different from those in the existing literature on Bertrand models with differentiated products, where uniqueness, supermodularity, the SCP, and LS usually hold under a linear market demand assumption, and best response functions slope upward. Our main results extend to a Stackelberg entry game where some established incumbents first set their prices and then a potential entrant sets its price.

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1 Introduction

In several markets, some firms may not be able to actively participate, and many decide to shut down. A large amount of literature has studied entry or exit decisions that are induced by information-based (i.e., signaling-based) limit pricing practiced by other firms.[1] However, the entry and exit behavior of firms might also be efficiency-based in highly competitive markets. Competitors' cost-reducing innovations, the inability to adapt to changing market conditions, a cost-efficient merger among rival firms, or firms' strategies to raise rivals' variable costs may induce a firm to exit or to remain idle temporarily. Nevertheless, an inactive firm might still be efficient enough to lead active firms to engage in efficiency-based limit pricing but not strong enough to enter the market itself.

In this paper, we study traditional static price-setting games among firms that have different levels of quality or cost efficiencies. The differences between these levels might be due to one of the above factors. Our main aim is to identify the set of active and inactive firms in any equilibrium and to provide a full characterization of the equilibrium behavior of firms. Such a characterization in static quantity-setting games is trivial. In particular, standard existence and uniqueness results for the Cournot equilibrium extend to environments where firms may prefer not to be active (Novshek, 1985 and Gaudet and Salant, 1991). However, the equilibrium behavior of firms constrained by non-negative production levels in Bertrand models has not been systematically addressed. We show that differentiated linear Bertrand oligopolies with constant unit costs and continuous best responses need not satisfy supermodularity (Topkis, 1979) or the single-crossing property (Milgrom and Shannon, 1994). Consequently, existence and uniqueness results for games that satisfy supermodularity or the single-crossing property do not apply in our framework. In particular, the Bertrand best responses might have negative slopes. When there are two firms or all firms are active, there is a unique equilibrium. However, there is a continuum of pure strategy Bertrand equilibria for a wide range of parameter values when the number of firms is more than two and $n^* \in [2, n-1]$ firms are active. We provide an iterative algorithm to

[1] Predatory pricing means that a firm charges a price that is below the firm's average costs with the sole intention of driving a rival out of the market. Such a behavior is deemed illegal by anti-trust authorities, such as the Federal Trade Commission (FTC) and the U.S. Department of Justice (DOJ).

find the set of active firms in any equilibrium and show that this set is the same in all equilibria. In each such equilibrium, the relatively more cost or quality efficient firms limit their prices to induce the exit of their rival(s). These results are very different from the existing literature on Bertrand models with differentiated products, where uniqueness, supermodularity, the single-crossing property, and log-supermodularity hold under a linear market demand assumption, and best response functions slope upward.[2]

To explain our results, consider a symmetric three-firm differentiated product Bertrand oligopoly where the marginal cost levels are $c_i = \xi$ for $i = 1, 2, 3$. All firms are active; that is, their equilibrium production levels are all strictly positive. Suppose that a process innovation is available for firms 1 and 2. Accordingly, their cost levels reduce to $\widehat{\xi} = \widehat{c}_1 = \widehat{c}_2 < \widehat{c}_3 = \xi$. If the initial cost level ξ is high enough, then there are two cutoff levels for $\widehat{\xi}$, say $\widehat{\xi}_1$ and $\widehat{\xi}_2$ with $0 < \widehat{\xi}_1 < \widehat{\xi}_2$, such that the firms' equilibrium strategies are qualitatively different when $\widehat{\xi}$ lies in the region $[0, \widehat{\xi}_1]$, $(\widehat{\xi}_1, \widehat{\xi}_2)$, or $[\widehat{\xi}_2, \xi)$. More specifically, if $\widehat{\xi} \in [\widehat{\xi}_2, \xi)$, then the level of innovation is not too high, and all three firms continue to be active in the market. At the other extreme, if $\widehat{\xi} \in [0, \widehat{\xi}_1]$, then firm 3 becomes very inefficient compared to firms 1 and 2 and leaves the market. Accordingly, firms 1 and 2 charge unconstrained duopoly prices. The most interesting region is the intermediate region here $\widehat{\xi} \in (\widehat{\xi}_1, \widehat{\xi}_2)$. This region involves efficiency-based limit pricing induced by firms 1 and 2 to keep firm 3 out of the market. If they ignored firm 3 and charged unconstrained duopoly prices, then firm 3 would continue to be active in the market.

In the case of linear demand, limit pricing takes a particularly simple form. Consider any price combination of firms 1 and 2 such that $p_1 + p_2 = M$ where M is uniquely determined by the parameters of the model. If either firm 1 or firm 2 charges a higher price, then firm 3 would start to produce, and the market would become a triopoly market. On the other hand, when either firm decreases its price, the market is a duopoly market. For this reason, the profit functions of firms 1 and 2 exhibit kinks at price combinations where $p_1 + p_2 = M$. Moreover, the fact that demand is more sensitive to a change in the price that a firm sets in the region where all three firms are active[3] implies that the right-hand derivative

[2]For instance, Friedman (1977) shows that when the best response functions are contractions, costs are nondecreasing, and all firms produce imperfectly substitutable products, then there is a unique Bertrand equilibrium.

[3]The reason is that when firm 1 changes its price in the duopoly region (i.e., where $p_1 + p_2 <$

of the profit of firm 1 with respect to p_1 is more negative (or less positive) than the left-hand derivative if $p_1 + p_2 = M$ as the demand drop is accelerated for prices where the third firm is active. At such price combinations, the optimality conditions for firm 1 require the left-hand derivative of the profit function to be positive, and the right-hand derivative to be negative, which can be satisfied by multiple combinations of p_1 and p_2 satisfying $p_1 + p_2 = M$. As a result, there is a host of equilibria in our price-setting game. Relatedly, the kink implies that the best response for firm 1 when firm 2 sets p_2 satisfies $p_1 = M - p_2$, so the price choices of firms 1 and 2 are strategic substitutes at such a point.

Our model has already been extensively studied in a two-firm set-up. For example, both Muto (1993) and Zanchettin (2006) show that when there are two firms, there is a unique limit pricing equilibrium, whereby the efficiency gap between the two firms is sufficiently high to rule out an interior equilibrium, where both firms are active, but not high enough to allow the most efficient firm to engage in (unconstrained) monopoly equilibrium. This paper generalizes the Bertrand equilibrium characterization results to an n-firm set-up when firms have any degree of cost and quality asymmetries. The generalization of the limit pricing equilibrium unveils a set of novel results such as the multiplicity of limit pricing equilibria result. There are several applications of the findings in the contexts of market exit after a cost-reducing process innovation or a cost efficient merger[4]; and of the comparisons of Cournot and Bertrand equilibria. For example, Zanchettin (2006) shows that both the efficient firm's and industry profits can be higher under Bertrand competition than under the Cournot competition in the limit pricing equilibrium region. This reverses Singh and Vives' (1984) ranking. It is clear from these arguments that the possibility of limit pricing and multiple equilibria might give rise to unexpected results in various contexts.

Our paper contributes to the literature on supermodularity in price-setting games. We show that under standard assumptions for demand and cost, a Bertrand game with differentiated substitutable products may not satisfy supermodularity, the single-crossing property, or log-supermodularity if some firms produce zero

M) then the firm's quantity responds relatively mildly since there is only one other firm (firm 2), to which customers divert. In the region where $p_1 + p_2 \geq M$, any increase in p_1 makes customers divert to both firms 2 and 3.

[4]Motta (2007) considers the possibility of a market exit after a cost-efficient Bertrand merger. Although a limit pricing region exists, it has not been pointed out.

output in equilibrium. This is in contrast with the previous literature that showed that if all firms are active in equilibrium, then the Bertrand oligopoly with differentiated substitutable products satisfies these properties for a wide variety of cost and demand functions.[5], [6] In particular, in Topkis (1979), Vives (1990), and Milgrom and Roberts (1990), demand function is assumed to be twice continuously differentiable. We argue that for standard demand functions, this assumption is only satisfied when all firms have positive production.

Given that our game is not supermodular (or log-supermodular), the question of pure strategy equilibrium existence and uniqueness arises naturally. Roberts and Sonnenschein (1977) and Friedman (1983) provide examples of non-supermodular differentiated Bertrand duopolies where an equilibrium does not exist in pure strategies.[7] Fortunately, a standard fixed-point theorem shows that a pure strategy Bertrand equilibrium exists in our game. However, when $n \geq 3$, the uniqueness of equilibrium fails, and there is a continuum of pure strategy limit pricing equilibria for a large configuration of parameters in our model.[8],[9] Such an equilib-

[5]Topkis (1979) shows that if the goods are substitutes with linear demand and costs and if the players' strategies are prices constrained to lie in an interval $[0, p]$, then the game is supermodular. Later, Vives (1990) extends the result to the case of convex costs. Building on Topkis (1979), Milgrom and Roberts (1990) show that there is a unique pure strategy Bertrand equilibrium with linear, constant elasticity of substitution (CES), logit, and translog demand functions and constant marginal costs.

[6]However, we might have Bertrand equilibria multiplicity in the case of homogeneous products. Dastidar (1995) shows that with identical, continuous, and convex cost functions, a Bertrand competition typically leads to multiple pure strategy Nash equilibria. Hoernig (2002) also finds there is a continuum of mixed strategy equilibria with continuous support. Moreover, there exists a unique and symmetric coalitional-proof Bertrand equilibrium if the firms possess an identical and increasing average cost (Chowdhury and Sengupta, 2004).

[7]Unlike our case, the non-supermodular examples of the above articles feature discontinuities in the best responses.

[8]Ledvina and Sircar (2011, 2012) and Federgruen and Hu (2015, 2016, 2017a, 2017b) study price-setting games that cover our set-up, where some firms may not produce in equilibrium. Ledvina and Sircar (2011, Theorem 2.1) show that there is a unique pure strategy Bertrand equilibrium. Similarly, Federgruen and Hu (2015, Theorem 3) show that when all firms are not active, each equilibrium of the Bertrand game is equivalent to a weakly dominated componentwise smallest price equilibrium (CWSE), where the inactive firm charges below its marginal cost. However, our result establishes that none of the equilibrium of this game is equivalent to this CWSE (See our comment Cumbul and Virág, 2017b) and it is necessary to assume positive production by all firms in order to assure supermodularity and the single-crossing property, and thus assure the uniqueness of the pure strategy Bertrand equilibrium. Our equilibrium multiplicity result have further implications in the dynamic Bertrand oligopoly games (Ledvina and Sircar, 2011) and mean field games (Chan and Sircar, 2015).

[9]Our multiplicity of (kinked demand) limit pricing equilibria result show that kinked demand equilibria are general, intuitive, and rationalize previous findings originally attributed to peculiar

rium multiplicity provides insights into how several firms may keep their competitors out together.[10] Last, Topkis (1998) argues that the log-supermodularity of demand is a critical sufficient condition for monotone best responses in Bertrand price-setting games if one takes a firm to be specified by its unit cost $(-\infty, \infty)$. However, our existence result of non-monotonic best responses in the case of linear demand shows that this is not the case if a firm has a unit cost in $[0, \infty)$. This has similarities to the findings of Amir and Grilo (2003).

We also study various extensions of the findings in the context of market entry. We consider an entry game with some established incumbents and a potential entrant. In the first stage, the incumbents simultaneously choose their prices. In the second stage, the entrant chooses its price. The limit pricing equilibria of the associated simultaneous move Bertrand game include the entry-deterring limit pricing equilibria of this sequential move game. Thus, our main findings are robust in Stackelberg price-setting games.

Last, among the set of limit pricing equilibrium price vectors, a consumer surplus-maximizing equilibrium price vector can minimize the total surplus (or total producer surplus) in our models. Moreover, our results in the Stackelberg game contribute to the ongoing debate of whether entry prevention can be seen as a public good or not. Each firm could free-ride on the entry-preventing activities of its competitors with the potential implication that there would be little entry deterrence. However, we show that each firm prefers the other firm to charge as high a price as is consistent with equilibrium. Thus, every incumbent would like to contribute to entry deterrence as much as the firm can given that the entry will be prevented.[11]

characteristics of specific models. In particular, Economidies (1994), Yin (2004), Cowan and Yin (2010), and Merel and Sexton (2010) characterize the set of kinked demand Bertrand equilibria between two active firms, which differ only in their locations, in a Hotelling model of horizontal differentiation. Different from this literature, for our multiplicity of equilibria result, we need at least three firms, where there is at least one relatively inefficient inactive firm. In each such equilibrium, the relatively more cost or quality efficient firms limit their prices to induce the exit of their rival(s).

[10] The existence of multiple predatory over-investment strategies has been found by Gilbert and Vives (1986) in a Stackelberg quantity-choosing entry game. This multiplicity is due to the presence of the entry costs of the entrant when there are discontinuities in the best replies. Moreover, efficiency-based limit pricing strategies are different from predatory pricing strategies as we stress throughout the paper. See also Iacobucci and Winter (2012) for a collusion based analysis on joint exclusion.

[11] This has similarities to the findings of Gilbert and Vives (1986), where each incumbent

In Section 2, we describe the model and provide the main theoretical analysis and our main results. In Section 3, we provide the connection between our results and the concepts of supermodularity, the single-crossing property and log-supermodularity. In Section 4, we discuss the extensions of the results in the case of sequential moves. We also provide implications for entry and exit models, and how firms may keep out rivals jointly in real-world markets. In Section 5, we study the welfare properties of the Bertrand and Stackelberg equilibria.

2 Bertrand Model

Let $N = \{1, 2, ..., n\}$ be a finite set of firms. Each firm $i \in N$ produces an imperfect substitutable product i (or provides such a service) at constant marginal cost c_i without incurring fixed costs.[12] Each firm $i \in N$ sets its price p_i simultaneously, knowing all the cost and demand parameters of the game.

Next, we describe the demand side of the economy. The representative consumer has an exogenous income I and maximizes consumer surplus:

$$CS = \sum_{k\in N} A_k q_k - \frac{\lambda}{2}\sum_{k\in N} q_k^2 - \lambda\theta \sum_{k\in N}\sum_{j>k} q_k q_j + (I - \sum_{k\in N} p_k q_k), \tag{1}$$

where A_i is the exogenously given measure of the quality of variety i in a vertical sense,[13] $\theta \in (0, 1)$ is an inverse measure of product differentiation, and $\lambda > 0$ is the slope of the demand curve. Note that U is concave at $\theta \in (-1/(n-1), 1)$ and $\lambda > 0$. The consumer will consume a strictly positive amount of some $s(\mathbf{p}_S)$ products, which are offered by the firms in set $S(\mathbf{p}) \subseteq N$, where $\mathbf{p}_S = (p_1, p_2, ..., p_s)$. The first-order condition of the consumer's problem yields that for all products that are consumed in a non-negative quantity, it holds that

$$p_i = A_i - \lambda q_i - \lambda\theta \sum_{j\in S\setminus i} q_j. \tag{2}$$

would like to over-invest to deter entry. On the other hand, some previous authors, who have highlighted the public good aspect of noncooperative entry prevention, would include Bernheim (1984), Waldman (1987,1991), Appelbaum and Weber (1992), and Kovenock and Roy (2005).

[12] Our main results will be there when we allow for avoidable fixed costs. A formal analysis can be provided to the reader upon request.

[13] For the interpretation of this parameter, we follow Häckner (2000) and Martin (2009). Other things being equal, an increase in A_i increases the marginal utility of consuming good i.

When $\theta = 1$ and $A_i = A_j$, $i \neq j$, all products are perfect substitutes and no longer differentiated. At the other extreme, when $\theta = 0$, each firm is a monopoly for the good the firm produces.

To obtain non-trivial results, we assume that for each $k \in N$, it holds that $A_k > c_k$. Let also the quality-cost differential be defined as $\delta_i = A_i - c_i$. Without loss of generality, assume that $\delta_1 > \delta_2 > .... > \delta_n$.[14] It is not easy to see at this step that the s products, which will be consumed by the consumer, have the highest quality-cost differentials, i.e., $S = \{1, 2, ..., s\}$, in *any* equilibrium of this game.[15]

Solving (2) for the quantities yields

$$q_i = D_i^S(\mathbf{p}_S) = a_{i,s} - b_s p_i + d_s \sum_{j \in S \setminus i} p_j, \tag{3}$$

where $a_{i,s} = (b_s + d_s)A_i - d_s \sum_{j \in S} A_j$, $b_s = \frac{1+\theta(s-2)}{\lambda(1-\theta)(1+\theta(s-1))}$, and $d_s = \frac{\theta}{\lambda(1-\theta)(1+\theta(s-1))}$.

Given a price vector $\mathbf{p} = (p_1, p_2, ..., p_n)$, one can calculate the profit of each firm $i \in N$ as follows. The profit of firm i, $\pi_i(\mathbf{p})$ is equal to 0 if $i \in N \setminus S(\mathbf{p})$. The profit of $i \in S(\mathbf{p})$ can be written as

$$\pi_i^S(\mathbf{p}) = (p_i - c_i)(a_{i,s(\mathbf{p})} - b_{s(\mathbf{p})} p_i + d_{s(\mathbf{p})} \sum_{j \in S(\mathbf{p}) \setminus i} p_j). \tag{4}$$

3 Equilibrium Analysis

A pure strategy *equilibrium* of the Bertrand game requires that for all $i \in N$ it holds that $p_i \in \arg\max_x \pi_i(x, \mathbf{p}_{-i})$ where we let $\mathbf{p}_{-i}$ be the vector of the prices set by all firms other than i. We argue that weakly dominated strategies are not credible in a one-shot Bertrand game. Thus, we assume $p_i \in [c_i, A_i]$ to characterize undominated Bertrand equilibria and ignore the actions below the marginal cost levels. Let $(\widehat{p}_i, \widehat{q}_i)$ denote an equilibrium price quantity vector of firm i.

Let $S' \subseteq N$ be the set of active firms with the cardinality of S' being s'

[14] All our results are valid for the case where some of the quality-cost differentials are equal, as we assume in some examples, but the notation becomes much more burdensome; therefore, we do not cover this case formally.

[15] A full characterization of S for any price vector is not necessary at this point. We use the relevant properties of S when we proceed with our analysis.

at a given price vector $\mathbf{p}_N = (p_j)_{j \in N}$, where $p_i \in [c_i, A_i]$, based on (3). Let $h = \arg\max_{i \in N \setminus S'} \delta_i$ and $S'' = S' \cup \{h\}$.

In the Appendix, we show that the profit of firm i is quasi-concave with respect to the price of firm i, and thus, a pure strategy equilibrium exists. In particular, when firm i is active, the first derivative of q_i and π_i with respect to p_i either exists and is decreasing or the right-hand derivative is less than the left-hand derivative (See Figures 1a and 1b). Moreover, the profit and demand become zero when p_i is sufficiently large. Therefore, the profit and demand functions are quasi-concave in p_i. This argument shows that both functions exhibit a *kink* at the point when a new firm becomes active because at such a point the demand of i becomes more sensitive to changes in i's price (that is, $b_{s'+1} > b_{s'}$) because the consumer may divert to more firms than before.

Lemma 1. *i) The demand and profit functions of the active firms are kinked when their prices hit the critical set where a new firm starts positive demand by pricing above marginal cost.*

ii) The profit function π_i is continuous, single-peaked, and quasi-concave in p_i when $p_i \in [c_i, A_i]$ and $q_i \geq 0$. Consequently, there exists a pure strategy Bertrand-Nash equilibrium.

Proof: All proofs are provided in the Appendix unless otherwise stated.

To find an equilibrium, one needs to check all possible combinations of firms that may be active. To facilitate the analysis, we first study a simpler game and ignore the non-negativity constraint for the output levels. In effect, we use (3) to calculate the demand even if $q_i < 0$ for some $i \in S$. We find the equilibrium of this modified game, which we call a relaxed equilibrium. In the next step, we impose the non-negativity constraints to find the necessary conditions for the equilibria of the original game. Then, we propose an iterative algorithm to find the firms that are active in the equilibrium of the original game. Finally, we characterize the equilibrium prices and quantities.

To provide a definition of a *relaxed equilibrium* we use (3). In the S-firm market, a price vector $\mathbf{p}^*_S(S) = (p^*_i(S))_{i \in S} \geq 0$ is a relaxed Bertrand-Nash equilibrium if for each $i \in S$ it holds that

$$p_i^*(S) = \arg\max_x (x - c_i)(a_{i,s} - b_s x + d_s \sum_{j \in S \setminus i} p_j). \tag{5}$$

Given our linearity assumptions, there is a unique relaxed equilibrium, which can be found by differentiating (5) with respect to x and setting the derivative to zero. The best response of firm $i \in S$ is then given as

$$BR_i^S : \mathbb{R}^{s-1} \to \mathbb{R} \ \text{ s.t. } \ BR_i^S(\mathbf{p}_{S \setminus i}) = \frac{a_{i,s} + d_s \sum_{S \setminus i} p_j + b_s c_i}{2b_s}, \tag{6}$$

where $\mathbf{p}_{S \setminus i} = (p_j)_{j \in S \setminus \{i\}}$ is the price vector that does not contain the i^{th} dimension.[16] Assuming that all firms best respond, we obtain the relaxed equilibrium price and quantity levels as stated in the following lemma.

Lemma 2. *Let $S \subseteq N$.*

i) The unique relaxed equilibrium price and the quantity strategies of firm $i \in S$ are given by

$$p_i^*(S) = \frac{\delta_i((1 + \theta(s-1))(2 + \theta(s-3))) - \theta(1 + \theta(s-2)) \sum_{j \in S} \delta_j}{(2 + \theta(s-3))(2 + \theta(2s-3))} + c_i \tag{7}$$

and

$$q_i^*(S) = b_s(p_i^*(S) - c_i). \tag{8}$$

ii) $q_i^*(S) > q_j^*(S)$ if and only if $\delta_i > \delta_j$.

An immediate conclusion from Lemma 2 is that firms that have higher quality-cost differences produce more than firms that have relatively lower quality-cost differences in the relaxed equilibrium. Moreover, if all firms are active, then this lemma uniquely characterize the price and quantity strategies of firms for $S = N$. We next derive the equilibrium strategies of firms when there is at least one inactive firm.

We now impose the constraint that the output of each firm is non-negative. First, we derive a condition that ensures that if the set of active firms in the market is S', then firm h does not want to enter. Our starting point is that when firm h is inactive, any firm $g \in N \setminus S'$ that is less efficient than firm h can be ignored for

[16] Throughout the paper, bold letters show that the considered variable is written in the vector form.

the analysis as those firms are also inactive. Consequently, the demand that firm h faces when it sets $p_h = c_h$ and takes $\mathbf{p}_{S'}$ as given follows from (3):

$$D_h^{S''}(\mathbf{p}_{S'}, p_h = c_h) = b_{s'+1}\delta_h + d_{s'+1}(\sum_{j \in S'} p_j - \sum_{j \in S'} A_j). \tag{9}$$

It is clear that firm h can be active (produce $q_h > 0$) if and only if $D_h^{S''}(\mathbf{p}_{S'}, p_h = c_h) > 0$ because otherwise even if firm h charges its break-even price c_h, the firm faces a non-positive demand.

Let us derive the necessary conditions for an equilibrium where only firms in S' are active.

Lemma 3. *If the set of active firms is S' (that is, $q_i > 0$ if and only if $i \in S'$) in an equilibrium, then one of i) or ii) holds*[17]*:*

i) (unconstrained equilibrium) If $D_h^{S''}(\mathbf{p}^*_{S''}(S'')) < 0$ and $D_h^{S''}(\mathbf{p}^*_{S'}(S'), p_h = c_h) \leq 0$, then for all $i \in S'$, $\widehat{q}_i = q_i^*(S')$, $\widehat{p}_i = p_i^*(S')$;

ii) (limit pricing equilibrium) If $D_h^{S''}(\mathbf{p}^*_{S''}(S'')) < 0$ and $D_h^{S''}(\mathbf{p}^*_{S'}(S'), p_h = c_h) > 0$, then $D_h^{S''}(\widehat{\mathbf{p}}_{S'}(S'), p_h = c_h) = 0$.

Lemma 3 shows that there are two possible types of equilibria of which exactly one type occurs for any parameter values. In an **unconstrained equilibrium**, the active firms, $S^{'}$, charge the prices they would if no firms other than the active firms existed in the market. If the most efficient inactive firm (firm h) receives a non-positive demand, then the active firms are unconstrained, and they charge their relaxed equilibrium quantities in the $S^{'}$-firm market, i.e., $\widehat{p}_i = p_i^*(S')$ (part i)). However, it might also be the case that firm h faces positive demand. In a **limit pricing (or constrained) equilibrium (LPE)**, the active firms are constrained by the presence of firm h. Thus, they limit their unconstrained equilibrium prices to some $\widehat{\mathbf{p}}_{S'}$ such that firm h receives exactly zero demand (part ii)). This eliminates the production incentive of firm h. The result is intuitive because if firm h was not on the verge of entering but was out of the market, then the active firms would not be constrained by firms not in S' when considering small

[17] A knife-edge case may also occur if firm h produces exactly zero when it interacts with firms in S' in the relaxed equilibrium (i.e., $D_h^{S''}(\mathbf{p}^*_{S''}(S'')) = 0$). In this case, all firms in S' are active when they charge their relaxed equilibrium prices in the market of firms in S' and h; that is, for all $i \in S^{''}$, $\widehat{p}_i = p_i^*(S'')$ and $\widehat{q}_i = q_i^*(S'')$. It can be shown from (8) that for each $i \in S^{'}$, $q_i^*(S^{'} \cup h) \neq q_i^*(S^{'})$, which explains why we consider this knife-edge case as a possibility.

deviations. In this case, the first-order conditions of the unconstrained equilibrium would apply, pinning down the equilibrium prices at the unconstrained equilibrium levels.

Next, we provide an algorithm that constructively finds the set of active firms, namely N^*, in any equilibrium of this game.

Lemma 4. *Apart from the knife-edge case, the set of active firms is N^* in any equilibrium*[18]*, where N^* is the set identified by the following Bertrand iteration algorithm (BIA) for $N_i = \{1, 2, ..., i\}$.*

STEP 1: *If $q_2^*(N_2) < 0$, then $N^* = N_1$. Otherwise, proceed to the next step.*

$\vdots$

STEP k: *If $q_{i+1}^*(N_{i+1}) < 0$, then $N^* = N_i$. Otherwise, proceed to the next step.*

$\vdots$

STEP n-1: *If $q_n^*(N) < 0$, then $N^* = N_{n-1}$. Otherwise, $N^* = N$.*

The algorithm explicitly assumes that the most efficient firms are active, a necessary condition for any equilibrium. Let the algorithm select the first n^* firms for set N. This means that there is a relaxed equilibrium in the market with n^* firms such that they are all active, but there is no such equilibrium in the market with the first n^*+1 firms. If the first n^* firms can play their unconstrained equilibrium without firm n^*+1 having an incentive to be active, then the result is immediate as all the other inactive firms can be safely ignored. If firm n^*+1 is not too inefficient, then it would be active if the first n^* firms played their unconstrained (relaxed) equilibrium strategies. In this case, it seems reasonable, and is suggested by our numerical example, that there is an equilibrium where the first n^* firms decrease the sum of their prices just to keep firm n^*+1 out.

This argument provides an intuition for why an equilibrium exists, in which the first n^* firms are active. It is more difficult to rule out equilibria where a different

[18] $q_{n^*}^* = 0$ is the knife-edge case. In such a case, the set of active firms is $N^* \setminus n^*$.

set of firms are active. First, it is clear that there cannot be two unconstrained equilibria with different sets of firms being active. This follows from comparing relaxed equilibria with different numbers of firms. The novelty is to prove that there cannot be multiple LPE or one unconstrained and at least one LPE with different numbers of firms being active. We cannot use supermodularity to argue this point (see Proposition 3), but we can show that equilibria with more active firms feature lower prices on aggregate. This property is sufficient to pin the set of active firms down.

We turn to the more interesting case where the equilibrium is constrained. By Proposition 3, if an equilibrium is not unconstrained or not at the knife edge, then the equilibrium can only be constrained. The set of active firms is N^* in such a LPE by Lemma 4. A necessary condition for a LPE to occur is that $\widehat{p}_{n^*+1} = c_{n^*+1}$ and $D^{\tilde{N}}_{n^*+1}(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*}, \widehat{p}_{n^*+1} = c_{n^*+1}) = 0$, or equivalently, by (9), the equilibrium prices of firms in N^* sum to a constant

$$\textbf{Condition 1:} \qquad \sum_{j \in N^*} \widehat{p}_j = M = \sum_{j \in N^*} A_j - \frac{(1 + \theta(n^* - 1))\delta_{n^*+1}}{\theta}, \qquad (10)$$

which means that firm $n^* + 1$ is indifferent about being active or not. Thus, if firm $i \in N^*$ decreases its price at $\widehat{p}_i$, then firm $n^* + 1$ does not produce, but if i slightly increases its price, then the production of firm $n^* + 1$ becomes positive. Hence, the profit function of firm $i \in N^*$ exhibits a kink in p_i at the candidate equilibrium price vector as we discussed before Lemma 1.

Unfortunately, Condition 1 is not sufficient for a LPE to exist. The condition eliminates only the deviation incentives of the most relatively inefficient firms (i.e., firm j, $j \geq n^* + 1$), which will not produce in a LPE. Now consider any price vector, $\widehat{\mathbf{p}}_{\tilde{N}} = (\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*}, \widehat{p}_{n^*+1})$, that satisfies Condition 1. We further need to make sure that the firms in N^* do not also have any incentives to deviate to charge a lower or higher price when fixing other firms' prices at $\widehat{\mathbf{p}}_{\tilde{N}}$. For instance, consider firm i ($i \in N^*$) that charges a slightly lower price than $\widehat{p}_i$. Then firm $n^* + 1$ does not deviate to produce, and the set of active firms in the market is still N^*. Therefore, if the left-hand derivative of firm i's profit in the N^*-firm market with respect to p_i is non-negative, then firm i does not have any incentive to charge a lower price than $\widehat{p}_i$. Similarly, consider firm i charging a slightly higher price than $\widehat{p}_i$. Thus, firm $n^* + 1$ deviates to produce, and the set of active firms becomes $\tilde{N} = N^* \cup \{n^*+1\}$. If the right-hand derivative of firm i's profit in the $\tilde{N}$-firm market with respect to p_i is non-positive (given that $\widehat{p}_{n^*+1} = c_{n^*+1}$), then firm i does not have any incentive to set a higher price than $\widehat{p}_1$. Altogether, for each $i \in N^*$, this condition for derivatives is

$$\textbf{Condition 2:} \qquad \frac{\partial {\pi_i}^{\tilde{N},R}}{\partial p_i} \,|_{\widehat{\mathbf{p}}_{N^*}, \widehat{p}_{n^*+1}=c_{n^*+1}} \leq 0 \quad \text{and} \quad \frac{\partial \pi_i^{N^*,L}}{\partial p_i} \,|_{\widehat{\mathbf{p}}_{N^*}} \geq 0, \tag{11}$$

where R and L denote the right- and left-hand derivatives, respectively, of the related function. The first derivative in Condition 2 translates to $p_i \geq \underline{p}_i^B$, while the second translates to $p_i \leq \overline{p}_i^B$, where

$$\overline{p}_i^B = \frac{(1+\theta(n^*-1))(\delta_i - \delta_{n^*+1})}{2+\theta(2n^*-3)} + c_i \tag{12}$$

and

$$\underline{p}_i^B = \frac{(1+\theta n^*)(\delta_i - \delta_{n^*+1})}{2+\theta(2n^*-1)} + c_i \tag{13}$$

as we show in the proof.

We also provide two critical cutoff values for δ_{n^*+1}, which allow the LPE to exist in the first place. Note that such an equilibrium exists if firm n^*+1 cannot be active in the presence of firms in N^* (i.e., $q^*_{n^*+1}(\tilde{N}) < 0$). This is equivalent to $\delta_{n^*+1} < \bar{\delta}^B_{n^*+1}$. Moreover, when firms in N^* do not consider firm n^*+1 and charge their unconstrained Bertrand prices, there is demand left for firm n^*+1 (i.e., $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) > 0$). This happens when $\delta_{n^*+1} > \underline{\delta}^B_{n^*+1}$. In the proof of the upcoming proposition, we determine these boundaries as

$$\underline{\delta}^B_{n^*+1} = \frac{\theta(1+\theta(n^*-2))\sum_{i\in N^*}\delta_i}{(1+\theta(n^*-1))(2+\theta(n^*-3))} \tag{14}$$

and

$$\bar{\delta}^B_{n^*+1} = \frac{\theta(1+\theta(n^*-1))\sum_{i\in N^*}\delta_i}{\theta^2+(1+\theta n^*)(2+\theta(n^*-3))}. \tag{15}$$

We now state the two (i.e., main) propositions of our paper. The first one will characterize all pure strategy equilibria of the game (both unconstrained and limit pricing) when there is at least one inactive firm.

Proposition 1.

i) A pure strategy unconstrained equilibrium exists if and only if $\delta_{n^*+1} \leq \underline{\delta}^B_{n^*+1}$. *In such an equilibrium, each firm* $i \in N^*$ *charges price* $\widehat{p}_i = p^*_i(N^*)$ *and produces* $\widehat{q}_i = q^*_i(N^*)$, *while each firm* $i \in N \setminus N^*$ *charges* $\widehat{p}_i \geq c_i$ *and produces* $\widehat{q}_i = 0$.

ii) A price vector $(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*})$ *is a pure strategy LPE price vector for active firms if and only if it satisfies (10) and* $\widehat{p}_i \in [\underline{p}^B_i, \overline{p}^B_i]$ *for all* $i \leq n^*$. *In each such equilibrium,* $\widehat{p}_{n^*+1} = c_{n^*+1}$ *and* $\widehat{q}_{n^*+1} = 0$*; and for each* $i > n^* + 1$, $\widehat{p}_i \geq c_i$ *and* $\widehat{q}_i = 0$.

iii) A pure strategy LPE exists if and only if $\delta_{n^*+1} \in I^B = (\underline{\delta}^B_{n^*+1}, \min\{\overline{\delta}^B_{n^*+1}, \delta_{n^*}\})$.

Recall that if all firms are active, then there is a unique equilibrium given by Lemma 2 for $S = N$. In part *i*), we prove the conditions under which an unconstrained equilibrium exists and provide full characterization when there is at least one inactive firm. When this firm's quality-cost gap is sufficiently low, the active firms play the same equilibrium strategies that they would in a world in which the inactive firms did not exist and thus disregard these. This case has the same flavour as the occurrence of "blockaded entry" in standard entry deterrence models. In this case, the FOCs in the N^*-firm market hold with equalities and the price decision of each active firm $i \in N^*$ is uniquely determined as $p^*_i(N^*)$.

In parts *ii*) and *iii*), we provide two characterizations of the LPE price vectors of any given game. In the first characterization, we show that conditions 1 and 2, which are stated in (10) and (11), respectively, are necessary and sufficient for a LPE to exist. For example, in the two-firm case, when $n^* = 1$, there is a unique LPE, which is given by $(\widehat{p}_1, \widehat{p}_2) = (A_1 - \frac{\delta_2}{\theta}, c_2)$, by part *ii*). This finding in the duopoly market coincides with the LPE characterizations of Muto (1993) and Zanchettin (2006) respectively when there are only cost asymmetries ($A_1 = A_2 = A$); and $\delta_1 = 1$, $\delta_2 \in (0, 1]$, and $\lambda = 1$. Our results generalize the ideas to an $n-$firm framework by allowing both cost and quality asymmetries. In part *iii*) of this proposition, we fix the quality-cost differences firms apart from firm $n^* + 1$ as $\delta_i > \delta_j$ for $i < j$. If $\delta_{n^*} \geq \overline{\delta}^B_{n^*+1}$, then a LPE exists if and only if $\delta_{n^*+1} \in (\underline{\delta}^B_{n^*+1}, \overline{\delta}^B_{n^*+1})$. This characterization result proves that limit pricing equilibria occur for a large set of parameter configurations. In the Appendix

(Proposition 9), we also provide various comparative statics about the sensitivity of limit pricing strategies to the degree of substitutability (θ).

Our second main result is the existence of multiple efficiency-based limit-pricing equilibria when there are at least three firms and $n^* \in [2, n-1]$ of them are active. This result follows from Proposition 1 but due to its importance, we state it as a separate result.

Proposition 2. *Assume that a LPE exists. Each firm $j > n^*$ is inactive. There is a continuum of LPE price vectors $(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*})$ for active firms N^* when $n \geq 3$ and $n^* \in [2, n-1]$. The LPE price vector is unique when $n \geq 2$ and $n^* = 1$.*

When all firms are active, the equilibrium is unconstrained and therefore it is unique. When there are one active firm and at least one inactive firm in the market ($n^* = 1$), there is a unique LPE such that the most efficient firm drives the relatively inefficient firm(s) out of the market. However, when there are at least two active firms and at least one inactive firm ($n^* \in [2, n-1]$), the set of limit pricing equilibria is multiple. The relatively more efficient firms limit their prices in multiple ways to induce the exit of their rival(s). This multiplicity is driven by the fact that, when more than (one) efficient firm engage in limit pricing, the strategic interaction among these active firms becomes dominated by the incentive of keeping the potential entrant out of the market while stealing most of the potential entrant's demand. The kinks in the profit functions (and hence the multiplicity of limit-pricing equilibria) arise from this incentive while prevailing on the standard one of stealing active rivals' demand at the highest possible own price. The existence of multiple equilibria in simple linear Bertrand-models is in sharp contrast with the previous literature, which found a unique Bertrand-equilibrium for a large class of demand functions[19].

Last, a simple numerical example helps fix these ideas.

Example: Let there be three firms, namely $N = \{1, 2, 3\}$. Let $(A_1, A_2) = (23, 23)$ and $(c_1, c_2) = (2, 2)$. Thus, the quality-cost differences of firms 1 and 2 are $\delta_1 = \delta_2 = 21$. Let the inverse demand be $p_i = A_i - q_i - 0.8(q_j + q_l)$, where $i, j, l = 1, 2, 3$ and $i \neq j \neq l$. Thus, the demand parameters of a two-firm and three-firm market are $a_{12} = a_{22} = \frac{115}{9}$, $b_2 = \frac{25}{9}$, $d_2 = \frac{20}{9}$ and for $A_3 = 25$,

[19] Ledvina and Sircar (2011, 2012) claim the uniqueness of Bertrand equilibrium in our set-up. They argue that firm $n^* + 1$ charges at its marginal cost in a limit pricing equilibrium. However, one also needs to make sure that firm $n^* + 1$ produces a zero output.

$a_{13} = a_{23} = \frac{75}{13}$, $a_{33} = \frac{205}{13}$, $b_3 = \frac{45}{13}$, $d_3 = \frac{20}{13}$ respectively. We differentiate three cases:

Case 1: (Unconstrained triopoly) Let $\delta_3 \geq \overline{\delta}_3^B = \frac{756}{47}$. Using (7) and (8), a three-firm equilibrium calls for $q_3^*(\{1,2,3\}) = \frac{9(47\delta_3 - 756)}{286}$. Thus, all firms are active in equilibrium and the equilibrium price and quantity strategies are uniquely given by Lemma 2.

Case 2: (Unconstrained duopoly) Let $\delta_3 < \underline{\delta}_3^B = \frac{140}{9}$ and Proposition 1-*i*) applies. If all firms operate, then the relaxed three-firm equilibrium would apply. However, $q_3^*(\{1,2,3\}) < 0$ at $\delta_3 < \frac{140}{9}$; therefore, firm 3 is not active in equilibrium. Is there an unconstrained duopoly equilibrium, where only firms 1 and 2 operate? Using (7) and (8), such an equilibrium would call for $p_1^*(\{1,2\}) = p_2^*(\{1,2\}) = 5.5$, and $q_1^*(\{1,2\}) = q_2^*(\{1,2\}) = \frac{175}{18}$. Then $D_3^N(p_1 = 5.5, p_2 = 5.5, p_3) = \frac{5(9(A_3 - p_3) - 140)}{13}$ by (3), and firm 3 does not have any incentive to produce by setting a price $p_3 \geq c_3$ as $\delta_3 < \frac{140}{9}$. Accordingly, the FOCs hold with equalities in the two-firm market and each price vector $(p_1 = 5.5, p_2 = 5.5, p_3 \geq c_3)$ constitutes an equilibrium as claimed by Lemma 1-*i*. See also Figure 1a) for an illustration of the demand and profit functions of active firms when $p_3 = c_3 = 10$. Observe that the equilibrium occurs at a price vector where both the demand and profit functions are differentiable.

Case 3: (Limit pricing) Let $A_3 = 25$ and $c_3 = 9$. Thus, $\delta_3 = 16 \in I^B = (\underline{\delta}_3^B, \overline{\delta}_3^B) = (\frac{140}{9}, \frac{756}{47})$ and Proposition 1-*ii* applies. By case 1, a three-firm equilibrium calls for $q_3^*(\{1,2,3\}) = -\frac{18}{143} < 0$; therefore, firm 3 is not active in equilibrium. Moreover, there cannot be an unconstrained duopoly equilibrium as $D_3^N(p_1 = 5.5, p_2 = 5.5, p_3) = \frac{425 - 45p_3}{13}$, and firm 3 has an incentive to produce by setting a price slightly higher than 9. Therefore, if such an equilibrium exists, it (denote it by $(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3)$) must be a LPE, where $\widehat{p}_3 = c_3$ and $D_3(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3 = c_3) = b_3\delta_3 + d_3(\widehat{p}_1 + \widehat{p}_2 - A_1 - A_2) = 0$ by ii) in Proposition 3. Then using $b_3 = \frac{45}{13}$, $d_3 = \frac{20}{13}$, $\delta_3 = 16$, and $A_1 + A_2 = 46$, we find that $\widehat{p}_1 + \widehat{p}_2 = 10$. Thus, firms 1 and 2 limit their total prices to 10 with respect to an unconstrained level of 11 to induce the exit of firm 3. We now show that there exists a continuum of LPE in which firms 1 and 2 are active and the equilibria are of the form[20]

$$\frac{109}{22} \leq \widehat{p}_1, \widehat{p}_2 \leq \frac{111}{22} \text{ and } \widehat{p}_1 + \widehat{p}_2 = 10 \text{ and } \widehat{p}_3 = c_3 = 9. \tag{16}$$

[20]There are only limit pricing equilibria in this case.

To see this, consider any price vector that satisfies (16). Firm 1 should not have an incentive to increase its price and let firm 3 in. Firm 1's profit in the three-firm market is $\pi_1^N = D_1^N(p_1, p_2, p_3)(p_1 - c_1) = (a_{1,3} - b_3 p_1 + d_3(p_2 + p_3))(p_1 - c_1)$, where $p_3 = c_3 = 9$. The negativeness of the related derivative is equivalent to $D_1^N(p_1, p_2, p_3 = 9) - b_3(p_1 - c_1) \le 0$, or equivalently $p_1 \ge \frac{109}{22}$ because $p_2 = 10 - p_1$. Similarly, firm 1 should not have an incentive to reduce its price and steal consumers from firm 2. Firm 1's profit in the two-firm market is $\pi_1^{\{1,2\}} = D_1^{\{1,2\}}(p_1, p_2)(p_1 - c_1) = (a_{1,2} - b_2 p_1 + d_2 p_2)(p_1 - c_1)$. The positiveness of the related derivative equals to $D_1^N(p_1, p_2, p_3 = 9) - b_3(p_1 - c_1) \ge 0$, or $p_1 \le \frac{73}{14}$ because $p_2 = 10 - p_1$. A symmetric argument shows that when $p_2 \in [\frac{109}{22}, \frac{73}{14}]$, firm 2 does not have an incentive to increase or decrease its price. Since $\widehat{p}_1 + \widehat{p}_2 = 10$, both firms' deviation incentives are eliminated when $\widehat{p}_i \in [\frac{109}{22}, \frac{111}{22}]$, $i = 1, 2$.

To explain equilibrium incentives, take the triple $(5, 5, 9)$, which constitutes a LPE. By construction, the left-hand derivative of the profit function π_i with respect to p_i is positive, while the right-hand derivative is negative. For firm 1, it is not worth charging a price lower than 5 because then only customers from firm 2 are attracted. It is not worth charging a higher price either because then customers may defect to *both* firms 2 and 3. A symmetric argument holds for firm 2. The kinks in the demand and profit functions of firms 1 and 2 are the key properties that make *multiple equilibria* possible (See Figure 1b). We will derive the associated best responses of firms 1 and 2 in the next section.

4 Supermodularity

The literature mostly assumed that all firms are active in equilibrium, and showed uniqueness by establishing that the Bertrand-game is supermodular. In this section, we show that supermodularity no longer holds if some firms may not be active in equilibrium, and best responses may not have positive slopes (or non-monotone).

Proposition 3. *Let $n \ge 3$. A linear Bertrand model with continuous best responses may not be supermodular or log-supermodular or satisfy the single-crossing property. Moreover, the best responses may be non-monotone.*

For a proof of this proposition, we refer to the Case 3 of the example of Section

3. Here, we show that the best response of each firm 1 and 2 is non-monotone and continuous as shown in Figure 2. The results follow after this observation as we show in the Appendix. In the absence of firm 3, the duopoly best response of firm 1 is $p_1 = BR_1^{\{1,2\}}(p_2) = \frac{33+4p_2}{10}$ by (6). By the symmetry between firms 1 and 2, the duopoly best responses intersect at $(p_1, p_2) = 5.5$, which corresponds to point O in Figure 2. However, firm 3 deviates to produce when its rivals both charge 5.5 as we have already shown. In order for firm 3 to not produce, the equilibrium prices of the active firms should satisfy $\widehat{p}_1 + \widehat{p}_2 = 10$. Thus, duopoly best responses can only be valid below the $p_1 + p_2 = 10$ line (or $p_2 < \frac{67}{14}$). On the above of this line (or $p_2 > \frac{111}{22}$), the projected-triopoly best responses of firms 1 and 2 are valid. By (6), the triopoly best response of firm 1 is $p_1 = BR_1^{\{N\}}(p_2, p_3) = \frac{33+4p_2+4p_3}{18}$. By setting $p_3 = c_3 = 9$, we project this best response on the $p_1 - p_2$ quadrant as $p_1 = BR_1^{\{N\}}(p_2, p_3 = 9) = \frac{69+4p_2}{18}$. Altogether, when $p_3 = c_3$, for each $i, j = 1, 2$, $i \neq j$, the best response of firm i is non-monotone and continuous and given by

$$p_i = BR_i(p_j) = \begin{cases} 3.3 + 0.4p_j & \text{if } p_j < \frac{67}{14} \\ 10 - p_j & \text{if } \frac{67}{14} \leq p_j \leq \frac{111}{22} \\ \frac{69+4p_j}{18} & \textit{otherwise.} \end{cases}$$

Observe Figure 2 that the best responses intersect along the segment $seg[CD] = \{(p_1, p_2) \in \mathbb{R}^2 : \frac{109}{22} \leq p_1 \leq \frac{111}{22} \text{ and } p_1 + p_2 = 10\}$. That is, each $\widehat{\mathbf{p}}_N \in \mathbb{R}^3_+$ such that $(\widehat{p}_1, \widehat{p}_2) \in seg[CD]$ and $\widehat{p}_3 = c_3$ is a pure-strategy equilibria of this game. This geometric finding is consistent with the previous algebraic finding.

5 Discussions and Extensions

5.1 Robustness of results

Our multiplicity of (kinked demand) limit pricing equilibria and the strategic substitutes mode of competition results are both driven by the fact that the strategic interaction among the active firms becomes dominated by the incentive of keeping the potential entrant out of the market. Both the kink in the demand and profit functions (hence the multiplicity of limit pricing equilibria) and non-supermodularity results drive from this incentive. We would expect a similar incentive effect to produce multiplicity of equilibria and competition in strategic modes of competition in any model where, in some parameter region, price competition between active firms (producing substitutable products) may cause an outflow of the total market demand served by the competing firms. This suggests that most of the results should (qualitatively) generalize to other specifications of the model, e.g., non-linear and less symmetric (θ_i rather than θ in (2)) specifications of demand, non-linear costs, non-strictly rankable quality-cost gaps, different order of players' moves (see the next section) or even to different models of horizontal differentiation (i.e., Hotelling model).

It is useful to point out the similarities between our results and the early results in the Hotelling literature. For example, Merel and Sexton (2010) characterize the kinked demand and profit Bertrand equilibria within the linear Hotelling duopoly model with fixed (extreme) firm's locations. They also show that price competition turns into competition in strategic substitutes under certain parameter conditions. The outflow in overall demand comes here from uncovering the market instead of a relatively less efficient firm, but the deep intuition of this and our results are quite similar. Both results are two applications of kinked demand theory in different models of horizontal differentiation. Our results are therefore general, intuitive, and rationalize previous findings attributed to peculiar characteristics of special models.

5.2 Sequential market entry

In this section, we test the robustness of our results to the order of moves. Let us also follow our original model's preliminary assumptions and notations. We consider the following sequential move incumbents and entrant game with complete information. Let $N^* = \{1, 2, ..., n^*\}$ denote the set of actively participating incumbents in an established Bertrand oligopoly, that is we assume $\delta_{n^*} > \frac{\theta(1+\theta(n^*-2)) \sum_{j \in N^*} \delta_j}{(1+\theta(n^*-1))(2+\theta(n^*-3))}$ by Lemma 2. Consider now that the threat of entry by firm $n^* + 1$ appears. The Stackelberg game has two stages.[21]

Stage 1: The incumbents simultaneously and independently set their price levels.

Stage 2: The potential entrant chooses whether or not to enter. If the firm decides to enter the market, it sets its price, taking the incumbents' prices as

[21] See Gilbert and Vives (1986) for a similar entry deterrence game, where the ex-ante symmetric incumbents choose outputs rather than prices in a homogeneous good set-up.

given. For simplicity, assume there are no fixed costs of entry.

We search for subgame perfect equilibria. Let us define two critical cutoff values for the quality-cost difference of the entrant as $\underline{\delta}^{SP}_{n^*+1}$ and $\overline{\delta}^{SP}_{n^*+1}$, where $\underline{\delta}^{SP}_{n^*+1} = \underline{\delta}^{B}_{n^*+1}$ and

$$\overline{\delta}^{SP}_{n^*+1} = \frac{\theta(2-\theta^2+2\theta(n^*-1)(2+\theta(n^*-1)))\sum_{i\in N^*}\delta_i}{(1+\theta(n^*-1))(4-3\theta^2+2\theta(n^*-1)(3+\theta(n^*-2)))}. \qquad (17)$$

In the Appendix, we show that there are three critical regions to consider assuming that $\overline{\delta}^{SP}_{n^*+1} < \delta_{n^*}$: i) entry is blocked if $\delta_{n^*+1} < \underline{\delta}^{SP}_{n^*+1}$, or ii) entry is prevented through efficiency-based limit pricing if $\delta_{n^*+1} \in I^{SP} = (\underline{\delta}^{SP}_{n^*+1}, \overline{\delta}^{SP}_{n^*+1})$, or iii) entry is allowed if $\delta_{n^*+1} > \overline{\delta}^{SP}_{n^*+1}$. We further show that a LPE exists if and only if condition 1 and a stricter condition compared to condition 2 holds. Thus, we obtain the following proposition.

Proposition 4. *Let $\delta_{n^*+1} \in I^{SP} = (\underline{\delta}^{SP}_{n^*+1}, \min\{\overline{\delta}^{SP}_{n^*+1}, \delta_{n^*}\})$. Each LPE of the above sequential move Stackelberg price-setting game is also a LPE of the simultaneous move Bertrand game among $N^* \bigcup \{n^*+1\}$ players.*

In light of Propositions 1 and 4, there is a continuum of Stackelberg limit pricing equilibria and the Stackelberg price-setting game may not be supermodular when $n^* \geq 2$ and the entrant is inactive by Proposition 7 of the Appendix.

6 Limit pricing and market performance

For the welfare analysis, it is important to know how equilibrium multiplicity affects consumer welfare, producer surplus, and total welfare in the described Bertrand and Stackelberg price-setting games. Based on Propositions 1 and 7, an equivalent way of writing the set of LPE prices of firm $i \in N^*$ is provided in the following corollary.

Corollary 1. *Let $n^* \geq 2$ and $w = B, SP$ represent the Bertrand or Stackelberg price-setting game played among $N^* \cup \{n^*+1\}$ players. Let M be provided by (10). The set of LPE prices of firm $i \in N^*$ in the w game is given by*

$$K_i^w = \{\widehat{p}_i \textit{ such that } \widehat{p}_i \in [\max\{\underline{p}_i^w, M - \sum_{j\in N^*\setminus i} \overline{p}_j^w\}, \min\{\overline{p}_i^w, M - \sum_{j\in N^*\setminus i} \underline{p}_j^w\}]\}.$$

The question is then which prices in the set of the LPE price vectors of K_i^w maximize the well-being of different agents of the economy.

6.1 The surplus of individual producers

An individual producer's surplus is equal to his profit. Our next result shows that in the set of LPE prices, each active firm prefers the equilibrium where the firm's price is the lowest (and the other firms' total price is the highest).

Proposition 5. *Let $w = B, SP$. Firm i prefers to charge its lowest LPE price, i.e.,* $\max\{\underline{p}_i^w, M - \sum_{j\in N^*\setminus i} \overline{p}_j^w\}$*, among the set of the LPE prices of firm i (K_i^w) in the w game.*

This result is intuitive as it states that each firm's profit is increasing in the other firms' total price (their products being substitutes). Therefore, in the set of equilibrium price vectors K_i^w, each firm naturally prefers the other firms to charge as high a price as is consistent with equilibrium. This observation implies that each firm has a strong interest in keeping out weaker rivals by charging a low price itself, and thus, a free-rider problem is not associated with this joint preemptive behavior.

This result in the Stackelberg game also contributes to the ongoing debate of whether entry prevention can be seen as a public good or not. If any firm sets its prices sufficiently low to prevent entry, all firms are protected from competition. Thus, each firm could free-ride on the entry-preventing activities of its competitors with the potential implication that there would be little entry deterrence. Our result shows that this is not the case in our model. Every incumbent would like to contribute to entry deterrence as much as the firm can given that the entry will be prevented.[22] This is in contrast to typical public good provision problems.

[22] This finding resembles the result of Gilbert and Vives (1986) in a homogenous good Stackelberg quantity-choosing game.

6.2 Consumer surplus (CS)

For simplicity, let $n^* = 2$ from now on. We rewrite the utility as a function of prices by plugging in $D_i(p_1, p_2)$ from (3) into (1). Thus, consumer surplus becomes

$$\begin{aligned} CS(p_1, p_2) = & \sum_{k \in \{1,2\}} A_k D_k(p_1, p_2) - \tfrac{\lambda}{2} \sum_{k \in \{1,2\}} (D_k(p_1, p_2))^2 - \ldots \\ & -\lambda\theta D_1(p_1, p_2) D_2(p_1, p_2) - p_1 D_1(p_1, p_2) - p_2 D_2(p_1, p_2). \end{aligned} \tag{18}$$

Proposition 6. *Let $w = B, SP$ and $n^* = 2$. Consumers would prefer either $\max\{\underline{p}_i^w, M - \overline{p}_j^w\}$ or $\min\{\overline{p}_i^w, M - \underline{p}_j^w\}$, $j \neq i$, over the other limit pricing equilibrium prices of firm i in the w game.*

This result states that consumers prefer the asymmetric prices of the active firms; that is, prices where one firm charges the lowest price consistent with a limit pricing equilibrium, while the other firm charges the highest price consistent with a limit pricing equilibrium. The reason is that the CS is convex in the prices; thus, extreme prices maximize the CS.

6.3 Total surplus (TS)

The results for the total producer surplus (TPS) and the total surplus (TS=the sum of CS and TPS) are less straightforward. The formal analysis for our welfare calculations are contained in Proposition 8, in the Appendix. In particular, we show in the Appendix that depending on the parameter values, the equilibrium that maximizes the TPS and the TS may be either a corner solution or an interior price vector where the two active firms charge prices that are more symmetric. Without going into details here, we would like to provide an intuition about why the TPS or TS may be maximized at an interior or at a corner solution in the set K_i^w, where $w = B, SP$. We also discuss how the total surplus and the consumer surplus may be maximized in K_i^w at similar or different prices.

Case 1. When the two active firms are symmetric ($A_1 = A_2$ and $c_1 = c_2$), then both the TPS and TS are maximized when the two firms charge an equal price (p^*, p^*). As we already argued, the CS is always maximized at a corner price vector of K_i^w. Moreover, we show in the Appendix that the CS is *minimized* at the symmetric price vector (p^*, p^*) (see Figure 3a).

Consequently, we have an interesting case where maximizing the CS and the TS (or the TPS) yields polar opposite recommendations. In particular, if the active firms are able to coordinate how they wish to keep out potential rivals, then they would choose the equilibrium with equal prices, which yields the lowest CS and the highest TS. We provide a geometric interpretation of this argument in Figure 3 with two additional remarks. First, if there are multiple equilibria when the active firms keep a rival out, then it is better for the CS if they choose different prices. For example, if the active firms alternate over time in terms of choosing price combinations so that the rival is kept out, this behavior enhances the CS. Second, as δ_3 changes in the range $(\underline{\delta}_3, \min\{\overline{\delta}_3, \delta_2\})$ where the equilibria are constrained, there are equilibria for a higher value of δ_3, which make the consumers better off than some equilibria that occur when δ_3 is lower.

Case 2. If the two active firms are asymmetric enough (i.e., $\delta_1 \in (\frac{(2-\theta)(1+2\theta)\delta_2}{2+\theta}, \frac{(2-\theta^2)\delta_2}{\theta})$),[23] then the above dichotomy disappears. All welfare measures are maximized at extreme prices as the firms are now different enough that it is better for all groups to let the more efficient (attractive) firm produce as much as possible (see Figure 3b). There are also some intermediate regions where two welfare measures are maximized at the same constrained price vector, while the other one is maximized at a different price vector.

7 Conclusion

Price-setting games are an important class of games that have been extensively studied in the literature. Most of the literature assumes that all firms are active and shows the uniqueness of equilibrium in a differentiated Bertrand oligopoly. However, firms might prefer not to be active in real-life situations. For instance, a cost-reducing innovation or cost-efficient mergers might induce firms to exit the market. Our analysis shows that when the number of firms is greater than two, the game need not satisfy supermodularity, log-supermodularity, or even the single-crossing property. Therefore, previous existence and uniqueness of equilibrium theorems regarding supermodular games do not apply in our framework. We argue that Bertrand best responses might have negative slopes and there is a continuum of pure strategy Bertrand-Nash equilibria on a wide range of parameter

[23] See Proposition 8 of the Appendix for more discussion for the Bertrand game.

values with more than two firms and $n^{**} \in [1, n-2]$ firms are inactive. Based on an iterative algorithm, we showed that the set of active firms is the same in all equilibria. As far as we know, our paper is the first that studies price-setting games in the context of active and inactive firms in a comprehensive way.

We also consider a Stackelberg entry game in which the incumbents first set their prices simultaneously in the first stage. In the second stage, an entrant decides to enter the market and chooses its price. We show that each limit pricing strategy of this Stackelberg price-choosing game is also a limit pricing strategy of the associated simultaneous move Bertrand game. Thus, our main results are robust when we change the order of players' moves.

When we characterized the set of pure strategy Bertrand or Stackelberg equilibria, we stated the importance of limit pricing strategies among firms to keep out their rivals from the market. These strategies are different from the predatory strategies the early literature mostly considers. If a firm does not engage in a predatory strategy, then entry is accommodated. However, an efficiency-based limit pricing strategy stems from the competitive nature of the problem. These strategies naturally emerge because all firms cannot be active at the same time in the first place because of some firms' relative inefficiency compared to the other(s). Moreover, the presence of some inactive firms cannot be safely ignored as they can still affect the pricing decisions of the active firms. Accordingly, in each limit pricing equilibrium, the active firms limit their prices to induce the exit of their competitors.

The possibility of multiple equilibria raises an equilibrium selection problem. In that regard, we investigated the consumer surplus, the producer surplus and the total surplus maximizing price vectors among the set of limit-pricing equilibria in a triopoly market. If active firms are completely symmetric, then the total- and producer surplus maximizing limit pricing equilibrium price vector minimizes consumer surplus. Thus, if the symmetric producers can coordinate their actions and choose an equilibrium price vector that maximize their total surplus, then they incidentally minimize the consumer surplus. However, if they are sufficiently asymmetric, then all welfare measures may be maximized at the same equilibrium price vector. Our results in the Stackelberg game also confirm that entry prevention is not a public good. Each incumbent prefers to limit its price as much as it can to prevent entry by the potential entrant.

In our n-firm oligopoly model, we worked with a very rich linear demand formulation that respects possible vertical or horizontal differentiation among firms' products. Less general versions of this demand formulation have been used in many IO papers to model product differentiation in price-setting games. Costs were also assumed to be linear. It should be obvious to the reader that our results do not stem from these linearity assumptions, but instead, the price-setting competition inevitably inherits limit pricing strategies. We predict that the same kind of results will be in non-linear demand or cost environments. However, the price combinations of the active firms that leave their rival indifferent about being active or not may form a curve in the price space (rather than a line). As future research, it would be also interesting to look for the existence and multiplicity of pure strategy limit pricing strategies when the Bertrand game is played repeatedly or the players face demand or cost uncertainties.

8 Appendix

Proof of Lemma 1: Consider any $\mathbf{p}_N = (p_j)_{j\in N}$, where $p_j \geq c_j$. Let $S' \subseteq N$ be the set of active firms at this price vector based on (3). Take any $i \in S'$. The derivative of $\pi_i^{S'}((p_j)_{j\in S'})$ with respect to p_i exists at all points where the set of active firms S' does not change in a neighborhood of p_i. In this case, the derivative is $D_i^{S'}(\mathbf{p}_{S'}) + (p_i - c_i)\frac{\partial q_i}{\partial p_i} = a_{i,s'} - 2b_{s'}p_i + d_{s'}\sum_{j\in S'\setminus i} p_j + b_{s'}c_i$ by (4), which is strictly decreasing in p_i.

Given $(p_j)_{j\in S'\setminus i}$, as p_i increases up to some level $\tilde{p}_i$, a new firm $h^{'} \in N \setminus S'$ may be on the verge of entering the market by charging its price $p_{h'}$. In such a situation, the prices of firms $\tilde{\mathbf{p}}_{S''} = (\tilde{p}_i, (p_j)_{j\in S''\setminus i})$ satisfy that $D_{h'}^{S''}(\tilde{\mathbf{p}}_{S''}) = 0$, where $S'' = S' \cup h^{'}$. At this price vector, the left-hand derivative is $D_i^{S'}(\tilde{\mathbf{p}}_{S'}) - b_{s'}(p_i - c_i)$, while the right-hand derivative becomes $D_i^{S''}(\tilde{\mathbf{p}}_{S''}) - b_{s'+1}(p_i - c_i)$, where $\tilde{\mathbf{p}}_{S'} = (\tilde{p}_i, (p_j)_{j\in S'\setminus i})$. Note that $D_i^{S''}(\tilde{\mathbf{p}}_{S''})$ equals to

$$D_i^{S''}(\tilde{\mathbf{p}}_{S''}) = (b_{s'+1} + d_{s'+1})(A_i - p_i) - d_{s'+1}(A_h - p_{h'}) + d_{s'+1}(\sum_{j\in S'} p_j - \sum_{j\in S'} A_j) \quad (19)$$

by (3). Moreover, $D_{h'}^{S''}(\tilde{\mathbf{p}}_{S''}) = 0$ implies that $A_h^{'} - p_h^{'} = -(d_{s'+1}/b_{s'+1})(\sum_{j\in S'} p_j - \sum_{j\in S'} A_j)$ by (9). Substituting the value of $A_{h'} - p_{h'}$ into (19) yields

$$(b_{s'+1} + d_{s'+1})(A_i - p_i) + d_{s'+1}(1 + \frac{d_{s'+1}}{b_{s'+1}})(\sum_{j\in S'} p_j - \sum_{j\in S'} A_j). \quad (20)$$

Straightforward calculations gives $b_{s'+1}+d_{s'+1} = b_{s'}+d_{s'}$ and $d_{s'+1}(1+d_{s'+1}/b_{s'+1}) = d_{s'}$. Hence, $D_i^{S''}(\tilde{\mathbf{p}}_{S''}) = D_i^{S'}(\tilde{\mathbf{p}}_{S'})$ by (3). Given that $q_i = D_i^{S'}(\tilde{\mathbf{p}}_{S'}) = D_i^{S''}(\tilde{\mathbf{p}}_{S''})$, $b_{s'+1} > b_{s'}$, and $p_i - c_i \geq 0$, we obtain that $q_i - b_{s'+1}(p_i - c_i) < q_i - b_{s'}(p_i - c_i)$ so the right-hand derivative is strictly lower than the left-hand derivative in the case of regime change. Therefore, as p_i increases and more and more inactive firms may become active, the derivative of both q_i and π_i with respect to p_i either exists and is decreasing in a neighborhood or it does not exist, but one sided derivatives always exist, and the right-hand derivative is always less than the left-hand derivative and thus the demand and profit functions are kinked. Therefore, as long as firm i remains active as the firm increases its price, its profit function is strictly concave in p_i. However, at a point where firm i becomes inactive its profit becomes zero, and the profit stays zero for any p_i higher than that. Therefore, the profit function is quasi-concave and single-peaked in p_i.

Existence of equilibrium follows from the standard results. In particular, note that $p_i > A_i$ and (2) together with the non-negativity of quantities implies that firm i cannot be active. Therefore, charging $p_i > A_i$ yields a zero profit, so such strategies can be ignored because a zero profit can be also achieved by charging c_i. So, the best reply of firm i always intersects with set $[c_i, A_i]$, and we can restrict the strategy space of firm i to $[c_i, A_i]$ without loss. Then we have a quasi-concave, continuous objective functions and convex, compact action spaces, so a pure strategy equilibrium exists.[24] $\square$

Proof of Lemma 2:

i) Let $S \subseteq N$ and consider any $j \in S$. We first show that if each $i \in S \backslash \{j\}$ uses $p_i^*(S) = \frac{\delta_i((1+\theta(s-1))(2+\theta(s-3)))-\theta(1+\theta(s-2))\sum_{k\in S}\delta_k}{(2+\theta(s-3))(2+\theta(2s-3))} + c_i$, then the unique best response for firm j is to use $p_j^*(S) = \frac{\delta_j((1+\theta(s-1))(2+\theta(s-3)))-\theta(1+\theta(s-2))\sum_{k\in S}\delta_k}{(2+\theta(s-3))(2+\theta(2s-3))} + c_j$. To see this, remark by (6) that the optimal choice of firm j is

$$p_j^*(S) = \frac{a_{j,s} + d_s \sum_{i\in S\backslash j} p_i + b_s c_j}{2b_s}. \tag{21}$$

By the initial supposition, first substitute $p_i = p_i^*(S)$, $i \in S \setminus \{j\}$, into (21). After some computations, $p_j^*(S)$ simplifies to $\frac{\delta_j((1+\theta(s-1))(2+\theta(s-3)))-\theta(1+\theta(s-2))\sum_{k\in S}\delta_k}{(2+\theta(s-3))(2+\theta(2s-3))} + c_j$, as desired. Substituting the equilibrium prices into (3) yields the equilibrium quantities as $q_i^*(S) = b_s(p_i^*(S) - c_i)$, upon some simplifications. As demand is linear and marginal costs are constant, uniqueness follows similarly as Vives (1999).

ii) Let $S \subseteq N$. Take any distinct $i, j \in S$. Subtracting $q_j^*(S)$ from $q_i^*(S)$ by using (7) and (8) gives

$$q_i^*(S) - q_j^*(S) = \frac{(1+\theta(s-2))(\delta_i - \delta_j)}{\lambda(1-\theta)(2+\theta(2s-3))}. \tag{22}$$

Since $\theta \in (0,1)$ and $\lambda > 0$, then $q_i^*(S) > q_j^*(S)$ if and only if $\delta_i > \delta_j$, as claimed. $\square$

[24]See for example Theorem 2.2 of Reny (2008).

Proof of Lemma 3:

i) It follows from the text.

ii) Take any $S' \subset N$ and let $\delta_h = \max_{j \in N \setminus S'} \delta_j$ and $S'' = S' \cup h$. Let $M' = \sum_{i \in S'} p_i^*(S')$. Suppose both that $D_h^{S''}(\mathbf{p}_{S'}^*(S'), p_h = c_h) > 0$ and there exists a limit pricing equilibrium in which only firms in S' are active. We claim that any equilibrium price vector of firms in S', say $\widehat{\mathbf{p}}_{S'}$, satisfies $D_h^{S''}(\widehat{\mathbf{p}}_{S'}, p_h = c_h) = 0$. Let $M = \sum_{i \in S'} \widehat{p}_i(S')$. It is clear it cannot be the case that $D_h^{S''}(\widehat{\mathbf{p}}_{S'}, p_h = c_h) > 0$ for firm h to be inactive. Therefore, suppose for a contradiction there exists an equilibrium price vector $\tilde{\mathbf{p}}_{S'}$ such that $D_h^{S''}(\tilde{\mathbf{p}}_{S'}, p_h = c_h) < 0$. That implies that $\sum_{i \in S'} \tilde{p}_i = M'' < M < M'$ by (9). Now take any $j \in S'$. Since $M'' < M$, then for sufficiently small ϵ, any price deviation in the $\epsilon-$neighbourhood of $\tilde{p}_j$ given $\tilde{\mathbf{p}}_{S \setminus j}$ is still associated with a market where only firms in S' actively produce. Hence S'-firm best responses, i.e., $BR_k^{S'}$, $k \in S'$, are valid below the $M = \sum_{i \in S'} \widehat{p}_i(S')$ hyperplane. But since $\bigcap_{l \in S'} Gr(BR_l^{S'}) = \mathbf{p}_{S'}^*(S')$ by the definition of unconstrained equilibrium and $M'' < M'$, then the best responses cannot intersect at $\tilde{\mathbf{p}}_{S'}$. Thus, $\tilde{\mathbf{p}}_{S'}$ cannot be an equilibrium price vector trivially, which is a contradiction. $\square$

Proof of Lemma 4:

Let $N^* = \{1, 2, ..., n^*\}$ be the set of firms found by the BIA. Define

$$V_L = \frac{\theta(1 + \theta(l-2)) \sum_{j \in L} \delta_j}{(1 + \theta(l-1))(2 + \theta(l-3))} \tag{23}$$

for some $L \subset N$ with $|L| = l$. We prove the result in three steps.

Step 1: Take any $H \subset N$ such that there exists a firm $i \in N \setminus H$ such that $\delta_i > \delta_j$ for some $j \in H$. WLOG, let $i = \arg\max_{k \in N \setminus H} \delta_k$. We claim that there cannot be an equilibrium where only firms in H are active, but firm i is not. Suppose on the contrary there exists such an equilibrium. Let $|H| = h$ and $G = H \cup i$. There are only two possible kinds of equilibrium: unconstrained and limit pricing.

1-i) Unconstrained equilibrium: An unconstrained equilibrium is characterized by the relaxed equilibrium prices, i.e., $(\widehat{p}_k, \widehat{q}_k)_{k \in H} = (p_k^*(H), q_k^*(H))_{k \in H}$, by Lemma 3-*i*). Using (7), (8), and (23), $q_j^*(H) > 0$ simplifies to $\delta_j > V_H$. Also remark that

$$D_i^G(\mathbf{p}_H^*(H), p_i = c_i) = b_{h+1}\delta_i + d_{h+1}(\sum_{k\in H} p_k^*(H) - \sum_{k\in H} A_k). \tag{24}$$

from (9). After substituting $p_k^*(H)$, $k \in H$, into (24) from (7), straightforward calculations yield that

$$D_i^G(\mathbf{p}_H^*(H), p_i = c_i) = b_{h+1}(\delta_i - V_H). \tag{25}$$

Since $\delta_j > V_H$ and $\delta_i > \delta_j$ by the initial supposition, $\delta_i > V_H$ as well. Hence, $D_i^G(\mathbf{p}_H^*(H), p_i = c_i) > 0$ as $b_{h+1} > 0$ for $\theta \in (0,1)$ in (25). Therefore, $q_i > 0$ by (3), a contradiction.

1-ii) LPE: In a LPE, it holds that $\widehat{q}_i = D_i^G(\widehat{\mathbf{p}}_H, \widehat{p}_i = c_i) = 0$ by Lemma 3-*ii*). A symmetric argument to Proposition 1-*ii* would tell that in order for firm $j \in H$ to not deviate to a lower or higher price from its limit price $\widehat{p}_j$ given $\widehat{p}_{-j}$, this limit price should lie in the interval $[\underline{p}_j, \overline{p}_j]$, where

$$\overline{p}_j = \frac{(1+\theta(h-1))(\delta_j - \delta_i)}{2+\theta(2h-3)} + c_j, \tag{26}$$

and

$$\underline{p}_j = \frac{(1+\theta h)(\delta_j - \delta_i)}{2+\theta(2h-1)} + c_j. \tag{27}$$

But $\overline{p}_j - \underline{p}_j$ simplifies to

$$\overline{p}_j - \underline{p}_j = \frac{\theta^2(\delta_j - \delta_i)}{(2+\theta(2h-3))(2+\theta(2h-1))}, \tag{28}$$

which is negative as $\delta_i > \delta_j$ by the initial assumption. Therefore, this case is not feasible, as desired.

Step 2: Let $1 \leq y < n^*$, and let $Y = \{1, 2, ..., y\}$ and $Z = Y \cup \{y+1\}$. We claim that there cannot be an equilibrium where only firms in Y are active. Assume by contradiction that there is.

2-i) Unconstrained equilibrium: As in Step 1-i, $(\widehat{p}_k, \widehat{q}_k)_{k\in Y} = (p_k^*(Y), q_k^*(Y))_{k\in Y}$. Further, note that $q_{y+1}^*(Z) > 0$ by the BIA. Hence using (7), (8), and

(23), we get $\delta_{y+1} > V_Z$. A symmetric calculation to the Step 1-i gives that

$$D^Z_{y+1}(\mathbf{p}^*_Y(Y), p_{y+1} = c_{y+1}) = b_{y+1}(\delta_{y+1} - V_Z) \tag{29}$$

from (9). As $\delta_{y+1} > V_Z$, (29) is positive and therefore firm $y+1$ is active, a contradiction.

2-ii) LPE: To ensure that firm $y+1$ is driven out of the market, the LPE prices of firms in Y satisfy $\sum\limits_{i\in Y} \widehat{p}_i = \sum_{i\in Y} A_i - \frac{b_{y+1}\delta_{y+1}}{d_{y+1}}$ and we have $D^Y_{y+1}(\widehat{\mathbf{p}}_Y) = 0$ and $\widehat{q}_{y+1} = 0$ from (3).

We claim that there exists a firm $j \in Y$ such that $\widehat{p}_j < \frac{(b_{y+1}+d_{y+1})(\delta_j - \delta_{y+1})}{2b_{y+1}+d_{y+1}} + c_j$. Otherwise, summing up equilibrium prices of all firms in Y yields

$$\sum_{i\in Y} \widehat{p}_i = \sum_{i\in Y} A_i - \frac{b_{y+1}\delta_{y+1}}{d_{y+1}} \geq \frac{(b_{y+1}+d_{y+1})(\sum_{i\in Y}\delta_i - y\delta_{y+1})}{2b_{y+1}+d_{y+1}} + \sum_{i\in Y} c_i. \tag{30}$$

But by Proposition 2, (30) can be rewritten as $q^*_{y+1}(Z) < 0$, a contradiction to the outcome of the BIA.

Using that there exists a firm $j \in Y$ such that $\widehat{p}_j < \frac{(b_{y+1}+d_{y+1})(\delta_j - \delta_{y+1})}{2b_{y+1}+d_{y+1}} + c_j$. Thus, the rightward derivative of firm j's profit is positive by using a similar argument to Step 2 of the proof of Proposition 1. That is, firm j has an incentive to increase its price. Hence, there cannot be a LPE where only firms in Y are active either.

Step 3: Suppose $n^* < n$, and let $1 < n^* < t$, with $T = \{1, 2, ..., t\}$. To finish the proof, we claim that there cannot be any equilibrium where only firms in T are active. Assume by contradiction that there is and we consider unconstrained and limit pricing equilibria again.

3-i) Unconstrained Equilibrium: It is sufficient to show that if $q^*_x(X) \leq 0$ for some $X = \{1, 2, ..., x\}$ then for all $X' = \{1, 2, ..., x'\}$ such that $x' > x$, it holds that $q^*_{x'}(X') < 0$. We prove the claim by induction. Assume that for some $X \subset N$, we have $q^*_x(X) \leq 0$. We need to show that $q^*_{x+1}(X \cup \{x+1\}) < 0$ holds. Using (7) and (8), $q^*_x(X) \leq 0$ implies that $\delta_x \leq V_x$. Now define

$$V'_{x+1} = \frac{\theta(1+\theta(x-1))\sum_{j\in X}\delta_j}{(1+\theta x)(2+\theta(x-2)) - \theta(1+\theta(x-1))}. \tag{31}$$

Subtracting V_x from V'_{x+1} yields

$$V'_{x+1} - V_x = \frac{\theta^3(1-\theta)\sum_{j\in X}\delta_j}{(1+\theta(x-1))(2+\theta(x-3))(2+3\theta(x-1)+\theta^2(1+x(x-3)))}, \tag{32}$$

which is positive at $\theta \in (0,1)$ and $x \geq 1$. In sum $\delta_x \leq V_x < V'_{x+1}$. But, as $\delta_{x+1} < \delta_x$ by assumption, $\delta_{x+1} < V'_{x+1}$. Thus, $q^*_{x+1}(X \cup \{x+1\}) < 0$ by Proposition 2. This completes the inductive step.

3-ii) LPE: Assume that there is a LPE where the set of active firms is $T = \{1, 2, ..., t\}$ with $t > n^*$. Let $T'' = T \cup \{t+1\}$. In a LPE, the equilibrium prices of firms in T satisfy $\sum\limits_{i\in T} \widehat{p}_i = \sum_{i\in T} A_i - \frac{b_{t+1}\delta_{t+1}}{d_{t+1}}$ so that $\widehat{q}_{t+1} = D^{X''}_{t+1}(\widehat{\mathbf{p}}_T, p_{t+1} = c_{t+1}) = 0$ by (3). Using (7), (8), and (23), $q^*_t(T) < 0$ simplifies to $\delta_t < V_T$ and thus $\delta_{t+1} < V_T$ as well.

We claim that there exists a firm $k \in T$ such that $\widehat{p}_k > F_k = \frac{(b_t+d_t)(\delta_k-\delta_{t+1})}{2b_t+d_t} + c_k$. Otherwise, summing up $\widehat{p}_i \leq F_i$ across $i \in T$ and reorganizing terms yields $\delta_{t+1} \geq V_T$, a contradiction. Firm k, for whom $\widehat{p}_k > \frac{(b_t+d_t)(\delta_k-\delta_{t+1})}{2b_t+d_t} + c_k$, has an incentive to decrease its price by following the similar arguments to Step 2 of the proof of Proposition 1. Hence, there is no LPE in which the set of active firms is T. Since t was an arbitrary integer such that $t > n^*$, our proof is now complete. $\square$

Proof of Proposition 1 :

i) In an unconstrained equilibrium, 1) $q^*_{n^*+1}(\tilde{N}) < 0$ and 2) $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) \leq 0$ by Lemma 3-*i*. Let $\overline{\delta}_{n^*+1}$ be such that $q^*_{n^*+1}(\tilde{N}) = 0$ if we had $\delta_{n^*+1} = \overline{\delta}^B_{n^*+1}$ ceteris paribus. But then $p^*_{n^*+1}(\tilde{N}) = c_{n^*+1}$ and solving for $\overline{\delta}^B_{n^*+1}$ by using (7) yields (15). Thus, $q^*_{n^*+1}(\tilde{N}) < 0$ implies that $\delta_{n^*+1} < \overline{\delta}^B_{n^*+1}$ from (8).

Similarly, let $\underline{\delta}^B_{n^*+1}$ be such that if we had $\delta_{n^*+1} = \underline{\delta}^B_{n^*+1} = \underline{A}_{n^*+1} - \underline{c}_{n^*+1}$, then $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = \underline{c}_{n^*+1}) = 0$. Accordingly, $\underline{\delta}^B_{n^*+1}$ solves $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = \underline{c}_{n^*+1}) = 0$ in (3). Related simplifications yield $\underline{\delta}^B_{n^*+1}$ as (14). But note that $\partial D^{\tilde{N}}_{n^*+1}(.)/\partial\delta_{n^*+1} = b_{n^*+1} > 0$ from (9) and $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) \leq 0$ by the initial supposition. Therefore, $\delta_{n^*+1} \leq \underline{\delta}^B_{n^*+1}$. Last, note that

$$\overline{\delta}^B_{n^*+1} - \underline{\delta}^B_{n^*+1} = \frac{\theta^3(1-\theta)\sum_{j\in N^*}\delta_j}{(1+\theta(n^*-1))(2+\theta(n^*-3))(\theta^2+(2+\theta(n^*-3))(1+\theta n^*))}, \tag{33}$$

which is positive at $\theta \in (0,1)$ and $n^* \geq 1$. Altogether, 1) and 2) hold when $\delta_{n^*+1} \leq \underline{\delta}^B_{n^*+1}$, as desired. By 2), any firm $k \in N \setminus N^*$ does not have any price deviation incentive when $p_i = p^*_i(N^*)$, $i \in N^*$.

Moreover, each firm $i \in N^*$'s FOC condition holds with equality in the N^*-firm market and thus firm i's profit attains a local maximum at $p_i = p_i^*(N^*)$ given $p_j = p_j^*(N^*)$, $j \in N^* \setminus i$ (see Figure 1a). But each local optimum is a global maximum as the profit is single-peaked by Lemma 1-*ii* and firm i's price deviation incentive is eliminated as well.

ii-iii) In order to prove all the claims, we proceed in three steps.

Step 1: In a LPE, $\delta_{n^*+1} \in (\underline{\delta}^B_{n^*+1}, \min\{\overline{\delta}^B_{n^*+1}, \delta_{n^*}\})$.

Proof of Step 1: In a LPE, 1) $q^*_{n^*+1}(\tilde{N}) < 0$ and 2) $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) > 0$ by Lemma 2-*ii*). By part i), 1) and 2 are respectively equivalent to $\delta_{n^*+1} < \overline{\delta}^B_{n^*+1}$ and $\delta_{n^*+1} > \underline{\delta}^B_{n^*+1}$. Moreover, $q^*_i(\tilde{N}) > 0$ for any $i \in N^*$ by the BIA. This translates to $\delta_i > \underline{\delta}^B_{n^*+1}$. Altogether, $\delta_{n^*} > \underline{\delta}^B_{n^*+1}$. In sum, $\delta_{n^*+1} \in I^B = (\underline{\delta}^B_{n^*+1}, \min\{\overline{\delta}^B_{n^*+1}, \delta_{n^*}\})$.

Step 2: The price vector $\widehat{\mathbf{p}}_{N^*}$ is a LPE if and only if $\sum_{i \in N^*} \widehat{p}_i = M$ and for each $i \in N^*$, $\widehat{p}_i \in [\underline{p}^B_i, \overline{p}^B_i]$.

Proof of Step 2: Take any price vector $\widehat{\mathbf{p}}_{N^*}$ such that $\sum_{i \in N^*} \widehat{p}_i = M$, where M is provided by (10). Thus, any firm that is less efficient than firm n^* receives zero demand and their deviation incentives are eliminated. It follows from the text that any active firm $i \in N^*$'s profit attains a local maximum at $\widehat{\mathbf{p}}_{N^*}$ if and only if condition 2 holds. But each local optimum is a global maximum as the profit is single-peaked by Lemma 1 and firm i's deviation incentive is eliminated when condition 2 holds. We first claim that $\frac{\partial \pi_i^{\tilde{N},R}}{\partial p_i} |_{\widehat{\mathbf{p}}_{N^*}, p_{n^*+1} = c_{n^*+1}} \leq 0$ if and only if $\widehat{p}_i \geq \underline{p}^B_i$. Taking the derivative of (4) w.r.t. p_i in the $\tilde{N}-$market and calculating it at $\mathbf{p}_{N^*} = \widehat{\mathbf{p}}_{N^*}$ and $p_{n^*+1} = c_{n^*+1}$ gives $\frac{\partial \pi_i^{\tilde{N},R}}{\partial p_i} |_{\widehat{\mathbf{p}}_{N^*}, p_{n^*+1} = c_{n^*+1}}$ as

$$d_{n^*+1}(\sum_{l \in N^*} (\widehat{p}_l - A_l) - \delta_{n^*+1}) + (b_{n^*+1} + d_{n^*+1})\delta_i - (2b_{n^*+1} + d_{n^*+1})(\widehat{p}_i - c_i). \quad (34)$$

Since $\sum_{i \in N^*} \widehat{p}_i = M$, then substituting $d_{n^*+1}(\sum_{l \in N^*}(\widehat{p}_l - A_l)) = -b_{n^*+1}\delta_{n^*+1}$ by (10) into (34) and rearranging terms yields

$$\frac{\partial \pi_i^{\tilde{N},R}}{\partial p_i} |_{\mathbf{p}_{N^*} = \widehat{\mathbf{p}}_{N^*}} = (b_{n^*+1} + d_{n^*+1})(\delta_i - \delta_{n^*+1}) - (2b_{n^*+1} + d_{n^*+1})(\widehat{p}_i - c_i). \quad (35)$$

This derivative is negative if and only if $\widehat{p}_i \geq \underline{p}_i^B$, where

$$\underline{p}_i^B = \frac{(b_{n^*+1} + d_{n^*+1})(\delta_i - \delta_{n^*+1})}{2b_{n^*+1} + d_{n^*+1}} + c_i. \tag{36}$$

The expression in (36) reduces to (13) after we substitute the values of b_{n^*+1} and d_{n^*+1}.

We next claim that $\frac{\partial \pi_i^{N^*,L}}{\partial p_i} |_{\mathbf{p}_{N^*}=\widehat{\mathbf{p}}_{N^*}} \geq 0$ if and only if $\widehat{p}_i \leq \bar{p}_i^B$. By (4), this derivative, in the N^*-firm market, equals to

$$d_{n^*}(\textstyle\sum_{l\in N^*} \widehat{p}_l - \sum_{l\in N^*} A_l) + (b_{n^*} + d_{n^*})\delta_i - (2b_{n^*} + d_{n^*})(\widehat{p}_i - c_i). \tag{37}$$

As $\sum_{i\in N^*} \widehat{p}_i = M$, first substitute $d_{n^*+1}(\sum_{l\in N^*}(\widehat{p}_l - A_l)) = -b_{n^*+1}\delta_{n^*+1}$ by (10) into (37). Then let $\frac{b_{n^*+1}d_{n^*}}{d_{n^*+1}} = \frac{1}{\lambda(1-\theta)} = b_{n^*} + d_{n^*}$ in the resulting equation to have

$$\frac{\partial \pi_i^{N^*,L}}{\partial p_i} |_{\mathbf{p}_{N^*}=\hat{\mathbf{p}}_{N^*}} = (b_{n^*} + d_{n^*})(\delta_i - \delta_{n^*+1}) - (2b_{n^*} + d_{n^*})(\widehat{p}_i - c_i). \tag{38}$$

But this derivative is positive if and only if $\widehat{p}_i \leq \bar{p}_i^B$, where

$$\bar{p}_i^B = \frac{(b_{n^*} + d_{n^*})(\delta_i - \delta_{n^*+1})}{2b_{n^*} + d_{n^*}} + c_i. \tag{39}$$

The result follows after we substitute the values of b_{n^*} and d_{n^*} into (39).

Step 3 (Firm Rationality Constraints): For each $i \in N^*$, $\bar{p}_i^B > \underline{p}_i^B > c_i$. Moreover, for each LPE price vector $\widehat{\mathbf{p}}_{N^*}$, for each $i \in N^*$, $\widehat{q}_i = D_i^{\tilde{N}}(\widehat{\mathbf{p}}_{N^*}, \widehat{p}_{n^*+1} = c_{n^*+1}) > 0$.

Proof of Step 3: Take any $i \in N^*$. First note that $\delta_i > \delta_{n^*+1}$ by the BIA. Hence, $\bar{p}_i^B > c_i$ and $\underline{p}_i^B > c_i$ by (12) and (13) respectively. Moreover, subtracting $\underline{p}_i^B$ from $\bar{p}_i^B$ gives:

$$\bar{p}_i^B - \underline{p}_i^B = \frac{\theta^2(\delta_i - \delta_{n^*+1})}{(2 + \theta(2n^* - 3))(2 + \theta(2n^* - 1))}, \tag{40}$$

which is positive at $\theta \in (0, 1)$, as claimed.

Now take any LPE price vector $\widehat{\mathbf{p}}_{N^*}$ and consider any $i \in N^*$. Straightforward calculations show that the difference $D_i^{\tilde{N}}(\widehat{\mathbf{p}}_{N^*}, \widehat{p}_{n^*+1} = c_{n^*+1}) - D_{n^*+1}^{\tilde{N}}(\widehat{\mathbf{p}}_{N^*}, \widehat{p}_{n^*+1} = c_{n^*+1})$ is equal to $(b_{n^*+1} + d_{n^*+1})(A_i - \widehat{p}_i - \delta_{n^*+1})$ by (3). But as $\sum_{i \in N^*} \widehat{p}_i = M$, $D_{n^*+1}^{\tilde{N}}(\widehat{\mathbf{p}}_{N^*}, p_{n^*+1} = c_{n^*+1}) = 0$. Therefore, we have $D_i^{\tilde{N}}(\widehat{\mathbf{p}}_{N^*}, \widehat{p}_{n^*+1} = c_{n^*+1}) = (b_{n^*+1} + d_{n^*+1})(A_i - \widehat{p}_i - \delta_{n^*+1})$. Note that $b_{n^*+1} + d_{n^*+1} = 1/(\lambda(1-\theta)) > 0$ for $\lambda > 0$ and $\theta \in (0,1)$. It is then sufficient to show that $\widehat{p}_i < A_i - \delta_{n^*+1}$ to conclude the proof. By Step 2, $\widehat{p}_i \leq \overline{p}_i^B$. Since $0 < 1 + \theta(n^* - 1) < 2 + \theta(2n^* - 3)$, then $\widehat{p}_i < \delta_i - \delta_{n^*+1} + c_i$ or $\widehat{p}_i < A_i - \delta_{n^*+1}$ by (12), as desired. □

Proof of Proposition 2: In a LPE, it holds that $\sum_{i \in N^*} \widehat{p}_i = M$ and $p_i \in [\underline{p}_i^B, \underline{p}_i^B]$ by Proposition 1-*ii*. When $n^* = 1$, $\widehat{p}_1 = M = A_1 - \delta_2/\theta$ by (10) and $\underline{p}_1^B < M < \overline{p}_1^B$ by (41) and (42). Thus, there is a unique LPE.

When $n^* \geq 2$, for each $i \in N^*$, $[\max\{\underline{p}_i^B, M - \sum_{j \in N^* \setminus i} \overline{p}_j^B\}, \min\{\overline{p}_i^B, M - \sum_{j \in N^* \setminus i} \underline{p}_j^B\}]$ is an equilibrium price vector of firm i by Corollary 1. Therefore, there is a continuum of LPE. □

Proof of Proposition 3: We prove the claims by following our example of Section 3. We have already shown in the text that best responses might have negative slopes, which is illustrated in Figure 2. A game with one-dimensional strategy choices is supermodular if it possesses the property of increasing differences. As the best responses of firms one and two are non-monotone, this example does not satisfy supermodularity. As the firms increase their prices, other entrants might start producing. Therefore, one also needs to consider the entry of other firms (i.e., firm 3) in the definition of supermodularity under the global domain. In the current example, consider (p_1^H, p_2^L) and (p_1^L, p_2^H) with $p_1^H \geq p_1^L$, $p_2^H \geq p_2^L$, $p_1^L + p_2^L < 10$, $p_1^H + p_2^H > 10$, and $p_1^H + p_2^L = p_1^L + p_2^H = 10$. The supermodularity condition is equivalent to the following requirement in our game, fixing the action of firm 3 at c_3 [25]:

$$\pi_1^{\{1,2,3\}}(p_1^H, p_2^H, p_3 = c_3) - \pi_1^{\{1,2,3\}}(p_1^H, p_2^L, p_3 = c_3) \geq \pi_1^{\{1,2\}}(p_1^L, p_2^H) - \pi_1^{\{1,2\}}(p_1^L, p_2^L).$$

The milder single-crossing condition (SSC) requires only that

$$\pi_1^{\{1,2\}}(p_1^H, p_2^L) \geq (>) \pi_1^{\{1,2\}}(p_1^L, p_2^L) \Rightarrow \pi_1^{\{1,2,3\}}(p_1^H, p_2^H, p_3 = c_3) \geq (>) \pi_1^{\{1,2,3\}}(p_1^L, p_2^H, p_3 = c_3).$$

[25] When $p_1 + p_2 = 10$, firm 3 is on the verge of entering the market. Therefore, $\pi_1^S(p_1, p_2) = \pi_1^N(p_1, p_2, p_3 = c_3)$. See the proof of Lemma 1 for a formal proof.

It is easy to see that supermodularity implies the SSC. One can expect that both supermodularity and the SSC fail along $seg[CD]$ as the segment has a slope of -1. Accordingly, let $(p_1^H, p_2^L) = (5.04, 4.96)$ and $(p_1^L, p_2^H) = (4.96, 5.04)$. We have $\pi_1^N(p_1^H, p_2^H, p_3 = c_3) - \pi_1^N(p_1^H, p_2^L, p_3 = c_3) = d_3(p_2^H - p_2^L)(p_1^H - c_1) = 0.37 > 0$. However, $\pi_1^S(p_1^L, p_2^H) - \pi_1^S(p_1^L, p_2^L) = d_2(p_2^H - p_2^L)(p_1^L - c_1) = 0.53 > 0$. But $0.37 < 0.53$, and therefore, supermodularity fails. Moreover, $\pi_1^N(p_1^H, p_2^H, p_3 = c_3) = 30.17$, $\pi_1^N(p_1^H, p_2^L, p_3 = c_3) = 29.79$, $\pi_1^S(p_1^L, p_2^H) = 30.19$, and $\pi_1^S(p_1^L, p_2^L) = 29.67$ by (3). Note that $\pi_1^N(p_1^H, p_2^L, p_3 = c_3) - \pi_1^S(p_1^L, p_2^L) = 0.12 > 0$. But, $\pi_1^N(p_1^H, p_2^H, p_3 = c_3) - \pi_1^S(p_1^L, p_2^H) = -0.02 < 0$. Hence, the SSC also fails.

Finally, we relate our findings to log-supermodularity. Log-supermodularity asks for

$$\pi_1^{\{1,2,3\}}(p_1^H, p_2^H, p_3 = c_3) * \pi_1^{\{1,2\}}(p_1^L, p_2^L) \geq \pi_1^{\{1,2,3\}}(p_1^H, p_2^L, p_3 = c_3) * \pi_1^{\{1,2\}}(p_1^L, p_2^H).$$

Athey (2001) shows that log-supermodularity implies the single-crossing property. As our example does not satisfy the single-crossing property, it does not satisfy log-supermodularity either by the Athey's result. In particular, $\pi_1^{\{1,2,3\}}(p_1^H, p_2^H, p_3 = c_3) * \pi_1^{\{1,2\}}(p_1^L, p_2^L) - \pi_1^{\{1,2,3\}}(p_1^H, p_2^L, p_3 = c_3) * \pi_1^{\{1,2\}}(p_1^L, p_2^H) = -4.22 < 0$. Thus, log-supermodularity fails. □

Proof of Corollary 1: Let $w = B, SP$ denote the type of competition as Bertrand or Stackelberg. In a LPE, it holds that $\underline{p}_i^w \leq \overline{p}_i^w$ by Propositions 1 and 7. For the interval $[\max\{\underline{p}_i^w, M - \sum_{j \in N^* \setminus i} \overline{p}_j^w\}, \min\{\overline{p}_i^w, M - \sum_{j \in N^* \setminus i} \underline{p}_j^w\}]$ to be well defined, it is then sufficient to show that $\sum_{i \in N^*} \underline{p}_i^w < M$ and $\sum_{i \in N^*} \overline{p}_i^w > M$, where M is given by (10). Elementary calculations show that

$$M - \sum_{i \in N^*} \underline{p}_i^B = \frac{\theta(1+\theta(n^*-1)) \sum_{i \in N^*} \delta_i - \delta_{n^*+1}(\theta^2 + (1+\theta n^*)(2+\theta(n^*-3)))}{\theta(2+\theta(2n^*-1))}, \tag{41}$$

which is positive as $\delta_{n^*+1} < \overline{\delta}_{n^*+1}^B$. Similarly,

$$\sum_{i \in N^*} \overline{p}_i^B - M = \frac{-\theta(1+\theta(n^*-2)) \sum_{i \in N^*} \delta_i + \delta_{n^*+1}(2+\theta(n^*-3))(1+\theta(n^*-1))}{\theta(2+\theta(2n^*-3))}, \tag{42}$$

which is also positive as $\delta_{n^*+1} > \underline{\delta}_{n^*+1}^B$. For the Stackelberg game, $\sum_{i \in N^*} \overline{p}_i^{SP} =$

$\sum_{i\in N^*}\overline{p}_i^B > M$ by the above finding. Similarly,

$$M-\sum_{i\in N^*}\underline{p}_i^{SP} = \frac{\theta(2-\theta^2+2\theta(n^*-1)(2+\theta(n^*-1)))\sum_{i\in N^*}\delta_i-(1+\theta(n^*-1))(4-3\theta^2+2\theta(n^*-1)(3+\theta(n^*-2))))\delta_{n^*+1}}{\theta(4+\theta(2-\theta)+2\theta(n^*-1)(4+\theta(2n^*-1)))},$$

which is positive as $\delta_{n^*+1} < \overline{\delta}_{n^*+1}^{SP}$, where $\overline{\delta}_{n^*+1}^{SP}$ is provided by (17).

When the n^*-tuple $(\widehat{p}_1^w, \widehat{p}_2^w, ..., \widehat{p}_{n^*}^w)$ is a LPE, we should have $\sum_{k\in N^*\setminus j}\widehat{p}_k^w \geq M-\overline{p}_j^w$ and $\sum_{k\in N^*\setminus j}\widehat{p}_k^w \leq M-\underline{p}_j^w$ by Propositions 1 and 7. Equivalently, for $j\neq i$, $\widehat{p}_i^w \geq M-\sum_{j\in N^*\setminus i}\overline{p}_j^w$ and $\widehat{p}_i^w \leq M-\sum_{j\in N^*\setminus i}\underline{p}_j^w$. Moreover, it should hold that $\widehat{p}_i^w \geq \underline{p}_i^w$ and $\widehat{p}_i \leq \overline{p}_i^w$. In sum, we should both have $\widehat{p}_i^w \geq \max\{\underline{p}_i^w, M-\sum_{j\in N^*\setminus i}\overline{p}_j^w\}$ and $\widehat{p}_i^w \leq \min\{\overline{p}_i^w, M-\sum_{j\in N^*\setminus i}\underline{p}_j^w\}$, as claimed. □

Lemma 5. *The described Stackelberg price-choosing game in Section 5.2 has a unique relaxed equilibrium. In this equilibrium, the entrant produces*

$$q_{n^*+1}^{*,SP} = \frac{b_{n^*+1}(4b_{n^*+1}^2-(2n^*+1)d_{n^*+1}^2-2b_{n^*+1}d_{n^*+1}(n^*-1))\delta_{n^*+1}-d_{n^*+1}(2b_{n^*+1}^2-d_{n^*+1}^2)\sum_{i\in N^*}\delta_i}{2(2(2b_{n^*+1}^2-d_{n^*+1}^2)-d_{n^*+1}(n^*-1)(2b_{n^*+1}+d_{n^*+1}))},$$

where SP denotes the Stackelberg price competition.

Proof of Lemma 5: We use backwards induction to determine the Stackelberg relaxed equilibrium. Inserting the entrant firm n^*+1's best response from (6) into the incumbent firm i's ($i\leq n^*$) problem gives

$$\max_{p_i}\pi_i = \frac{(p_i-c_i)(-(2b_{n^*+1}^2-d_{n^*+1}^2)p_i + d_{n^*+1}(2b_{n^*+1}+d_{n^*+1})\sum_{j\in N^*\setminus i}p_j + x_i)}{2b_{n^*+1}}, \tag{43}$$

where $x_i = 2b_{n^*+1}a_{i,n^*+1} + d_{n^*+1}(a_{n^*+1,n^*+1}+b_{n^*+1}c_{n^*+1})$. F.O.C.'s boil down to

$$p_i = \frac{c_i}{2} + \frac{d_{n^*+1}(2b_{n^*+1}+d_{n^*+1})\sum_{j\in N^*\setminus i}p_j + x_i}{2(2b_{n^*+1}^2-d_{n^*+1}^2)}. \tag{44}$$

Sum across firms in N^* and rearrange terms to have

$$\sum_{k\in N^*}p_k = \frac{(2b_{n^*+1}^2-d_{n^*+1}^2)\sum_{k\in N^*}c_k + \sum_{k\in N^*}x_k}{2(2b_{n^*+1}^2-d_{n^*+1}^2)-(n^*-1)d_{n^*+1}(2b_{n^*+1}+d_{n^*+1})}. \tag{45}$$

After substituting (6) into the demand formula of the entrant, its relaxed equilibrium quantity satisfies

$$q_{n^*+1}^{*,SP} = \frac{a_{n^*+1,n^*+1} - b_{n^*+1}c_{n^*+1} + d_{n^*+1}\sum_{k\in N^*} p_k}{2}. \tag{46}$$

Finally, insert (45) into (46) to get the relaxed equilibrium of the entrant as stated in the lemma. □

Lemma 6. *Consider the Bertrand game. Let $n^* = 2$ and $\delta_1 \in (\delta_2, \frac{(2-\theta^2)\delta_2}{\theta})$. Let the limit pricing equilibrium price vectors of firm i (K_i^B) be given by Corollary 1. Let also $\check{p}_i^{CS} = \arg\min_{p_i} CS(p_i, p_j = M - p_i)$ and $\hat{p}_i^{TPS} = \arg\max_{p_i} TPS(p_i, p_j = M - p_i)$ and $\hat{p}_i^{TS} = \arg\max_{p_i} TS(p_i, p_j = M - p_i)$, where M is given by (10).*

i) $\check{p}_1^{CS} \in K_1^B$ if and only if $\delta_3 \in [\underline{U}_1^{CS}, \overline{U}_2^{CS}]$,

ii) Let $t \in \{TPS, TS\}$. $\hat{p}_1^t \in K_1^B$ if and only if $\delta_3 \in [\underline{U}_2^t, \overline{U}_1^t]$,

iii) $\check{p}_1^{CS} \notin K_1^B$ implies that $\check{p}_1^{CS} > \min\{\overline{p}_1^B, M - \underline{p}_2^B\}$,

iv) Let $t \in \{TPS, TS\}$. $\hat{p}_1^t \notin K_1^B$ implies that $\hat{p}_1^t < \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$,

where the boundaries are defined in the proof.

Proof of Lemma 6:

Let $i = 1, 2$. If $p_i \in K_i^B$, then it needs to hold that *i)* $p_1 \in [\underline{p}_1^B, \overline{p}_1^B]$, *ii)* $p_2 \in [\underline{p}_2^B, \overline{p}_2^B]$,and *iii)* $p_1 + p_2 = M$ by Proposition 1-*ii*, where M is given by (10), $\overline{p}_i^B$ and $\underline{p}_i^B$ are provided from (12) and (13), respectively, at $n^* = 2$ as

$$\overline{p}_i^B = \frac{(1+\theta)(\delta_i - \delta_3)}{2+\theta} + c_i \text{ and } \underline{p}_i^B = \frac{(1+2\theta)(\delta_i - \delta_3)}{2+3\theta} + c_i. \tag{47}$$

i) First consider consumers. Recall that the *constrained* consumer surplus is convex in the price of either firm by Proposition 6. It follows that $CS(p_i, p_j = M - p_i)$ is *minimized* at $(\check{p}_i^{CS}, \check{p}_j^{CS} = M - \check{p}_i^{CS})$, where

$$\check{p}_i^{CS} = A_i - \frac{\delta_3(1+\theta)}{2\theta}. \tag{48}$$

Comparing (47) and (48), $\check{p}_i^{CS} \le \overline{p}_i^B$ if and only if $\delta_3 \ge \underline{U}_i^{CS}$ and $\check{p}_i^{CS} \ge \underline{p}_i^B$ if and only if $\delta_3 \le \overline{U}_i^{CS}$, where

$$\underline{U}_i^{CS} = \frac{2\theta\delta_i}{2+\theta-\theta^2} \text{ and } \overline{U}_i^{CS} = \frac{2\theta(1+\theta)\delta_i}{2+3\theta-\theta^2}. \tag{49}$$

As $\delta_1 > \delta_2$, we have $\underline{U}_2^{CS} < \underline{U}_1^{CS}$ and $\overline{U}_2^{CS} < \overline{U}_1^{CS}$ at $\theta \in (0,1)$ by (49). In sum, $\check{p}_1^{CS} \in K_1^B$ if and only if for $i = 1, 2$, $\check{p}_i^{CS} \in [\underline{p}_i^B, \overline{p}_i^B]$, which implies that $\delta_3 \in [\underline{U}_1^{CS}, \overline{U}_2^{CS}]$ by the above findings, as claimed.

ii) Summing up $\pi_k(p_1, p_2)$ across $k = 1, 2$ yields the total producer surplus as

$$TPS = \sum_{k=1,2} (p_k - c_k) D_k(p_1, p_2). \tag{50}$$

Similar to the above analysis, first substitute $p_j = M - p_i$ into (50). Note that $\frac{\partial^2 TPS(p_i, p_j = M - p_i)}{\partial^2 p_i} = -\frac{4}{\lambda(1-\theta)} < 0$. Accordingly, maximizing $TPS(p_i, p_j = M - p_i)$ from (50) with respect to p_i gives

$$\hat{p}_i^{TPS} = \frac{3A_i + A_j + c_i - c_j}{4} - \frac{\delta_3(1+\theta)}{2\theta}. \tag{51}$$

Similarly, the total surplus is expressed as the summation of producer surplus and consumer surplus. Accordingly, summing up (50) and (18) gives

$$\begin{aligned} TS(p_1, p_2) = &\sum_{k=1,2} A_k D_k(p_1, p_2) - \tfrac{\lambda}{2} \sum_{k=1,2} (D_k(p_1, p_2))^2 - ... \\ &-\lambda\theta D_1(p_1, p_2) D_2(p_1, p_2) - \sum_{k=1,2} c_k D_k(p_1, p_2). \end{aligned} \tag{52}$$

First substitute $p_j = M - p_i$ from (10) into (52). As $\frac{\partial^2 TS(p_i, p_j = M - p_i)}{\partial^2 p_i} = \frac{-2}{\lambda(1-\theta)} < 0$, $TS(p_i, p_j = M - p_i)$ from (52) is maximized at

$$\hat{p}_i^{TS} = \frac{A_i + A_j + c_i - c_j}{2} - \frac{\delta_3(1+\theta)}{2\theta}. \tag{53}$$

Comparing (47) and (53), $\hat{p}_i^{TS} \leq \overline{p}_i^B$ if and only if $\delta_3 \geq \underline{U}_i^{TS}$ and $\hat{p}_i^{TS} \geq \underline{p}_i^B$ if and only if $\delta_3 \leq \overline{U}_i^{TS}$, where the critical cutoff values of δ_3 in total welfare calculations are

$$\underline{U}_i^{TS} = \frac{\theta(\delta_j(2+\theta) - \theta\delta_i)}{2 + \theta - \theta^2} \text{ and } \overline{U}_i^{TS} = \frac{\theta(\delta_j(2+3\theta) - \theta\delta_i)}{2 + 3\theta - \theta^2}; \;\; i \neq j. \tag{54}$$

Note that $\underline{U}_2^{TS} - \underline{U}_1^{TS} = \frac{2\theta(\delta_1 - \delta_2)}{2+\theta-\theta^2}$ and $\overline{U}_2^{TS} - \overline{U}_1^{TS} = \frac{2\theta(1+\theta)(\delta_1-\delta_2)}{2+3\theta-\theta^2}$. As $\delta_1 > \delta_2$, $\underline{U}_2^{TS} > \underline{U}_1^{TS}$ and $\overline{U}_2^{TS} > \overline{U}_1^{TS}$ at $\theta \in (0, 1)$. Therefore, for each $i = 1, 2$, $\hat{p}_i^{TS} \in [\underline{p}_i^B, \underline{p}_i^B]$ if and only if $\delta_3 \in [\underline{U}_2^{TS}, \overline{U}_1^{TS}]$, as desired.

Finally, consider the total producer surplus. The critical cutoff values for δ_3

in the total producer surplus calculations are given by

$$\underline{U}_i^{TPS} = \frac{\underline{U}_i^{CS} + \underline{U}_i^{TS}}{2} \text{ and } \overline{U}_i^{TPS} = \frac{\overline{U}_i^{CS} + \overline{U}_i^{TS}}{2}. \tag{55}$$

Similar calculations to above show that $\underline{U}_2^{TPS} > \underline{U}_1^{TPS}$ and $\overline{U}_2^{TPS} > \overline{U}_1^{TPS}$ at $\theta \in (0,1)$ and $\delta_1 > \delta_2$. Hence, $\hat{p}_1^{TPS} \in K_1$ if and only if for each $i = 1,2$, $\hat{p}_i^{TPS} \in [\underline{p}_i^B, \underline{p}_i^B]$, or say $\delta_3 \in [\underline{U}_2^{TPS}, \overline{U}_1^{TPS}]$, as claimed.

iii) First note that by using (15), (14), and (49),

$$\overline{U}_i^{CS} - \overline{\delta}_3^B = \frac{\theta(1+\theta)(\delta_i - \delta_j)}{2+3\theta-\theta^2} \text{ and } \underline{\delta}_3^B - \underline{U}_j^{CS} = \frac{\theta(\delta_i - \delta_j)}{2+\theta-\theta^2}. \tag{56}$$

As $\delta_1 > \delta_2$, $\overline{U}_1^{CS} > \overline{\delta}_3^B$ and $\underline{\delta}_3^B > \underline{U}_2^{CS}$.

If $\check{p}_1^{CS} \notin K_1^B$, then it is either the case that $\check{p}_1^{CS} > \min\{\overline{p}_1^B, M - \underline{p}_2^B\}$ or $\check{p}_1^{CS} < \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$ by Corollary 1. To prove the claim, it is sufficient to show that $\check{p}_1^{CS} \geq \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$. Suppose not. By part i), $\check{p}_1^{CS} < \underline{p}_1^B$ implies that $\delta_3 > \overline{U}_1^{CS}$. Similarly, $\check{p}_1^{CS} < M - \overline{p}_2^B$ implies that $\delta_3 < \underline{U}_2^{CS}$. Moreover, $\overline{U}_1^{CS} > \overline{\delta}_3^B$ and $\underline{\delta}_3^B > \underline{U}_2^{CS}$ by above. In sum, $\delta_3 > \overline{\delta}_3^B$ or $\delta_3 < \underline{\delta}_3^B$, which does not hold in the limit pricing region by Proposition 1-*iii*.

iv) By a symmetric argument to part iii), when $\hat{p}_1^f \notin K_1^B$, it is sufficient to show that $\hat{p}_1^f \leq \min\{\overline{p}_1^B, M - \underline{p}_2^B\}$ to prove the claim. Suppose not. By part ii), $\hat{p}_1^f > \overline{p}_1^B$ implies that $\delta_3 < \underline{U}_1^f$. Moreover, $\hat{p}_1^f > M - \underline{p}_2^B$ implies that $\delta_3 > \overline{U}_2^f$. It can be further shown that $\underline{U}_1^f < \underline{\delta}_3^B$ and $\overline{\delta}_3^B < \overline{U}_2^f$ at $\delta_1 > \delta_2$. In sum, $\delta_3 > \overline{\delta}_3^B$ or $\delta_3 < \underline{\delta}_3^B$ holds, an impossibility in the limit pricing region. □

Proof of Proposition 4: By (15) and (17),

$$\overline{\delta}_{n^*+1}^B - \overline{\delta}_{n^*+1}^{SP} = \frac{\theta^3(1-\theta)(1+\theta n^*)\sum_{i\in N^*}\delta_i}{(1+\theta(n^*-1))(\theta^2+(2+\theta(n^*-3))(1+\theta n^*))(4-3\theta^2+2\theta(n^*-1)(3+\theta(n^*-2)))} > 0$$

at $\theta \in (0,1)$. Moreover, $\underline{p}_i^{SP} > \underline{p}_i^B$ by Proposition 7-ii) of the Appendix. Also remark that $\overline{p}_i^{SP} = \overline{p}_i^B$ and $\underline{\delta}_{n^*+1}^{SP} = \underline{\delta}_{n^*+1}^B$. In sum, $I^{SP} \subset I^B$ and $[\underline{p}_i^{SP}, \overline{p}_i^{SP}] \subset [\underline{p}_i^B, \overline{p}_i^B]$. Thus, when $\delta_{n^*+1} \in I^{SP}$, all firms in N^* are active and firm n^*+1 is inactive in both games by Assumption 1. If $n^* = 1$ and $n = 2$, $\widehat{p}_1^B = \widetilde{p}_1^{SP} = M =$

$A_1 - b_2\delta_2/d_2$ is the unique limit pricing equilibrium in both games by Proposition 2. If, however, $n^* = n-1$ and $n \geq 3$, there is a continuum of limit pricing equilibria in both games. Nevertheless, since $[\underline{p}_i^{SP}, \overline{p}_i^{SP}] \subset [\underline{p}_i^{B}, \overline{p}_i^{B}]$, the set of limit pricing equilibrium prices in the Bertrand game includes the set of limit pricing equilibrium prices in the Stackelberg game by Propositions 1-ii) and 7-ii). $\square$

Proof of Proposition 5: We claim that each active firm's profit is decreasing in its price in the limit pricing region. Formally, for each $i \in N^*$, for each price $\widehat{p}_i^w \in [\underline{p}_i^w, \overline{p}_i^w]$, we have $\frac{\partial \pi_i(p_i, \sum_{j \in N^* \setminus i} p_j = M - p_i)}{\partial p_i} < 0$, where $w = B, SP$ and M is provided by Condition 1. This profit is equivalent to

$$\pi_i(p_i, \sum_{j \in N^* \setminus i} p_j = M - p_i) = q_i(p_i - c_i) = (a_{i,n^*} - b_{n^*}p_i + d_{n^*}(M - p_i))(p_i - c_i). \quad (57)$$

The derivative of $\pi_i(.)$ with respect to p_i is smaller than zero if

$$\widehat{p}_i^w > \Upsilon = \frac{A_i + c_i - \delta_{n^*+1}}{2} \quad (58)$$

in the w game. But note that

$$\underline{p}_i^B - \Upsilon = \frac{\theta(\delta_i - \delta_{n^*+1})}{4 + \theta(4n^* - 2)}, \quad (59)$$

which is positive as $\delta_i > \delta_{n^*+1}$, $\theta \in (0,1)$, and $n^* \geq 1$. The claim follows by noting that $\widehat{p}_i^B \geq \underline{p}_i^B$ in a limit pricing equilibrium and thus $\widehat{p}_i^B > \Upsilon$. Similarly, for the Stackelberg price setting game, as $\widehat{p}_i^{SP} \geq \underline{p}_i^{SP} > \underline{p}_i^B$ by the proof of Proposition 7, $\widehat{p}_i^{SP} \geq \underline{p}_i^{SP} > \underline{p}_i^B > \Upsilon$ as well. $\square$

Proof of Proposition 6: Let $n^* = 2$. The claim follows after we show that the constrained consumer surplus, $CS(p_i, p_j = M - p_i)$, is convex in p_i. To see that, constrained consumer surplus is, for $i = 1, 2$,

$$CS(p_i, p_j = M - p_i) = \frac{-(\delta_3)^2 + \delta_3(2A_i - 2p_i - \delta_3)\theta + 2(A_i - p_i)(-A_i + p_i + \delta_3)\theta^2}{2\lambda(-1+\theta)\theta^2},$$

where we substitute the demand formulas from (3) into (18). It can be shown that $\frac{\partial^2 CS(p_i, p_j = M - p_i)}{\partial^2 p_i} = \frac{2}{\lambda(1-\theta)} > 0$ at $\lambda > 0$ and $\theta \in (0,1)$. Thus, the *constrained* consumer surplus is convex in the price of firm i, as desired. $\square$

Proposition 7. *Let Assumption 1 hold,* $\overline{p}_i^{SP} = \overline{p}_i^B$ *and*

$$\underline{p}_i^{SP} = \frac{2(1+\theta(n^*-1))(1+\theta n^*)(\delta_i - \delta_{n^*+1})}{4+\theta(2-\theta)+2\theta(n^*-1)(4+\theta(2n^*-1))} + c_i. \tag{60}$$

i) A Stackelberg entry-preventing limit pricing equilibrium in which only incumbents are active exists if and only if $\delta_{n^*+1} \in I^{SP} = (\underline{\delta}_{n^*+1}^{SP}, \min\{\overline{\delta}_{n^*+1}^{SP}, \delta_{n^*}^{SP}\})$.

ii) A price vector $(\widetilde{p}_1, \widetilde{p}_2, ..., \widetilde{p}_{n^*})$ *is a limit pricing Stackelberg equilibrium price vector for active incumbent firms if and only if it satisfies (10) and* $\widetilde{p}_i \in [\underline{p}_i^{SP}, \overline{p}_i^{SP}]$ *for all* $i \leq n^*$. *In each such equilibrium,* $\widetilde{p}_{n^*+1} = c_{n^*+1}$ *and* $\widetilde{q}_{n^*+1} = 0$.

iii) When $n^* = 1$ *and* $n \geq 2$, *there is a unique limit pricing Stackelberg equilibrium price vector. When* $n^* \in [2, n-1]$ *and* $n \geq 3$, *there is a continuum of limit pricing Stackelberg equilibrium price vectors.*

iv) Let $n \geq 3$ *and* $n^* \geq 2$. *A linear Stackelberg price-setting model with continuous best responses may not be supermodular or log-supermodular or satisfy the single-crossing property. Moreover, the best responses may be non-monotone.*

Proof of Proposition 7: i) An equilibrium is constrained if a) the potential entrant is inactive in the relaxed Stackelberg equilibrium ($q_{n^*+1}^{*,SP} < 0$) and b) there is demand left for the entrant if the incumbents play the Bertrand game. While a) equals to $\delta_{n^*+1} < \overline{\delta}_{n^*+1}^{SP}$, where $\overline{\delta}_{n^*+1}^{SP}$ is given by (17), by Lemma 5 of the Appendix, b) equals to $\delta_{n^*+1} > \underline{\delta}_{n^*+1}^{B} = \underline{\delta}_{n^*+1}^{SP}$ by Proposition 1-i). Note that both $\overline{\delta}_{n^*+1}^{SP}$ and $\underline{\delta}_{n^*+1}^{SP}$ are positive as $\theta \in (0,1)$ and $n^* \geq 1$. Straightforward calculations show that by (14) and (17),

$$\overline{\delta}_{n^*+1}^{SP} - \underline{\delta}_{n^*+1}^{B} = \frac{\theta^3(1-\theta)\sum_{i\in N^*}\delta_i}{(1+\theta(n^*-1))(2+\theta(n^*-3))(4-3\theta^2+2\theta(n^*-1)(3+\theta(n^*-2)))} > 0$$

at $\theta \in (0,1)$. Thus, $\overline{\delta}_{n^*+1}^{SP} > \underline{\delta}_{n^*+1}^{SP}$, as desired.

ii) For each $\delta_{n^*+1} \in I^{SP} = (\underline{\delta}_{n^*+1}^{SP}, \min\{\overline{\delta}_{n^*+1}^{SP}, \delta_{n^*}^{SP}\})$, the equilibrium is constrained by part i). The incumbents limit their total equilibrium prices to

$$\sum_{j\in N^*} \widetilde{p}_j = \frac{b_{n^*+1}c_{n^*+1} - a_{n^*+1,n^*+1}}{d_{n^*+1}} \tag{61}$$

by Condition 1 so that the entrant is inactive. Thus, in a limit pricing equilibrium, the entrant charges at its marginal cost and produces zero. Now let $\tilde{\mathbf{p}}$ denotes an arbitrary limit pricing equilibrium vector for the incumbents. At this price vector, the leftward derivative of each incumbent $i \in N^*$'s profit should be positive while the rightward derivative of its profit should be negative. The leftward derivatives between the Bertrand and Stackelberg price setting games are identical to each other as firm n^*+1 is inactive in both games. Therefore, the positiveness of the leftward derivative requires that $p_i < \overline{p}_i^{SP} = \overline{p}_i^{B}$. The rightward derivative of incumbent i's profit is negative in the $\tilde{N} = N^* \bigcup \{n^*+1\}-$firm market (or $\frac{\partial \pi_i^{\tilde{N},R}}{\partial p_i} |_{\tilde{\mathbf{p}}_{N^*}, \tilde{p}_{n^*+1} = BR_{n^*+1}(\tilde{\mathbf{p}}_{N^*})} < 0$) if

$$\widetilde{p}_i > \frac{c_i}{2} + \frac{d_{n^*+1}(2b_{n^*+1} + d_{n^*+1})(\sum_{j \in N^*} \widetilde{p}_j - \widetilde{p}_i) + x_i}{2(2b_{n^*+1}^2 - d_{n^*+1}^2)}, \tag{62}$$

where $x_i = 2b_{n^*+1}a_{i,n^*+1} + d_{n^*+1}(a_{n^*+1,n^*+1} + b_{n^*+1}c_{n^*+1})$ by (44). After substituting the value of $\sum\limits_{j \in N^*} \widetilde{p}_j$ from (61) into (62), we obtain $\widetilde{p}_i > \underline{p}_i^{SP}$, where $\underline{p}_i^{SP}$ is stated by (60), as claimed. Moreover,

$$\underline{p}_i^{SP} - \underline{p}_i^{B} = \frac{\theta^2(1 + \theta n^*)}{(2 + \theta(2n^* - 1))(4 + \theta(2 - \theta) + 2\theta(n^* - 1)(4 + \theta(2n^* - 1)))}, \tag{63}$$

which is positive at $\theta > 0$. As $\underline{p}_i^{B} > c_i$, $\underline{p}_i^{SP} > c_i$ as well. Last, as $\widetilde{p}_i < \overline{p}_i^{B} = \overline{p}_i^{SP}$, $\widetilde{q}_i > 0$ for each $i \in N^*$ by the proof of Step 3 of Proposition 1.

iii) These claims follow from Corollary 1.

iv) Consider a modified version of the example of Section 2. Let $N = \{1, 2, 3\}$ and firm three be a Stackelberg follower while firms one and two are the leaders. Assume that $p_i = A_i - q_i - 0.8 \sum_{j \in N \setminus i} q_j$, $(A_1, A_2, A_3) = (23, 23, 25)$, $(c_1, c_2, c_3) = (2, 2, 9.2)$, and $(\delta_1, \delta_2, \delta_3) = (21, 21, 15.8)$. Note that $\delta_3 = 15.8 \in (\underline{\delta}_3^{SP}, \overline{\delta}_3^{SP}) \equiv [15.56, 15.845]$, and thus a limit pricing equilibrium exists by part i). Firms one and two limit their total prices to $p_1 + p_2 = 10.45$ so that $q_3 = 0$ by (61). $p_1 + p_2 = 10.45$ line is a border for firm 3's production. For price combinations above this line, firms one and two have Stackelberg best responses. For $i, j = 1, 2$, the Stackelberg best response of firm i is $p_i = 3.62 + 0.3p_j$ by (44). For price combinations below this line, firms one and two have Bertrand best responses. For $i, j = 1, 2$, the best response of firm i is non-monotone and continuous and given by

$$p_i = BR_i(p_j) = \begin{cases} 3.3 + 0.4p_j & \text{if } p_j < 4.8 \\ 10.45 - p_j & \text{if } 4.8 \le p_j \le 5.5 \\ 3.62 + 0.3p_j & \textit{otherwise.} \end{cases}$$

Thus, the Stackelberg price-setting game need not to satisfy supermodularity, log-supermodularity and the single crossing property similarly as the Bertrand game.

□

Proposition 8. *Consider the Bertrand game and let* $\delta_1 \in (\underline{\delta}_1, \overline{\delta}_1) \equiv (\delta_2, \frac{(2-\theta^2)\delta_2}{\theta})$*. For* $t \in \{CS, TPS, TS\}$*, let* $\tilde{p}_1^t = \arg\max_{p_1 \in K_1^B} t(p_1, p_2 = M - p_1)$*. Let* $\check{p}_1^{CS} = \arg\min_{p_1} CS(p_1, p_2 = M - p_1)$ *and for* $f \in \{TPS, TS\}$*,* $\hat{p}_1^f = \arg\max_{p_1} f(p_1, p_2 = M - p_1)$*. Let* $\overline{\delta}_1^{TPS} = \frac{\delta_2(2-\theta)(1+2\theta)}{2+\theta}$*,* $\overline{\delta}_1^{CS} = \frac{\delta_2(2-\theta)(1+\theta)^2}{2+3\theta-\theta^2}$*, and* $\overline{\delta}_1^{TS} = \frac{\delta_2(2+\theta(5+2\theta(1-\theta)))}{2+\theta(5+\theta-\theta^2)}$*.*

i) If $\delta_1 \in (\overline{\delta}_1^{TPS}, \overline{\delta}_1)$*, then* $\tilde{p}_1^{PS} = \tilde{p}_1^{CS} = \tilde{p}_1^{TS} = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$*.*

ii) If $\delta_1 \in (\overline{\delta}_1^{CS}, \overline{\delta}_1)$*, then* $\tilde{p}_1^{CS} = \tilde{p}_1^{TS} = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$*.*

iii) If $\delta_1 \in (\overline{\delta}_1^{TS}, \overline{\delta}_1)$*, then* $\tilde{p}_1^{TS} = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$*.*

iv) Let $\delta_1 \in (\underline{\delta}_1, \overline{\delta}_1^{CS})$*. If* $\delta_3 \in [\underline{U}_1^{CS}, \overline{U}_2^{CS}]$*, then* $\tilde{p}_1^{CS} \in \{\max\{\underline{p}_1^B, M - \overline{p}_2^B\}, \min\{\overline{p}_1^B, M - \underline{p}_2^B\}\}$*. Otherwise,* $\tilde{p}_1^{CS} = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$*.*

v) Let $f \in \{TPS, TS\}$*. Let* $\delta_1 \in (\underline{\delta}_1, \overline{\delta}_1^f)$*. If* $\delta_3 \in [\underline{U}_2^f, \overline{U}_1^f]$*, then* $\tilde{p}_1^f = \hat{p}_1^f$*. Otherwise,* $\tilde{p}_1^f = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$*.*

Proof of Proposition 8:

Note first that since $\delta_1 \in (\delta_2, \frac{(2-\theta^2)\delta_2}{\theta})$, $q_i^*(\{1,2\}) > 0$, $i = 1, 2$ by Proposition 2. By Lemma 6-*i*, $\check{p}_1^{CS} \in K_1^B$ if and only if $\delta_3 \in [\underline{U}_1^{CS}, \overline{U}_2^{CS}]$. Moreover, for each $f \in \{TPS, TS\}$, $\hat{p}_1^f \in K_1^B$ if and only if $\delta_3 \in [\underline{U}_2^f, \overline{U}_1^f]$. First, we derive the necessary conditions that these defined intervals are well-defined in the first place. Afterwards, we prove the proposition.

A) Necessary Conditions: Subtracting $\underline{U}_1^{CS}$ from $\overline{U}_2^{CS}$ equals to

$$\overline{U}_2^{CS} - \underline{U}_1^{CS} = \frac{2\theta(\delta_2(2-\theta)(1+\theta)^2 - \delta_1(2+3\theta-\theta^2))}{(2+3\theta-\theta^2)(2+\theta-\theta^2)}. \tag{64}$$

by (49). Similarly, subtracting $\underline{U}_j^{TPS}$ from $\overline{U}_i^{TPS}$ yields

$$\overline{U}_1^{TPS} - \underline{U}_2^{TPS} = \frac{\theta^2(\delta_2(2-\theta)(1+2\theta) - \delta_1(2+\theta))}{(2+3\theta-\theta^2)(2+\theta-\theta^2)}. \tag{65}$$

Last, subtracting $\underline{U}_2^{TS}$ from $\overline{U}_1^{TS}$ gives

$$\overline{U}_1^{TS} - \underline{U}_2^{TS} = \frac{2\theta(\delta_2(2+\theta(5+2\theta(1-\theta))) - \delta_1(2+\theta(5+\theta(1-\theta))))}{(2+3\theta-\theta^2)(2+\theta-\theta^2)}. \tag{66}$$

Now define the following cutoff values

$$\overline{\delta}_1^{TPS} = \frac{\delta_2(2-\theta)(1+2\theta)}{2+\theta}; \ \overline{\delta}_1^{CS} = \frac{\delta_2(2-\theta)(1+\theta)^2}{2+3\theta-\theta^2}; \ \overline{\delta}_1^{TS} = \frac{\delta_2(2+\theta(5+2\theta(1-\theta)))}{2+\theta(5+\theta-\theta^2)}.$$

Straightforward calculations yield

$$\overline{\delta}_1^{TPS} - \overline{\delta}_1^{CS} = \frac{\delta_2\theta(1-\theta)(4+4\theta-3\theta^2)}{4+\theta(8+\theta-\theta^2)}; \overline{\delta}_1^{CS} - \overline{\delta}_1^{TS} = \frac{\delta_2\theta^3(1-\theta)(2+2\theta-\theta^2)}{(2+3\theta-\theta^2)(2+\theta(5+\theta-\theta^2))}.$$

Therefore, $\overline{\delta}_1^{TPS} > \overline{\delta}_1^{CS} > \overline{\delta}_1^{TS}$ at $\theta \in (0,1)$. Further note that $\overline{\delta}_1 > \overline{\delta}_1^{TPS}$ and $\overline{\delta}_1^{TS} > \underline{\delta}_1$ because

$$\overline{\delta}_1 - \overline{\delta}_1^{TPS} = \frac{\delta_2(1-\theta)(4+4\theta-\theta^2)}{\theta(2+\theta)} > 0; \ \overline{\delta}_1^{TS} - \underline{\delta}_1 = \frac{2\delta_2(1+\theta)}{2+\theta} > 0$$

for $\theta \in (0,1)$. Finally observe that for $f \in \{TPS, TS\}$, $\overline{U}_1^f > \underline{U}_2^f$ if and only if $\delta_1 \leq \delta_1^f$ by (65) and (66). Moreover, $\overline{U}_2^{CS} > \underline{U}_1^{CS}$ if and only if $\delta_1 \leq \delta_1^{CS}$ from (64).

B) The proofs of the parts of the proposition:

Parts i), ii), and iii): Assume first that $\delta_1 \in (\overline{\delta}_1^{TPS}, \overline{\delta}_1)$. As $\overline{\delta}_1^{TPS} > \overline{\delta}_1^{CS} > \overline{\delta}_1^{TS}$, $\delta_1 > \overline{\delta}_1^{CS}$ and $\delta_1 > \overline{\delta}_1^{TS}$ as well. Therefore, $\underline{U}_1^{CS} > \overline{U}_2^{CS}$, $\underline{U}_2^{TPS} > \overline{U}_1^{TPS}$, and $\underline{U}_2^{TS} > \overline{U}_1^{TS}$ from the above proof. This implies that $\check{p}_1^{CS}, \hat{p}_1^{TPS}, \hat{p}_1^{TS} \notin K_1^B$ by Lemma 6-$i), ii)$ and corner solutions emerge. Thus, by Lemma 6-iii, iv,

$\hat{p}_1^{TS}, \hat{p}_1^{TPS} < \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$ and $\check{p}_1^{CS} > \min\{\overline{p}_1^B, M - \underline{p}_2^B\}$. Since total welfare and total producer surplus are concave in p_1 but consumer surplus is convex in p_1, then $\tilde{p}_1^{TPS} = \tilde{p}_1^{TS} = \tilde{p}_1^{CS} = \max\{\underline{p}_1, M - \overline{p}_2\}$, as claimed in part i). By a similar argument, when $\delta_1 \in (\overline{\delta}_1^{CS}, \overline{\delta}_1)$, we have $\tilde{p}_1^{CS} = \tilde{p}_1^{TS} = \max\{\underline{p}_1, M - \overline{p}_2\}$ as in part ii). Finally, when $\delta_1 > (\overline{\delta}_1^{TS}, \overline{\delta}_1)$ as assumed in part iii), we have $\tilde{p}_1^{TS} = \max\{\underline{p}_1, M - \overline{p}_2\}$, as desired.

Parts iv), and v): First suppose that $\delta_1 \in (\underline{\delta}_1, \overline{\delta}_1^{CS})$. Then $\underline{U}_1^{CS} \leq \overline{U}_2^{CS}$ by the necessary conditions stated in stage A. Therefore, if $\delta_3 \in [\underline{U}_1^{CS}, \overline{U}_2^{CS}]$, then $\check{p}_1^{CS} \in K_1^B$ by Lemma 6-i). So, $\tilde{p}_1^{CS} \in \{\max\{\underline{p}_1^B, M - \overline{p}_2^B\}, \min\{\overline{p}_1^B, M - \underline{p}_2^B\}\}$ by Proposition 6 as claimed in part iv). However, if $\delta_3 \notin [\underline{U}_1^{CS}, \overline{U}_2^{CS}]$, then $\check{p}_1^{CS} \notin K_1^B$. Thus, $\check{p}_1^{CS} > \min\{\overline{p}_1^B, M - \underline{p}_2^B\}$ by Lemma 6-iii). In such a corner solution case, we have already concluded that $\tilde{p}_1^{CS} = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$.

Finally, let $f \in \{TPS, TS\}$ and $\delta_1 \in (\underline{\delta}_1, \overline{\delta}_1^f)$. Thus, $\underline{U}_2^f \leq \overline{U}_1^f$. Therefore, if $\delta_3 \in [\underline{U}_2^f, \overline{U}_1^f]$, then $\delta_3 \in K_1^B$. Hence, $\hat{p}_1^f \in K_1^B$ by Lemma 6-ii. So interior solutions arise and $\tilde{p}_1^f = \hat{p}_1^f$ as claimed in part v). But if $\delta_3 \notin [\underline{U}_2^f, \overline{U}_1^f]$, then $\check{p}_1^f \notin K_1^B$. So, by Lemma 6-$iv$), $\hat{p}_1^f < \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$. Therefore, we are bounded with corner solutions and should have $\tilde{p}_1^f = \max\{\underline{p}_1^B, M - \overline{p}_2^B\}$, as claimed. □

Last, it is useful to study the sensitivity of limit pricing strategies to the degree of substitutability (θ). Our results are summarized in the following proposition:

Proposition 9.

i) For $i \in N^*$, $\frac{\partial \overline{p}_i^B}{\partial \theta} > \frac{\partial \underline{p}_i^B}{\partial \theta} > 0$.

ii) $\frac{\partial \underline{\delta}_{n^*+1}^B}{\partial \theta} > 0$ and $\frac{\partial \overline{\delta}_{n^*+1}^B}{\partial \theta} > 0$.

iii) For $n^* \geq 1$, $\lim_{\theta \to 0} \overline{\delta}_{n^*+1}^B - \underline{\delta}_{n^*+1}^B = 0$. For $n^* = 1$, $\lim_{\theta \to 1} \overline{\delta}_2^B - \underline{\delta}_2^B > 0$. For $n^* \geq 2$, $\lim_{\theta \to 1} \overline{\delta}_{n^*+1}^B - \underline{\delta}_{n^*+1}^B = 0$.

iv) The sum of the limit equilibrium equilibrium prices of the active firms increases in the degree of substitutability.

Proof of Proposition 9:

i) Let $i \in N^*$. Note that $\frac{\partial \overline{p}_i^B}{\partial \theta} - \frac{\partial \underline{p}_i^B}{\partial \theta} = \frac{8\theta(1+(n^*-1)\theta)(\delta_i - \delta_{n^*+1})}{(2+\theta(2n^*-3))^2(2+\theta(2n^*-1))^2}$ and $\frac{\partial \underline{p}_i^B}{\partial \theta} = \frac{\delta_i - \delta_{n^*+1}}{(2+\theta(2n^*-1))^2}$ by (12) and (13). The claims follow after noting that $\delta_i > \delta_{n^*+1}$ and $\theta \in (0,1)$.

ii) and **iii)** Consider (15) and (14). We have

$$\frac{\partial \bar{\delta}^B_{n^*+1}}{\partial \theta} = \frac{(2 + 4(n^* - 1)\theta + (2 + n^*(2n^* - 3))\theta^2) \sum_{j \in N^*} \delta_j}{(2 + \theta(-3 + \theta + n^*(3 + \theta(n - 3))))^2}$$

$$\frac{\partial \underline{\delta}^B_{n^*+1}}{\partial \theta} = \frac{(2 + 4(n^* - 2)\theta + (7 + n^*(2n^* - 7))\theta^2) \sum_{j \in N^*} \delta_j}{(2 + \theta(n^* - 3))^2 (1 + \theta(n^* - 1))^2},$$

which are both positive at $\theta \in (0,1)$. Moreover, for $n^* \geq 1$, $\lim_{\theta \to 0} \overline{\delta}^B_{n^*+1} - \underline{\delta}^B_{n^*+1} = 0$ by (33). Further, $\lim_{\theta \to 1} \overline{\delta}^B_2 - \underline{\delta}^B_2 = \delta_1/2 > 0$, and for $n^* \geq 2$, $\lim_{\theta \to 1} \overline{\delta}^B_{n^*+1} - \underline{\delta}^B_{n^*+1} = 0$ by (33).

iv) Note that condition 1, which is labelled by (10), is a necessary condition for a limit pricing equilibrium to exist. Thus, the claim follows from this condition.

□

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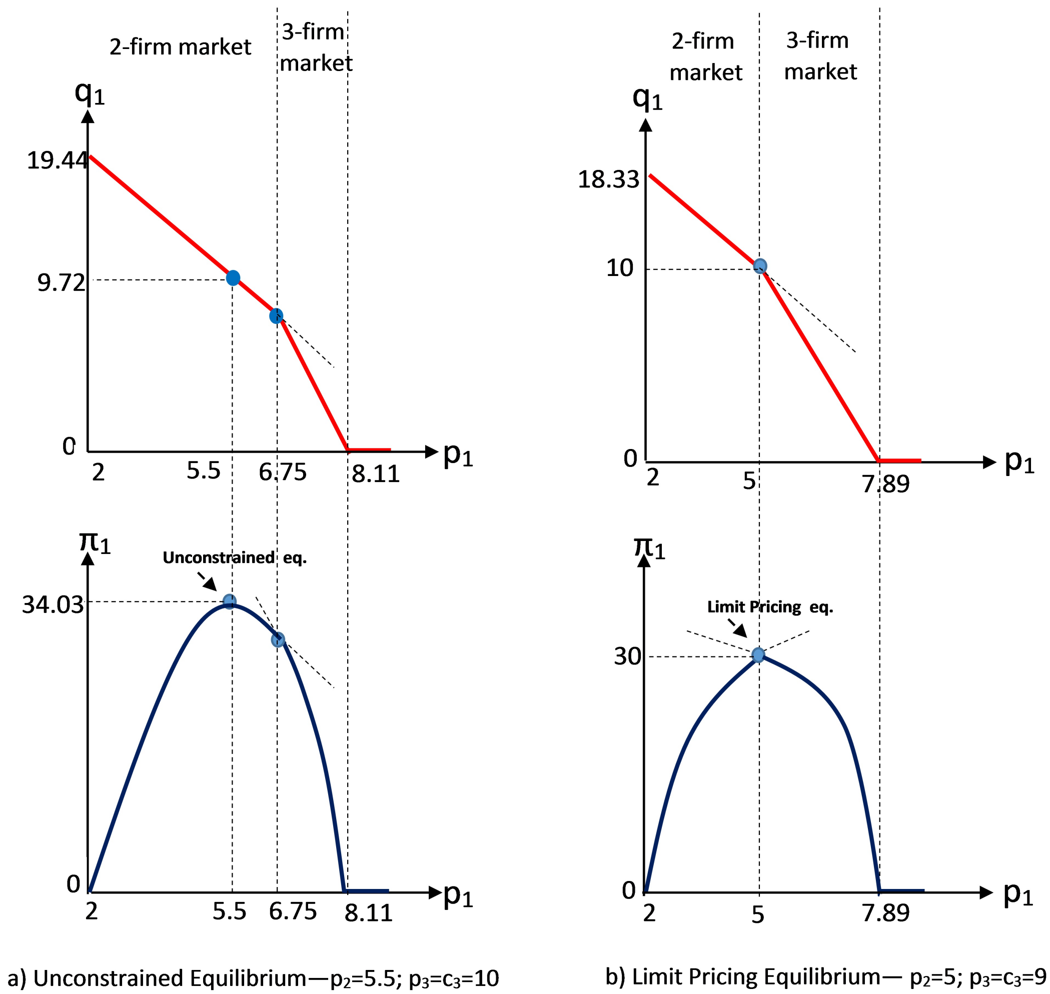


Figure 1: **Unconstrained and Limit-pricing Equilibria** Let $N = \{1, 2, 3\}$ and for each $i \in N$, let $p_i = A_i - q_i - 0.8 \sum_{j \in N \setminus i} q_j$, $(A_1, A_2, A_3) = (23, 23, 25)$, and $(c_1, c_2) = (2, 2)$. In these figures, we draw both the demand and profit curves of firm 1 ($\equiv$ firm 2 by symmetry) for different set of parameters. Observe that both curves are quasi-concave in the global domain. In part a), we let $p_3 = c_3 = 10$ and $p_2 = 5.5$. When $c_3 = 10$, the equilibrium is unconstrained and only firms 1 and 2 are active by Proposition 1-i. They both charge their unconstrained duopoly prices of 5.5. In particular, when $p_3 = 10$, and $p_2 = 5.5$, firm 1's profit is globally maximized at $p_1 = 5.5$ at which it is differentiable. In part b), we let $p_3 = c_3 = 9$ and $p_2 = 5$. When $c_3 = 9$, there are multiple limit pricing equilibria and only firms 1 and 2 are active by Proposition 1-ii. For instance, the price vector $(5, 5, 9)$ constitutes an equilibrium. In particular, when $p_3 = 9$, and $p_2 = 5$, firm 1's profit is globally maximized at $p_1 = 5$. In this case, the equilibrium occurs when both the demand and profit functions have kinks.

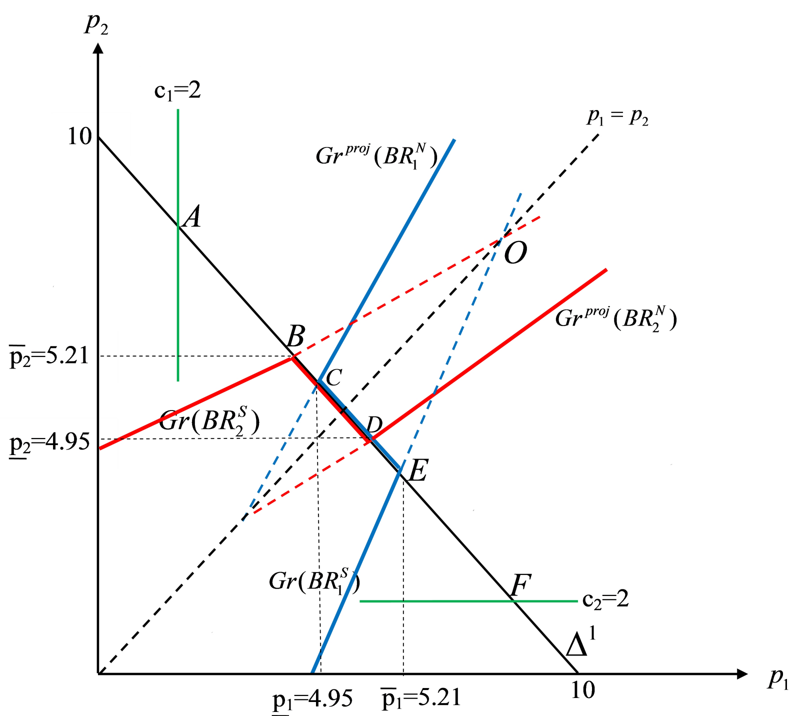
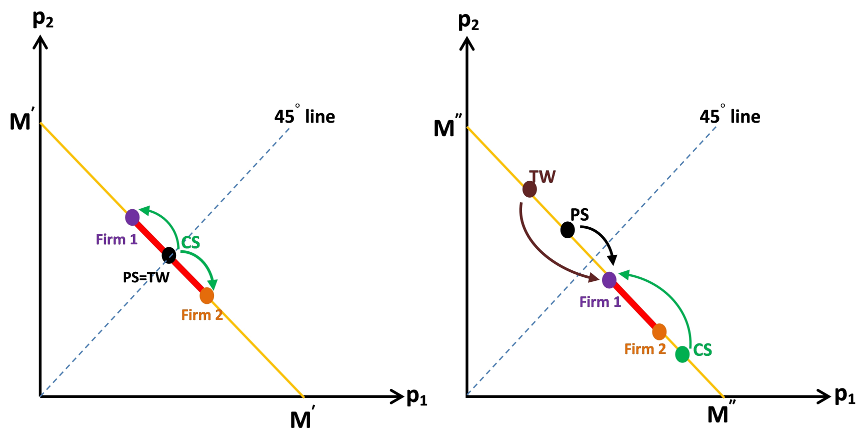


Figure 2: **The Sketch of the Proof of Proposition 3:** Let $N = \{1, 2, 3\}$ and $S = \{1, 2\}$. Let $p_i = A_i - q_i - 0.8 \sum_{j \in N \setminus i} q_j$, $(A_1, A_2, A_3) = (23, 23, 25)$, and $(c_1, c_2, c_3) = (2, 2, 9)$. We draw the best responses of firms 1 and 2 when $p_3 = c_3$, which are piecewise linear and non-monotone as shown in the figure. Moreover, they intersect at multiple points showing that each $\hat{\mathbf{p}} \in \{\mathbf{p} \in \mathbb{R}^3 : (p_1, p_2) \in seg[CD]$ and $p_3 = c_3\}$ is a pure-strategy Bertrand equilibria.



$A_1 = A_2$ and $c_1 = c_2$ $\qquad$ $\delta_1 \in \left(\frac{(2-\theta)(1+2\theta)\delta_2}{2+\theta}, \frac{(2-\theta^2)\delta_2}{\theta}\right)$

Figure 3: In these figures, we provide the individual firm profit, consumer surplus, total producer surplus and total welfare maximizing price bundles of firms in the limit pricing Bertrand equilibrium region, which is drawn by the red line. In the first picture, the active firms are completely symmetric. In such a case, total producer surplus and welfare maximizing limit price bundle is the one where firms have balanced prices. However, the same bundle minimizes consumer surplus. Therefore, consumers prefer the extreme prices (drawn by purple and orange points) over any limit pricing equilibrium price. In the second picture, we consider part i) of Proposition 8 of the Appendix, where the quality-cost difference of firm 1 is sufficiently higher than the quality-cost difference of firm 2. In such a case, while the total surplus and the total producer surplus are both maximized at the left of the limit pricing equilibrium set (red region), the consumer surplus minimizing point (green point) is at the right of the red region. In this case, corner solutions arise and both consumers and total producers prefer the same equilibrium price bundle (purple point). In both figures, each firm prefers to charge the lowest possible limit equilibrium price. Thus, firms 1 and 2's most preferred price bundles are drawn by purple and orange points respectively.