Joint Dynamic Pricing of Multiple Perishable Products Under Consumer Choice

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In response to competitive pressures, firms are increasingly adopting revenue management opportunities afforded by advances in information and communication technologies. Motivated by revenue management initiatives in industry, we consider a dynamic pricing problem facing a firm that sells given initial inventories of multiple substitutable and perishable products over a finite selling horizon. Since the products are substitutable, individual product demands are linked through consumer choice processes. Hence, the seller must formulate a joint dynamic pricing strategy while explicitly incorporating consumer behavior. For a general model of consumer choice, we model this multi-product dynamic pricing problem as a stochastic dynamic program and characterize its optimal prices. In addition, since consumer behavior depends on the nature of product differentiation, we introduce a linear random utility framework that captures the cases of vertical and horizontal product differentiation. When products are vertically differentiated, our results show monotonicity properties (with respect to quality, inventory, and time) of the optimal prices and reveal that the optimal price of a product depends on higher quality product inventories only through their aggregate inventory rather than individual availabilities. Further, this aggregate inventory solely determines the product’s markup over an adjacent lower quality product in the assortment. We exploit these properties to develop a polynomial-time exact algorithm for determining the optimal prices. When products are horizontally differentiated, we find that analogous monotonicity properties do not hold. Additionally, we find that individual, rather than aggregate, product inventory availability drives pricing in this case.

Key words: dynamic pricing, revenue management, perishable products, consumer choice, vertical and horizontal product assortments, polynomial algorithm

1. Introduction

Increasing global competition is forcing companies to rethink their existing pricing and sales strategies and explore new opportunities afforded by advances in information and communication technologies. With these technologies, firms have extensive reach to customers and to information systems throughout their organizations, allowing them to collect market data, learn about cus-
tomer behavior, understand market segments, and change prices dynamically. As a result, sellers can now continuously monitor availability and demand, and adjust prices dynamically to maximize profits. Such demand management principles offer firms an opportunity to make substantial gains – a recent McKinsey study (Marn et al. 2003) estimates that for a typical S&P 1500 company, a 1% improvement in pricing can lead to an 8% improvement in profits. Recognizing these opportunities, managers in several industries including travel, hospitality, public utility, and equipment rental (see Talluri and van Ryzin (2004b) for a detailed profile of revenue management applications in these and other industries) are increasingly using revenue management tools such as dynamic pricing to maximize profits when facing uncertain demand.

In many industrial applications, firms offer their customers a line of differentiated products. Often, these products are perishable (e.g. room inventories for a particular day expire at the end of the day) and substitutable (e.g., either a standard or a deluxe room can meet consumer demand). To meet uncertain demand in this context, firms must adopt a proactive demand management strategy such as dynamic pricing to maximize revenues. Since product demands for substitutable products are linked by consumer choice processes, managers must determine a joint pricing strategy based on consumer behavior. For instance, a hotel might have various types of rooms (e.g., standard vs. deluxe rooms) that differ in the amenities and facilities available for the guests. In this case, the demand for an individual room type depends not only on the price and non-price characteristics of that room type, but also on those of the other room types. As a result, the hotel must understand the choices that consumers make when facing such a product assortment and determine the prices for different room-types jointly. Similarly, a discount airline that offers parallel flights (with different arrival and departure times within the same day for a particular origin-destination flight leg) must determine the fares for these itineraries jointly and model how consumers choose these flight legs.

While both examples require a joint dynamic pricing strategy, consumer behavior in each context is different and follows the nature of product differentiation. In the hotel example, since room types can be ordered based on their quality, they are vertically differentiated; consequently, if all the room types are priced the same, consumers would prefer deluxe rooms over standard rooms. In contrast, in the airline example, customer preferences are not uniformly ordered and the different flight legs
are horizontally differentiated. That is, if all flight legs are priced the same, some consumers may prefer an early departure while others may choose a later departure.

Motivated by these revenue management applications, this paper studies a dynamic pricing problem facing a firm that sells given initial inventories of multiple substitutable and perishable products over a finite selling season. The demand for each product depends on the price and non-price characteristics of all products in the assortment and the behavior of consumers induced by the nature of product differentiation. We refer to this problem as the Multi-Product Dynamic Pricing (MPDP) problem. We formulate a stochastic dynamic program (DP) for the MPDP problem and characterize the optimal prices for a general consumer choice model. Then, we introduce a linear random utility framework that is the basis of choice models for both vertically and horizontally differentiated product assortments. For each case of product differentiation, our analysis examines the structure of the optimal policies. Our analysis also leads to a polynomial-time exact algorithm for the MPDP problem with vertically differentiated products. Overall, our structural results offer valuable managerial insights and facilitate actionable multi-product dynamic pricing policies that are easy to understand, compute and implement. We describe the specific contributions of our work next.

The MPDP models we study have applications in several industries. For example, the MPDP model for vertically differentiated products applies to the hospitality (e.g., pricing hotel rooms that offer different amenities), entertainment (e.g., pricing of event tickets for different seat locations), agriculture (e.g., pricing perishable agriculture goods of different grades), and information technology (e.g., pricing of advertisement slots at different positions on web pages) industries. In addition, our analysis of the MPDP model leads to several important contributions. For the MPDP problem with a general model of consumer choice, which we refer to as the $G$ model, we characterize the structure of the optimal prices. To investigate how the nature of product differentiation affects a firm’s pricing policy, we examine the special cases of vertical and horizontal product differentiation. Starting with the linear random utility framework in each case to build an appropriate choice model, we formulate MPDP models for vertically and horizontally differentiated products and refer to them as the $V$ and $H$ models, respectively.
In the $V$ model, assuming a commonly used choice model of vertical demand, we provide a complete analysis of the structure of optimal prices. We show that the optimal prices exhibit (1) 
*quality monotonicity*: the optimal price of a high quality product is always higher than that of a lower quality product; (2) *inventory monotonicity*: when a product’s inventory level increases, the firm must set a lower price not only for that product, but also for the other products in the assortment; and (3) *time monotonicity*: as the end of the sales horizon approaches, the firm must reduce the prices for all its products. Further, we show that the optimal price of a product is governed by an *aggregation* of the inventories of higher quality products (rather than their individual availabilities) and the *individual* inventory of lower quality products. In addition, the aggregate inventory of higher quality products *alone* determines the *price difference* between two adjacent products. When the product inventories are in surplus (that is, when the total inventory of all products is greater than the maximum possible future demand), we also prove that the surplus units, starting from the lowest quality, are of no value to the firm and hence can be removed from inventory. These results imply that the pricing policy for the $V$ model should be driven by the aggregate, rather than the individual, inventory availability. We exploit these structural properties to develop a polynomial-time and exact algorithm that decomposes the multi-dimensional DP for the $V$ model into a series of one dimensional DPs.

In the $H$ model, we model consumer behavior using the *multinomial logit model* (MNL). We illustrate with a counterexample that the optimal prices do not conform to monotonicity properties described earlier. We show that the optimal prices depend on the individual inventory availability, instead of the aggregate inventory availability as in the $V$ model. We also prove that the firm should charge a uniform price for all the products with inventory surplus (i.e., the inventory of each product by itself can meet the maximum possible future demand), and should charge a premium over the uniform price for any product with an inventory shortfall (i.e., the inventory of each product by itself cannot meet the maximum possible future demand). Consequently, unlike the $V$ model, the firm extracts greater value from a product with inventory shortfall than a product with inventory surplus, regardless of their respective attribute ratings.

To the best of our knowledge, this is the first paper that incorporates the notion of product
differentiation explicitly in a dynamic pricing context and uncovers important analytical properties and their managerial implications, helping understand the impact of product differentiation on optimal joint pricing policies for multiple products.

The remainder of this paper is organized as follows. Section 2 provides a review of related literature and positions our work relative to others. Section 3 presents a DP formulation for the $G$ model, derives its optimal prices, and introduces a linear random utility framework. We build on this framework in Sections 4 and 5 to study the MPDP problems specific to vertical and horizontal product assortments. Section 6 uses numerical examples to highlight contrasts between the $V$ and $H$ models. Finally, Section 7 describes managerial insights and offers future research directions.

2. Related Literature

As the importance of revenue management has grown in practice, so has academic research on dynamic pricing and capacity management. The papers by McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003), and the recent book by Talluri and van Ryzin (2004b) provide comprehensive surveys of the literature in revenue management. We focus our review on models that consider dynamic pricing over a finite selling horizon with no replenishment opportunities and classify this literature into two categories – single and multi-product dynamic pricing models. For a single-product setting, Gallego and van Ryzin (1994) and Zhao and Zheng (2000) study dynamic pricing models with a continuous-time formulation. They show that the optimal price decreases with increasing inventory and decreasing remaining time.

Focusing on contexts where continuous updating of prices might not be feasible, Feng and Gallego (1995), Bitran and Mondschein (1997), and Feng and Xiao (2000) consider models that allow for only a finite number of price changes.

In the studies that develop pricing models for multiple products, pricing decisions are linked because of joint capacity constraints and/or due to demand correlations. Given starting inventories of components, Gallego and van Ryzin (1997) model the problem of determining the price for multiple products over a finite selling horizon. Since their model is difficult to solve, they develop heuristics based on the deterministic solution to the problem and show that these are asymptot-
ically optimal. Karaesmen and van Ryzin (2004) consider the substitutability of inventories to determine overbooking limits in a two-period model. When a firm uses a single resource to produce multiple products, Maglaras and Meissner (2006) explore the relation between dynamic pricing and capacity control and show that the dynamic pricing problem in Gallego and van Ryzin (1997) and the capacity control approach (for example, Lee and Hersh 1993) can be reduced to a common formulation. Liu and Milner (2006) study the multi-product pricing problem with a common price constraint. However, these papers do not explicitly model individual consumer choices.

Talluri and van Ryzin (2004a), and Zhang and Cooper (2005) model consumer choice behavior explicitly when considering booking limit (capacity control) policies for airline revenue management. Focusing on a single-leg yield management problem with exogenous fares, Talluri and van Ryzin (2004a) model how consumers choose from multiple fare products in determining the booking limits for various fare classes. Zhang and Cooper (2005) extend their model and consider capacity control for parallel flights. For an airline revenue management application that considers parallel flights and consumer choice, Zhang and Cooper (2007) model a pricing control problem. They acknowledge the complexity of the DP, construct heuristics, and test performance using a numerical study. Dong et al. (2008) examine both the initial inventory and subsequent dynamic pricing decisions. Through numerical experiments, they demonstrate the value of dynamic pricing and show that their approach determines near-optimal initial inventories. Using a modification of a budget-constrained choice model in Hauser and Urban (1986), Bitran et al. (2005) formulate a continuous-time problem to determine the optimal prices of product sub-families. They derive pricing policies based on a deterministic approximation of the retailer’s pricing problem.

We also address a pricing control problem; however, unlike previous approaches, we explicitly model product differentiation through suitable consumer choice models and derive strong analytical results. The analytical results for the $V$ model translate into an efficient multi-product pricing algorithm. Our analysis reveals strikingly different pricing characteristics for the $V$ and $H$ models, underscoring the importance of studying the impact of the nature of product differentiation on the MPDP problem with appropriate consumer choice models.
3. Pricing for a General Model of Consumer Behavior

We begin this section with a formulation of the MPDP for a general consumer choice model and derive the structure of optimal prices in this case. Then, we introduce a linear random utility framework, explore the resulting choice model, and examine its optimal prices.

3.1. Dynamic Programming Formulation

Consider the tactical pricing problem facing a firm that sells \( n \) substitutable products with indices \( j = 1, 2, \ldots, n \), over a finite selling season. The firm starts the selling season with a given initial inventory \( \kappa_j \) of product \( j \) and is unable to replenish these inventories during the season. Moreover, the inventories are perishable and any inventory that remains at the end of the season expires.

To model the multi-product demand process, we divide the selling season into \( T \) time periods such that each period has at most one customer arrival, and assume that each arriving customer requires no more than one unit of inventory. This demand arrival model is similar to others in the revenue management literature (for instance, Gerchak et al. 1985, Talluri and van Ryzin 2004a, Zhang and Cooper 2007). Let \( \lambda_t \) denote the probability of a customer arrival in period \( t \). We index the time periods in reverse chronological order: that is, \( t = 0 \) and \( t = T \) correspond to the end and the beginning of the selling season. The firm’s objective is to maximize the total revenues from the selling season by selecting an appropriate price for each product in every period. That is, in each period \( t \), for every current inventory level \( x = (x_1, x_2, \ldots, x_n) \), the firm must determine a price \( p_{jt} \) for each product \( j, j = 1, \ldots, n \). Depending on the prices that the firm quotes, customers may purchase one of the products in the assortment or not purchase any product at all. To capture the latter no-purchase option, we define a dummy product \( n + 1 \) in the assortment. Given price vector \( \mathbf{p}_t = (p_{1t}, p_{2t}, \ldots, p_{jt}, \ldots, p_{nt}) \), a consumer chooses to buy a product \( j \) with probability \( \alpha_j(\mathbf{p}_t) \), and has a no-purchase probability of \( \alpha_{n+1}(\mathbf{p}_t) = 1 - \sum_{j=1}^{n} \alpha_j(\mathbf{p}_t) \). At a particular inventory level \( x \) at time \( t \), since some products may have zero inventory, we define a state-dependent action space

\[
\mathcal{P}_x = \{ \mathbf{p}_t \geq 0 : \alpha_j(\mathbf{p}_t) = 0 \text{ if } x_j = 0, j = 1, \ldots, n \}. 
\]  

The exact form of \( \mathcal{P}_x \) depends on the choice probabilities, which we will specify in Sections 4 and 5. Given inventory vector \( x \), let \( V_t(x) \) denote the optimal expected revenue from period \( t \) to the
end of the season. Then, we formulate the MPDP problem as the following dynamic program:

$$V_t(x) = \max_{p_t \in P_x} \left\{ \sum_{j=1}^{n} \lambda_t \alpha_j(p_t)(p_{jt} + V_{t-1}(x - e_j)) + \lambda_t \alpha_{n+1}(p_t)V_{t-1}(x) + (1 - \lambda_t)V_{t-1}(x) \right\},$$  \hspace{1cm} (2)

with boundary conditions $V_t(0) = 0$ for $t = 0, 1, \ldots, T$, and $V_0(x) = 0$ for all $x$, where $e_j$ is a vector of size $n$ with 1 at the $j^{th}$ entry and zeros elsewhere. Since $\sum_{j=1}^{n} \alpha_j(p_t) = 1$, we can rewrite the optimality equation (2) as:

$$V_t(x) = \max_{p_t \in P_x} \left\{ \sum_{j=1}^{n} \lambda_t \alpha_j(p_t)(p_{jt} + V_{t-1}(x - e_j) - V_{t-1}(x)) \right\} + V_{t-1}(x).$$  \hspace{1cm} (3)

To further clarify the structure of the MPDP problem in (3), we define the difference functions of $V_t(x)$ with respect to $t$ and $x_j$. Let

$$\Delta_t V_t(x) = V_t(x) - V_{t-1}(x) \hspace{1cm} \text{for} \hspace{1cm} t=0,1,\ldots,T,$$

and

$$\Delta_{x_j} V_t(x) = V_t(x) - V_t(x - e_j) \hspace{1cm} \text{for} \hspace{1cm} j=1,2,\ldots,n.$$

Here, $\Delta_t V_t(x)$ represents the maximum expected gain, with inventory level $x$ in period $t$, if the firm had one additional selling period (marginal value of time). Similarly, $\Delta_{x_j} V_t(x)$ is the maximum expected gain, with inventory level $x$ in period $t$, if the firm had one more unit of product $j$ inventory to sell (marginal value of inventory). Using this notation, we rewrite (3) as:

$$\Delta_t V_t(x) = V_t(x) - V_{t-1}(x) = \max_{p_t \in P_x} \left\{ \sum_{j=1}^{n} \lambda_t \alpha_j(p_t)(p_{jt} - \Delta_{x_j} V_{t-1}(x)) \right\}.$$  \hspace{1cm} (4)

Note that in (4), the purchase probabilities $\alpha_j(p_t)$ do not incorporate any specific assumptions about the nature of product differentiation. Therefore, we refer to it as the $G$ (general) model.

Next, we derive the optimal prices for this model.

### 3.2. Optimal Pricing for a General Consumer Choice Model

From the optimality equation in (4), we define

$$G_t(x, p_t) = \sum_{j=1}^{n} \lambda_t \alpha_j(p_t)(p_{jt} - \Delta_{x_j} V_{t-1}(x)), \hspace{1cm} p_t \in P_x.$$  \hspace{1cm} (5)

To determine the optimal prices, we first examine the first order conditions for $G_t(x, p_t)$, assuming that $G_t(x, p_t)$ is strictly quasi-concave in $p_t$. Setting the partial derivatives of $G_t(x, p_t)$ with respect to $p_{jt}$ to zero gives:
\[
\frac{\partial G_t(x, p_t)}{\partial p_{jt}} = \sum_{k=1}^{n} \lambda_t \frac{\partial \alpha_k(p_t)}{\partial p_{jt}} (p_{kt} - \Delta_{x_k} V_{t-1}(x)) + \lambda_t \alpha_j(p_t) = 0, \quad j = 1, 2, \ldots, n. \tag{6}
\]

The conditions in (6) form a system of equations that the optimal price vector \(p_t\) must satisfy, which we can write in the matrix form as:

\[
p_t = -\alpha(p_t) \left( \frac{\partial \alpha(p_t)}{\partial p_t} \right)^{-1} + \Delta_x V_{t-1}(x), \tag{7}
\]

where \(\frac{\partial \alpha(p_t)}{\partial p_t}\) is the Jacobian matrix of \(\alpha(p_t) = (\alpha_1(p_t), \alpha_2(p_t), \ldots, \alpha_n(p_t))\) when prices are set at \(p_t\), with \((i, j)\) element \(\frac{\partial \alpha_i(p_t)}{\partial p_j}\), and \(\left( \frac{\partial \alpha(p_t)}{\partial p_t} \right)^{-1}\) is the inverse of the Jacobian matrix. Since \(G_t(x, p_t)\) is strictly quasi-concave, the unique solution, \(p_t\), of the above expression maximizes function \(G_t(x, p_t)\).

We formally state this result in the following theorem.

**Theorem 1** For given \(x\) and \(t\), suppose \(G_t(x, p_t)\) is strictly quasi-concave in \(p_t\). Then, the optimal price vector, denoted by \(p_t(x) = (p_{t1}(x), \ldots, p_{tn}(x))\), satisfies:

\[
p_t(x) = h(p_t(x)) + \Delta_x V_{t-1}(x), \tag{8}
\]

where \(h(p_t) = (h_1(p_t), \ldots, h_n(p_t))\) is an \(n\)-dimensional vector given by

\[
h(p_t) = -\alpha(p_t) \left( \frac{\partial \alpha(p_t)}{\partial p_t} \right)^{-1}, \tag{9}
\]

and \(\Delta_x V_t(x) = (\Delta_{x_1} V_t(x), \ldots, \Delta_{x_n} V_t(x))\). Specifically, the \(j^{th}\) element of \(p_t(x)\) is given by

\[
p_{jt}(x) = h_j(p_t(x)) + \Delta_{x_j} V_{t-1}(x), \quad x_j > 0. \tag{10}
\]

Theorem 1 suggests an intuitive interpretation of the optimal price of a product. Equation (10) shows that the optimal price of product \(j\) at inventory level \(x\) in period \(t\) is composed of two terms — the first term \(h_j(p_t)\) is the current marginal value of the product \(j\)'s inventory, and the second term \(\Delta_{x_j} V_{t-1}(x)\) is the future marginal value of the product \(j\)'s inventory, at inventory level \(x\). These expressions highlight the decision maker’s need to balance short- and long-term considerations in determining the optimal prices.

For the special case of a firm offering a single product (when consumers must choose between this product and the outside option), we can show that \(V_t(x)\) is a supermodular function of \(t\) and...
\(x\), a concave function of \(x\), and a concave function of \(t\) if \(\lambda_t\) is non-increasing in \(t\) (see EC.1 for proofs). We use these properties to develop results for the multi-product problems.

While Theorem 1 applies to any choice model, to gain insight into the impact of product differentiation on optimal pricing, we must develop choice models that explicitly incorporate the nature of product differentiation. Developing a consumer choice model begins with a formulation of consumer utility; accordingly, in Section 3.3, we develop a linear random utility model that forms the basis for both the vertical and horizontal differentiation choice models.

3.3. A Linear Random Utility Model for Differentiated Products

To model individual consumer choice from a differentiated assortment, we adopt a random utility approach that is consistent with the literature on discrete choice models (notably influenced by Luce (1959) and McFadden (1974), and described in detail in Anderson et al. (1992) and Roberts and Lilien (1993)). This approach treats products as bundles of attributes (such as price and quality) and captures consumer preferences for the attributes through associated random parameters to determine a consumer utility function. In some cases, modelers may include an independent random component that incorporates consumers’ idiosyncratic taste preferences (for the product as a whole, rather than its individual attributes). Building on the assumption that each consumer chooses a single variant that maximizes his/her utility, these models then construct choice probabilities for the variant, based on the probability distributions of the appropriate random parameters.

We consider a model from the economic literature (Caplin and Nalebuff 1991, Train 2003, Hensher and Greene 2003) that assumes that a consumer’s utility is linear in random parameters. Specifically, we describe a typical consumer’s utility, \(u_j\), from the purchase of variant \(j\) with attribute value \(q_j\) at price \(p_j\) as:

\[
 u_j = \theta q_j - p_j + \mu \xi_j, \tag{11}
\]

where \(\theta\) and \(\xi_j\) are independent draws from given distributions, and \(\mu \geq 0\) is a scalar. In (11), all consumers have a common rating \(q_j\) of variant \(j\), and each draw of random variable \(\theta\) represents an individual consumer’s sensitivity (strength of preference) to \(q_j\), with the distribution of \(\theta\) capturing the heterogeneity of consumer sensitivities to \(q_j\). On the other hand, consumers may have
and brand image. The evaluation of these aspects varies widely across consumers and the random term \( \mu \xi_j \) captures this variability, where \( \mu \) measures the degree (strength) of such preferences.

Researchers often make suitable assumptions regarding the random parameters for analytical tractability. Let \( \mu > 0, \xi_j \) follow the standard Gumbel distribution, and \( \theta \) follow a general distribution \( F \) with support \([0, \infty]\). Then, the linear random utility model yields the highly flexible mixed logit model (McFadden and Train 2000, Train 2003), whose choice probability is given by

\[
\alpha_j(p_t) = \int_0^\infty \frac{e^{(\theta_0 - \theta_j)/\mu}}{1 + \sum_{j=1}^n e^{(\theta_0 - \theta_j)/\mu}} dF(\theta), \quad j = 1, 2, \ldots, n. \tag{12}
\]

The mixed logit model embeds the two extremes of product differentiation – vertical and horizontal differentiation – as special cases. Berry and Pakes (2007) show that the mixed logit model approaches the demand model for vertically differentiated products as \( \mu \) approaches zero.

Next, we apply Theorem 1 to find the optimal prices for this choice model. First, we find the Jacobian matrix of \( \alpha(p_t) \). Let \( \epsilon_j(\theta) = e^{(\theta_0 - \theta_j)/\mu}, \Psi(\theta) = 1 + \sum_{j=1}^n \epsilon_j(\theta) \), and \( \Psi_j^{-1}(\theta) = 1 + \sum_{j \neq j} \epsilon_i(\theta) \), for \( j = 1, 2, \ldots, n \). Taking the partial derivative of \( \alpha_j(p_t) \) with respect to \( p_{jt} \), we obtain

\[
\frac{\partial \alpha_j(p_t)}{\partial p_t} = \left( \begin{array}{cccc}
- \int_0^\infty \frac{c_1(\theta) \Psi_j^{-1}(\theta) dF(\theta)}{\mu \Psi^2(\theta)} & \int_0^\infty \frac{c_1(\theta) c_2(\theta) dF(\theta)}{\mu \Psi^2(\theta)} & \cdots & \int_0^\infty \frac{c_1(\theta) c_n(\theta) dF(\theta)}{\mu \Psi^2(\theta)} \\
\int_0^\infty \frac{c_1(\theta) c_2(\theta) dF(\theta)}{\mu \Psi^2(\theta)} & - \int_0^\infty \frac{c_2(\theta) \Psi_j^{-2}(\theta) dF(\theta)}{\mu \Psi^2(\theta)} & \cdots & \int_0^\infty \frac{c_2(\theta) c_n(\theta) dF(\theta)}{\mu \Psi^2(\theta)} \\
\vdots & \vdots & \ddots & \vdots \\
\int_0^\infty \frac{c_1(\theta) c_n(\theta) dF(\theta)}{\mu \Psi^2(\theta)} & \int_0^\infty \frac{c_n(\theta) \Psi_j^{-2}(\theta) dF(\theta)}{\mu \Psi^2(\theta)} & \cdots & - \int_0^\infty \frac{c_n(\theta) c_j(\theta) dF(\theta)}{\mu \Psi^2(\theta)}
\end{array} \right). \tag{13}
\]

The determinant of the above matrix is \( \left| \frac{\partial \alpha_j(p_t)}{\partial p_t} \right| = \sum_{j=1}^n \frac{n!}{\prod_{i=1}^n c_{ij}} C_{ij} \), where \( C_{ij} = (-1)^{i+j} M_{ij} \) is the cofactor of the above matrix associated with entry \( \frac{\partial \alpha_j(p_t)}{\partial p_t} \), and \( M_{ij} \) is the determinant of the sub-matrix obtained by removing the \( i \)th row and \( j \)th column of the above matrix. Since the Jacobian matrix is symmetric, we have \( C_{ij} = C_{ji} \). Hence, the inverse of the Jacobian matrix (13) is

\[
\left( \frac{\partial \alpha_j(p_t)}{\partial p_t} \right)^{-1} = \left( \frac{C_{ij}}{n!} \right), \quad \text{where} \quad (C_{ij}) \quad \text{is the cofactor matrix of the Jacobian.}
\]

Following Theorem 1, the optimal price of product \( j \) in period \( t \) is, for \( j = 1, 2, \ldots, n \),

\[
p_{jt}(x) = h_j(p_t) + \Delta x_j V_{t-1}(x) = \frac{-\sum_{i=1}^n \alpha_i(p_t) C_{ij}}{\sum_{i=1}^n \frac{\partial c_i(p_t)}{\partial p_t} C_{ij}} + \Delta x_j V_{t-1}(x)
\]

\[
= \frac{-\sum_{i=1}^n \int_0^\infty \frac{c_i(\theta) \epsilon_i(\theta) dF(\theta)}{\Psi(\theta)}}{\sum_{i \neq j} \int_0^\infty \frac{c_i(\theta) \epsilon_i(\theta) dF(\theta)}{\Psi(\theta)}} - \int_0^\infty \frac{c_j(\theta) \Psi_j^{-1}(\theta) dF(\theta)}{\Psi(\theta)} + \Delta x_j V_{t-1}(x). \tag{14}
\]
The choice probabilities in (12) for the mixed logit model do not have closed-form expressions for a general distribution of $\theta$. Therefore, the optimal price expression in (14) is unwieldy and not conducive for generating analytical insights. Following our goal of understanding the impact of product differentiation on the firm’s pricing strategy, next we extract two utility specifications from (11) that allow us to construct choice models specifically for vertical and horizontal differentiation.

When $\mu$ is zero, the linear random utility function (11) does not have the random term $\mu \xi_j$, and directly reflects consumers’ valuation of product attributes. This case gives a consumer choice model for vertically differentiated products where consumers agree on product attribute values. The resultant model is a pure characteristics demand model (Berry and Pakes 2007) and has been widely used to describe vertical demand (e.g., Bresnahan 1987, Tirole 1988, Wauthy 1996, and Bhargava and Choudhary 2001). In Section 4, we study the MPDP problem for vertically differentiated products using a special case of the linear random utility model in (11), with $\mu$ equal to 0 and $\theta$ following a uniform distribution between 0 and 1.

When $\mu$ is positive, the random term $\mu \xi_j$ in (11) captures idiosyncratic customer preferences for products. In this case, since products cannot be universally ordered in terms of their attribute values, the resultant model captures the behavior of consumers facing horizontally differentiated products. However, as we demonstrated previously, when $\theta$ follows a general distribution, the resulting model is unlikely to yield insights. A simplifying approach is to assume that $\theta$ follows a degenerate distribution whose support consists of a single value, which often represents the average sensitivity of customers to attribute value $q_j$. The resulting choice model is the multinomial logit (MNL) model, whose choice probabilities are easily specified and conducive for analysis (Anderson and Palma (1992)). In Section 5, we use the MNL model to study the MPDP problem for a horizontally differentiated assortment.

Finally, we note that researchers have developed effective estimation procedures for model parameters of both vertical and horizontal differentiation-based choice models (see Train 2003, Berry and Pakes 2007, and Song 2007), allowing for their wide applicability. In the following discussions, we assume the availability of reliable estimates of the relevant parameters of our choice models.
4. Pricing of Vertically Differentiated Products

Building on the utility function (11) with $\mu = 0$ and $\theta$ as a uniform [0,1] random variable, we first develop a quality-based choice model for vertically differentiated products in Section 4.1. Then, we formulate the MPDP problem with this particular choice model, which we refer to as the $V$ model, in Section 4.2. We derive structural properties of the optimal prices and discuss managerial implications in Section 4.3. Finally, in Section 4.4, we synthesize these results to develop an effective and exact algorithm to solve this problem.

4.1. A Quality-Based Choice Model for Vertical Differentiation

Consider a consumer who must choose from $n$ products that have different quality ratings. Let product $j$ have a quality index $q_j$ (common to all consumers) and assume that product quality can be ordered as $q_1 > q_2 > \cdots > q_n$. We model consumer utility as a variant of the linear random utility function in (11) with random coefficient $\theta$ being uniformly distributed between 0 and 1, and the idiosyncratic error term $\mu \xi_j$ excluded ($\mu = 0$). Accordingly, we can express a typical consumer’s utility from the purchase of product $j$ at price $p_{jt}$ as $u_{jt} = \theta q_j - p_{jt}$. In this setup, if any two products $i$ and $j$ have the same price, then a consumer would prefer product $j$ over product $i$ when $q_j > q_i$. A consumer can also choose the no-purchase option, which is denoted as product $n+1$ in the assortment. We normalize the value of the no-purchase option at 0 by setting $q_{n+1} \equiv p_{n+1,t} \equiv 0$, for all $0 \leq t \leq T$.

A utility maximizing customer facing price $p_t$ in period $t$ would choose product $j$ at time $t$ if $u_{jt} \geq \max_{k \neq j} \{u_{kt}\}$, or equivalently if $\theta q_j - p_{jt} \geq \theta q_k - p_{kt}$, $\forall k \neq j$. This further implies that a consumer will choose product $j$ with probability

$$\alpha_j(p_t) = \begin{cases} P \left( \max_{k > j} \frac{p_{kt} - p_{jt}}{q_k - q_j} \leq \theta \right), & j = 1, \\ P \left( \max_{k > j} \frac{p_{kt} - p_{jt}}{q_k - q_j} \leq \theta \leq \min_{k < j} \frac{p_{jt} - p_{kt}}{q_k - q_j} \right), & j = 2, \ldots, n. \end{cases} \tag{15}$$

To obtain an explicit expression of these choice probabilities, first note that for product $j = n$, (15) implies that it is sufficient to consider the candidate prices $p_{n-1,t}$ and $p_{nt}$ satisfying

$$0 \leq \frac{p_{nt} - p_{n+1,t}}{q_n - q_{n+1}} \leq \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n}. \tag{16}$$

In (16), the first inequality is self-explanatory; the second inequality is valid, for otherwise the
choice probability in (15) for product \( n \) would be zero, the same as the null probability when the
second inequality in (16) assumes an equality. Similarly, for product \( j = n - 1 \), (15) implies that it
is sufficient to consider the candidate prices \( p_{n-2,t}, p_{n-1,t} \) and \( p_{n,t} \) such that

\[
\max \left\{ \frac{p_{n-1,t} - p_{n,t}}{q_{n-1} - q_n}, \frac{p_{n-1,t} - p_{n+1,t}}{q_{n-1} - q_{n+1}} \right\} = \frac{p_{n-1,t} - p_{n,t}}{q_{n-1} - q_n} \leq \frac{p_{n-2,t} - p_{n-1,t}}{q_{n-2} - q_{n-1}},
\]

where the equality in the above expression is the result of (16). Following a similar argument, we
can show inductively that for any product \( j \), we only need to consider the prices that satisfy

\[
\max_{k>j} \left\{ \frac{p_{jt} - p_{kt}}{q_j - q_k} \right\} = \frac{p_{j,t} - p_{j+1,t}}{q_j - q_{j+1}} \leq \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j}, \quad \text{for } j = 2, \ldots, n,
\]

\[
\max_{k=1} \left\{ \frac{p_{jt} - p_{kt}}{q_j - q_k} \right\} = \frac{p_{1,t} - p_{jt}}{q_1 - q_j} \leq 1, \quad \text{for } j = 1.
\]

Furthermore, to ensure \( p_t \in \mathcal{P}_x \), we need to force (17) to an equality so that \( \alpha_j(p_t) = 0 \) whenever
\( x_j = 0, j = 1, \ldots, n \). Together, our observations imply that, given \( x \), it is sufficient to restrict the
candidate prices to the following set of quality-aligned prices, which we denote as \( \tilde{\mathcal{P}}_x \):

\[
\tilde{\mathcal{P}}_x = \left\{ p_t : \frac{1}{\sum_{k=1}^n \frac{p_{jt} - p_{kt}}{q_j - q_k}} = \frac{1}{\sum_{k=1}^n \frac{p_{jt} - p_{kt}}{q_j - q_k}} = \frac{1}{\sum_{k=1}^n \frac{p_{jt} - p_{kt}}{q_j - q_k}} \right\},
\]

Interestingly, observe that for the \( V \) model, it is sufficient to consider the candidate prices \( \{p_{jt}\} \) that
form an increasing convex mapping of the quality values \( \{q_j\} \) over the interval \([0,1]\). A managerial
implication of this result is that, for vertically differentiated products, a higher quality product
should always be priced higher than a lower quality product (quality monotonicity), regardless
of its individual inventory level; furthermore, the “price increment per unit quality”, \( \frac{p_{jt} - p_{jt+1}}{q_j - q_{j+1}} \), is
decreasing in quality index \( j \). In other words, a firm has an increasing pricing power (per unit
quality) for higher quality products. In fact, this observation applies to any distribution of \( \theta \).

By restricting the prices \( p_t \) to the set \( \tilde{\mathcal{P}}_x \), we have \( \max_{k>j} \left\{ \frac{p_{jt} - p_{kt}}{q_j - q_k} \right\} = \frac{p_{j,t} - p_{j+1,t}}{q_j - q_{j+1}} \) and
\( \min_{k<j} \left\{ \frac{p_{jt} - p_{kt}}{q_j - q_k} \right\} = \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j} \). Therefore, we can write the choice probabilities (15) as

\[
\alpha_j(p_t) = \begin{cases} 1 - \frac{p_{jt} - p_{kt}}{q_j - q_k}, & j = 1, \\ \frac{p_{j,t} - p_{j+1,t}}{q_j - q_{j+1}}, & j = 2, \ldots, n, \\ 1 - \sum_{i=1}^{n-1} \alpha_i(p_t), & j = n + 1. \end{cases}
\]

### 4.2. Optimal Pricing for Vertical Differentiation

By substituting the purchase probabilities from (19) into (4), we have

\[
G_t(x, p_t) = \lambda_t \left( 1 - \frac{p_{1t} - p_{2t}}{q_1 - q_2} \right) (p_{1t} - \Delta x, V_{t-1}(x))
\]
the non-tridiagonal entries are zero, an infinitesimal change in the price of product $j$ at a linear rate, i.e., when the price of product $j$ at a linear rate, i.e., when the price of product $j$, decreases at a linear rate. Also, $\Delta_{jt} \in \mathbb{R}^{n \times n}$ is a tridiagonal symmetric matrix with negative diagonal entries and nonnegative nondiagonal entries, with each entry independent of the price vector $p$. Since all the non-tridiagonal entries are zero, an infinitesimal change in the price of product $j$ will result in demand substitution, but only to an adjacent product. Moreover, this demand substitution occurs at a linear rate, i.e., when the price of product $j$ increases, the likelihood that a customer will buy $j$ decreases at a linear rate and the probability that the customer will buy $j - 1$ or $j + 1$ increases at a linear rate. Also, $\frac{\partial G_{jt}(p)}{\partial p_{jt}} = 1$ and $\frac{\partial G_{jt}(p)}{\partial p_{jt}} = 0$ for $j \neq 1, \ldots, n$, implying that the total purchase probability (market share) depends only on $p_{jt}$, the price of the lowest quality product.

We first assume $x > 0$. Clearly, $G_{jt}(x, p)$ in (20) is a quadratic and concave function of $p_{jt}$; consequently, Theorem 1 applies to the $V$ problem. While it is possible to obtain the optimal prices from the results in Section 3.3 (by letting $\mu \to 0$), it involves more complex notation and technicality of sequence convergence; therefore, we choose to directly derive the optimal prices using Theorem 1.

The Jacobian matrix of probability vector $\alpha(p)$ for given $p \in \tilde{P}_x$ is

$$\frac{\partial \alpha(p)}{\partial p} = \begin{pmatrix} q_{1-1} q_{2-2} q_{3-3} & 0 & 0 & 0 \\ 0 & q_{2-2} q_{3-3} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & q_{n-1} q_{n-1} q_{n-1} q_{n-1} q_{n-1} \end{pmatrix}. \quad (21)$$

The Jacobian of $\alpha(p)$ in (21) is a tridiagonal symmetric matrix with negative diagonal entries and nonnegative nondiagonal entries, with each entry independent of the price vector $p$. Since all the non-tridiagonal entries are zero, an infinitesimal change in the price of product $j$ will result in demand substitution, but only to an adjacent product. Moreover, this demand substitution occurs at a linear rate, i.e., when the price of product $j$ increases, the likelihood that a customer will buy $j$ decreases at a linear rate and the probability that the customer will buy $j - 1$ or $j + 1$ increases at a linear rate. Also, $\frac{\partial \alpha_{n+1}(p)}{\partial p_{n+1}} = 1$ and $\frac{\partial \alpha_{n+1}(p)}{\partial p_{n+1}} = 0$ for $j \neq 1, \ldots, n$, implying that the total purchase probability (market share) depends only on $p_{nt}$, the price of the lowest quality product.

We can derive the inverse of the Jacobian matrix in (21) as:

$$\left(\frac{\partial \alpha(p)}{\partial p}\right)^{-1} = -\begin{pmatrix} q_{1} q_{2} & \cdots & q_{n-1} q_{n} \\ q_{2} q_{2} & \cdots & q_{n-1} q_{n} \\ \vdots & \vdots & \vdots \\ q_{n-1} q_{n-1} & \cdots & q_{n-1} q_{n} \\ q_{n} q_{n} & \cdots & q_{n} q_{n} \end{pmatrix}. \quad (22)$$

Substituting (21) and (22) into $h(p) = -\alpha(p)\left(\frac{\partial \alpha(p)}{\partial p}\right)^{-1}$, we get:

$$h_{j}(p) = \left(1 - \frac{p_{1t} - p_{2t}}{q_{1} q_{2}}, \ldots, \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_{j}}, \frac{p_{j+1,t} - p_{jt}}{q_{j+1} - q_{j}}, \ldots, \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_{n}}, \frac{p_{nt}}{q_{n}}\right) \times \begin{pmatrix} q_{1} \\ \vdots \\ q_{j} \\ \vdots \\ q_{n} \end{pmatrix}. \quad (23)$$
\[ p_{jt}(x) = q_j - p_{jt}, \quad j = 1, 2, \ldots, n. \]

Since the optimal prices satisfy \( p_{jt}(x) = q_j - p_{jt}(x) + \Delta_{x_j} V_{t-1}(x) \) (from Theorem 1), we obtain

\[ p_{jt}(x) = \frac{1}{2} (q_j + \Delta_{x_j} V_{t-1}(x)), \quad j = 1, 2, \ldots, n. \quad (24) \]

Thus, if \( x_j > 0 \), the optimal price \( p_{jt}(x) \) is the average of the quality rating \( q_j \) of product \( j \) and the future marginal value of product \( j \) inventory, \( \Delta_{x_j} V_{t-1}(x) \). Using (20), we obtain

\[ V_t(x) = G_t(x, p_t(x)) + V_{t-1}(x) \]
\[ = \lambda_t \left( q_1 - 2p_{1t}(x) + \sum_{k=1}^{n-1} \left( \frac{p_{kt}(x) - p_{k+1,t}(x)}{q_k - q_{k+1}} + \frac{p_{n,t}^2(x)}{q_n} \right) \right) + V_{t-1}(x). \quad (25) \]

Now, suppose \( x_j = 0 \) in \( x \) for some \( j \). Then, we adopt the following two steps to determine the price for each product. In the first step, we ignore the zero-inventory products and solve the problem for the positive-inventory products alone using the procedure described in (24) and (25). In the second step, we set the prices for the zero-inventory products as follows. Recall that if \( x_j = 0 \), the price \( p_{jt}(x) \) needs to satisfy

\[ \frac{p_{tj}(x) - p_{2t}(x)}{q_1 - q_2} = 1 \quad \text{for} \quad j = 1 \quad \text{and} \quad \frac{p_{jt}(x) - p_{j+1,t}(x)}{q_j - q_{j+1}} = \frac{p_{j-1,t}(x) - p_{jt}(x)}{q_{j-1} - q_j} \quad \text{for} \quad j = 2, \ldots, n; \]

or, equivalently,

\[ p_{jt}(x) = \begin{cases} p_{2t}(x) + (q_1 - q_2), & j = 1, \\ \frac{p_{jt}(x) - (q_j - q_{j+1})p_{j-1,t}(x) + (q_{j-1} - q_j)p_{j+1,t}(x)}{q_{j-1} - q_j}, & j = 2, \ldots, n. \end{cases} \quad (26) \]

This provides us with a recursive relation to determine the price of a zero-inventory product from that of its adjacent products. By using the recursion (iteratively, if necessary), we can express the price of each zero-inventory product by the price(s) of its adjacent, positive-inventory product(s).

Since the prices of the positive-inventory products have been determined in the first step, the prices for the zero-inventory products are determined uniquely. The value function in (25) still holds, except that the summation is over index \( j \) such that \( x_j > 0 \). Therefore, we can assume \( x > 0 \) without loss of generality and do so for the remainder of this section.

Equations (24)-(25) allow us to derive structural properties of the optimal price \( p_t(x) \) and the optimal value function \( V_t(x) \). We describe these properties next.

### 4.3. Structural Properties of the \( V \) Model

In this section, we first uncover special structures of the \( V \) model that highlight the importance of aggregate product inventories in determining the optimal prices. To facilitate this discussion, let
\|x\|_j = \sum_{i=1}^{j} x_i$ denote the aggregate inventory of the $j$ highest quality products in state $x$ and let $p_{jt}(x)$ represent the optimal price of product $j$ in period $t$ given $x$. The next theorem describes the behavior of the optimal prices as a function of the aggregate inventory level $|x|_j$.

**Theorem 2** For the $V$ model, the optimal prices $p_{i}(x)$ have the following properties:

(a) If $|x|_j \geq t$, then $p_{kt}(x) = \frac{q_k}{2}$ for $k \geq j$ and $\alpha_k(p_i(x)) = 0$ for $k > j$.

(b) If $|x|_j \geq t$, then $p_{it}(x) = p_{it}(x + e_k)$ for $i \leq j \leq k$ and $V_i(x) = V_i(x + e_k)$ for $k \geq j$.

(c) $p_{jt}(x)$ depends on the inventory levels of the first $j$ products only through their sum $|x|_j$. Specifically, for $i' < i \leq j$,

$$p_{jt}(x) = p_{jt}(x - e_{i'} + e_i) = \cdots = p_{jt}(0, \ldots, 0, |x|_j, x_{j+1}, \ldots, x_n). \quad (27)$$

(d) Let products $i$ and $j$ be two adjacent products that have positive inventories. Then the price difference $p_{it}(x) - p_{jt}(x)$ is strictly positive and depends only on the total inventory of the first $i$ products $|x|_i$. Specifically, for $i = 1, \ldots, n-1$ and $j > i$,

$$p_{it}(x) - p_{jt}(x) = \frac{1}{2} (q_i - q_j) + V_{i-1}(0, \ldots, 0, |x|_i, 0, \ldots, 0) - V_{i-1}(0, \ldots, 0, |x|_i - 1, 0, \ldots, 1, 0, \ldots, 0)), \quad (28)$$

where $|x|_i$ and $|x|_i - 1$ are in the $i^{th}$ position of the two vectors and 1 is in the $j^{th}$ position.

**Proof.** See Appendix EC.2.

Next, we provide a brief interpretation of Theorem 2. We refer to $|x|_j$ as product $j$’s aggregate inventory, and say that product $j$ has an aggregate inventory surplus if $|x|_j \geq t$, i.e., if product $j$’s aggregate inventory is sufficient to satisfy the potential remaining demand. Part (a) states that when the firm has an aggregate inventory surplus for some product $j$, then any lower quality product’s inventory has no future marginal value (i.e., $\Delta_{q_k} V_{t-1}(x) = 0$, for $k \geq j$); as such, its price should be set as $p_{kt}(x) = \frac{q_k}{2}$, $k \geq j$, for the rest of the selling season. At this constant price, we have $\frac{p_{k-1,t}(x) - p_{k,t}(x)}{q_{k-1} - q_k} = \frac{p_{k,t}(x) - p_{k+1,t}(x)}{q_k - q_{k+1}}$, $k > j$, implying that the purchase probability for any lower quality product $k > j$ is zero. For this scenario, part (b) states that inventories of lower quality products have no effect on the firm’s pricing strategy for higher quality products. On the other
hand, both parts (c) and (d) rest on an intuitively appealing key result (see the proof of (EC.5) in
the electronic companion), that is,

\[ V_i(x) - V_i(x - e_i + e_j) \text{ is independent of } (x_j, \ldots, x_n), \text{ for any } i < j. \] (29)

In economic terms, the above statement means that the marginal benefit of having one more unit
of a higher quality product \(i\) and one less unit of a lower quality product \(j\) at inventory level \(x\) is
independent of inventory levels of products \(j \geq i\). Result (29) has two implications. First, it implies
\[ \Delta x_j V_i(x) = \Delta x_j V_i(x - e_i + e_j) = \cdots = \Delta x_j V_i(0, \ldots, 0, |x|_j, x_{j+1}, \ldots, x_n), \]
i.e., the future marginal value of product \(j\)’s inventory depends on the inventory levels of the
first \(j\) products only at the aggregate level \(|x|_j\) (note that the future marginal value of product
\(j\) still depends on the individual inventory levels of lower quality products). Since \(p_{jt}(x) = \frac{1}{2}(q_i + \Delta x_j V_i(x))\), part (c) follows immediately. Result (29) also leads to part (d), which allows us to
compute the price \(p_{it}(x)\) as a markup over the price of an adjacent lower quality product. Moreover,
this markup depends solely on product \(i\)’s aggregate inventory. To see this, note that the price
difference of these products is equal to the average of the difference of their quality ratings, \(q_i - q_j\),
and the marginal benefit of having one less unit of product \(j\) versus one less unit of product \(i\) at
inventory level \(x\), \(V_i(x - e_j) - V_i(x - e_i)\), which is independent of \((x_j, \ldots, x_n)\) according to (29).
We exploit these properties to develop a polynomial-time exact algorithm in Section 4.4.

Theorem 2 has several important managerial and operational implications that highlight the
importance of aggregate inventory properties in determining optimal prices. When inventory is
abundant relative to demand (i.e., when demand is lower than that anticipated by strategic capacity
choices or when the end of the season approaches), Theorem 2 provides the firm with a simple rule,
via the notion of aggregate inventory surplus of a product, to determine which product inventories
to market and which to ignore. Even when inventories are not abundant, inventory aggregation is
a key principle. In this case, the principle suggests that in determining the price of any product \(j\),
managers can treat \(j\) as the product with the highest quality in the assortment with a corresponding
inventory of \(|x|_j\). By accounting for inventories of products \(i < j\) through the aggregate inventory
this principle emphasizes that it is the echelon-type inventory position \(|x|_j\) of product \(j\), rather than individual availabilities of higher quality products, that determines the price of the product. Moreover, the price of product \(j\) can be determined as a markup over the adjacent lower quality product, where the markup itself is determined solely by the the aggregate inventory \(|x|_j\) of product \(j\). Together, these results (based on the aggregation principle) provide great insights into the manager’s problem of jointly pricing multi-product inventories over time.

**Theorem 3** For the V model,

(a) \(\Delta_{x_j} V_t(x)\) is non-decreasing in \(t\) (or equivalently, \(\Delta_t V_t(x)\) is non-decreasing in \(x_j\)).

(b) \(\Delta_{x_j} V_t(x)\) is non-increasing in \(x_i, i \neq j\).

(c) \(\Delta_{x_j} V_t(x)\) is non-increasing in \(x_j\).

(d) If \(\lambda_t \geq \lambda_{t+1}\), then \(\Delta_t V_t(x)\) is non-increasing in \(t\).

**Proof.** See Appendix EC.3.

Theorem 3 implies that \(V_t(x)\) is supermodular in time \(t\) and product \(j\)’s inventory \(x_j\), submodular in inventory levels of products \(i\) and \(j\) (\(x_i\) and \(x_j\) respectively), and concave in product \(j\)’s inventory \(x_j\). Further, \(V_t(x)\) is also concave in time \(t\) if the arrival probability \(\lambda_t\) is non-increasing in \(t\). Since the optimal price satisfies \(p_{jt}(x) = \frac{1}{2} (q_j + \Delta_{x_j} V_{t-1}(x))\), we know that \(p_{jt}(x)\) carries all the structural properties of \(\Delta_{x_j} V_{t-1}(x)\). The next corollary follows directly from Theorem 3.

**Corollary 4** For the V model, the optimal price \(p_{jt}(x)\) is a non-decreasing function of \(t\), a strictly increasing function of \(j\), a non-increasing function of \(x_j\), and a non-increasing function of \(x_i\) for \(i \neq j\).

Corollary 4 generalizes the monotonicity results from a single product (given in Appendix EC.1) to an assortment of vertically differentiated products. Specifically, Corollary 4 shows that the optimal prices exhibit (1) quality monotonicity: a higher quality product is always priced higher than a lower quality product (as mentioned earlier, this property follows from the fact that prices \(\{p_{jt}\}\) must form an increasing convex mapping of quality rating \(\{q_j\}\)); (2) inventory monotonicity: \(p_{jt}(x)\) becomes lower if the inventory of any product becomes higher; and (3) time monotonicity: the price
for any product is non-increasing when the end of the sales horizon approaches. As we will see in Section 6, these monotonicity properties do not necessarily hold for the horizontally differentiated products.

4.4. An Efficient and Exact Algorithm for the V Model

We can effectively translate the structural results in Section 4.3 into an exact, polynomial-time computational algorithm. Exploiting the structure of the optimal prices, this algorithm decomposes the multi-dimensional state and action spaces of the V model into a sequence of single-dimensional state and action space DPs, thereby drastically reducing the computational effort, both in memory requirements and running time. In the following discussion, we first highlight the key insights that drive our algorithm and then present a formal description of the procedure.

To describe this algorithm, we first define the following notation that helps us identify those products that have positive inventories at a given inventory level. Suppose inventory level $x$ has $m(x) \leq n$ products with positive inventories. Let $k_r(x)$, $r = 1, \ldots, m(x)$, be the product with the $r$th highest quality rating among the products with non-zero inventory levels. Accordingly, \{ $k_1(x), \ldots, k_{m(x)}(x)$ \} represents the set of products with non-zero inventories at inventory level $x$. For convenience, hereafter, we express $k_r(x)$ simply as $k_r$, and $m(x)$ as $m$.

Our algorithm recursively uses the price difference function (28), stated in Theorem 2(d), to obtain optimal prices $p_{k_r,t}(x)$, starting from that of the lowest quality product $p_{k_m,t}(x)$. In order to use (28), our first step is to solve a sequence of single-dimensional DP problems,

$$V_i^x(t) := V_i(0, \ldots, 0, x, 0, \ldots, 0), \text{ for } i = 1, \ldots, n,$$

where $x$ appears in the $i$th position, and

$$V_i^{x,1}(x-1, 1) := V_i(0, \ldots, 0, x-1, 0, \ldots, 0, 1, 0, \ldots, 0), \text{ for } i = 1, \ldots, n-1, j > i,$$

where $x-1$ appears in the $i$th position and 1 in the $j$th position.

In the second step, we determine the optimal price for each product, computing the prices for positive inventory products first and then using (26) for the remaining zero inventory products. Recall from Theorem 2(c) that the price of product $k_m$ depends only on its aggregate inventory
\[ |x|_{k,m}. \] Accordingly, the optimal price \( p_{k,m,t}(x) \) of the lowest quality product \( k_m \) with positive inventory can be obtained as \( p_{k,m,t}(x) = \frac{1}{2}(q_{km} + V_{t-1}^{k,m}(|x|_{k,m}) - V_{t-1}^{k,m}(|x|_{k,m} - 1)) \). Next, our algorithm progresses from the lowest quality product \( k_m \) to the adjacent higher quality product \( k_{m-1} \) using (28) to obtain the price \( p_{k_{m-1},t}(x) \) as

\[
p_{k_{m-1},t}(x) = p_{k,m,t}(x) + \frac{1}{2} \left( q_{k,m-1} - q_{km} + V_{t-1}^{k,m-1}(|x|_{k,m-1}) - V_{t-1}^{k,m-1,k,m}(|x|_{k,m-1}, 1) \right).
\]

Since we have already established the price \( p_{k,m,t}(x) \) and computed the optimal values \( V_{t-1}^{k,m-1,k,m}(x−1,1) \) and \( V_{t-1}^{k,m-1}(x) \), we can readily compute \( p_{k_{m-1},t}(x) \). Recursively, for any two adjacent products \( i \) and \( j \) with positive inventories, starting with the optimal single dimensional DP solutions \( V_{t-1}^{i,j}(x−1,1) \) and \( V_{t-1}^{i,j}(x) \) and the computed price \( p_{ij}(x) \), we can determine price \( p_{ij}(x) \) using the corresponding price equation.

We can also derive the value function \( V_t(x) \) directly from the single dimensional DPs \( V_{t-1}^{i,j}(x−1,1) \) and \( V_{t-1}^{i,j}(x) \). Using the result (29), for the two highest quality products \( k_1 \) and \( k_2 \) with positive inventories,

\[
V_t(x) - V_t(x - e_{k_1} + e_{k_2}) = V_t(0,\ldots,0, x_{k_1}, 0,\ldots,0) - V_t(0,\ldots,0, x_{k_1} - 1, 0,\ldots,0, 1, 0,\ldots,0)
\]

\[
= V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1} - 1, 1).
\]

Using this relationship recursively, we obtain:

\[
V_t(x) = V_t(x - e_{k_1} + e_{k_2}) + V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1} - 1, 1)
\]

\[
= V_t(x - 2e_{k_1} + 2e_{k_2}) + V_t^{k_1}(x_{k_1} - 1) - V_t^{k_1,k_2}(x_{k_1} - 2, 1) + V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1} - 1, 1)
\]

\[
= V_t(0,\ldots,0, |x|_{k_2}, x_{k_2+1},\ldots,x_n) + \sum_{x=1}^{\frac{|x|_{k_1}}{1}} V_t^{k_1}(x) - \sum_{x=0}^{\frac{|x|_{k_1} - 1}{1}} V_t^{k_1,k_2}(x, 1)
\]

\[
= V_t(0,\ldots,0, |x|_{k_3}, x_{k_3+1},\ldots,x_n) + \sum_{x=1}^{\frac{|x|_{k_2}}{2}} V_t^{k_2}(x) - \sum_{x=0}^{\frac{|x|_{k_2} - 1}{2}} V_t^{k_2,k_3}(x, 1) + \sum_{x=1}^{\frac{|x|_{k_1}}{1}} V_t^{k_3}(x) - \sum_{x=0}^{\frac{|x|_{k_1} - 1}{1}} V_t^{k_1,k_2}(x, 1)
\]

\[
= \ldots
\]

\[
= \sum_{r=1}^{m} \sum_{x=1}^{\frac{|x|_{k_r}}{r}} V_t^{k_r}(x) - \sum_{j=2}^{m} \sum_{x=0}^{\frac{|x|_{k_{r-1}} - 1}{r-1}} V_t^{k_{r-1},k_r}(x - 1, 1), \quad x \leq \kappa, \quad 1 \leq t \leq T.
\]

We formally describe the steps in our exact algorithm next.

**Algorithm 5 (An Exact Algorithm for Model V)**
Step 1 Determine the value functions $V_i^i(x)$ and $V_i^{i,j}(x,1)$:

For all $t$, compute $V_i^i(x)$ for $i = 1, \ldots, n$, and $V_i^{i,j}(x,1)$ for all $i = 1, \ldots, n-1$ and $j > i$.

Step 2 Determine the optimal prices $p_{jt}(x)$, $j = 1, \ldots, n$:

2a For all $t$, set price $p_{km,t}(x)$ for the lowest quality product with positive inventory as

$$p_{km,t}(x) = \frac{1}{2} \left( q_{km} + V_{t-1}^{km} (|x|_{km} - V_{t-1}^{km} (|x|_{km} - 1)) \right) \quad (31)$$

2b For all $t$, starting with $r = m$ and $2 \leq r \leq m$, recursively set the price $p_{kr-1,t}(x)$ of the adjacent lower quality product with positive inventory as

$$p_{kr-1,t}(x) = p_{kr,t}(x) + \frac{1}{2} \left( q_{kr-1} - q_{kr} + V_{t-1}^{kr-1} (|x|_{kr-1}) \right) - V_{t-1}^{kr-1} (|x|_{kr-1} - 1, 1) \quad (32)$$

2c For all $t$ and any $j$ with $x_j = 0$, set $p_{jt}(x)$ as

$$p_{jt}(x) = \begin{cases} 
    p_{2j}(x) + (q_1 - q_2), & j = 1, \\
    \frac{(q_j - q_{j+1})p_{j-1,t}(x) + (q_{j-1} - q_j)p_{j+1,t}(x)}{q_{j-1} - q_{j+1}}, & j = 2, \ldots, n. 
\end{cases} \quad (33)$$

Step 3 Compute the value function $V_i(x)$:

For all $t$, $1 \leq r < m$ and $0 \leq x \leq \kappa$, set

$$V_i(x) = \sum_{r=1}^{m} \sum_{z=1}^{|x|_{kr}} V_t^{kr}(x) - \sum_{r=2}^{m} \sum_{z=1}^{|x|_{kr-1}} V_t^{kr-1,kr}(x,1). \quad (34)$$

We close this section with a brief discussion of the complexity of our algorithm versus that of the standard backward induction algorithm. Suppose $\kappa = \max_i \{\kappa_i\}$ is the maximum of the initial product inventory levels. Then, solving for the value function $V_i(x)$ for $1 \leq t \leq T$ using backward induction would need an exponential number of state evaluations, resulting in a running time of $O(\kappa^n T)$. In contrast, the computational complexity of our algorithm depends on that of the single dimensional DPs $V_t^i$ for all $i = 1, \ldots, n$ and $V_t^{i,j}$ for all $i < j$ and $j = 2, \ldots, n$. Since there are $n^2 \frac{n+1}{2}$ such DPs, each requiring a running time of $O(n \kappa T)$, the total running time is $O(n^3 \kappa T)$. Therefore, Algorithm 5 is a polynomial-time exact algorithm. We can also show that the savings in memory requirements are similar to the savings in the running time. Clearly, our exact algorithm
drastically reduces the computational complexity of the backward induction algorithm and can be implemented for large-sized problems.

5. Pricing of Horizontally Differentiated Products

Starting from the utility model in (11) with $\mu > 0$ and $\theta$ as constant, we first describe a choice model that captures consumer behavior when facing horizontally differentiated products. We refer to the MPDP problem with this particular horizontal demand model as the $H$ model. We next obtain the optimal price expressions and derive some properties for this model.

Consider a firm that offers $n$ horizontally differentiated products each with a rating $q_j$ for product $j$ that is common to all the consumers in the market. Building on (11), we express the utility function of a typical consumer from the purchase of product $j$ by the well-known multinomial logit (MNL) discrete choice model (see McFadden 1980, Anderson et al. 1992 and Anderson and Palma 1992),

$$u_j = \theta q_j - p_j + \mu \xi_j,$$

where $\theta$ is a constant, denoting the mean sensitivity of consumers to value $q_j$, random variable $\xi_j$ is a draw from the standard Gumbel distribution with zero mean and unit variance, and $\mu$ is a positive parameter. Suppose the utility of an outside option is normalized to zero. In this case, the mixed logit model in Section 3.3 reduces to the MNL model, and the choice probability (12) becomes

$$\alpha_j(p_t) = \frac{e^{(\theta q_j - p_{jt})/\mu}}{1 + \sum_{j=1}^{n} e^{(\theta q_j - p_{jt})/\mu}}, \quad j = 1, 2, \ldots, n. \quad (35)$$

The MNL model has been extensively used in econometrics to describe consumer choice (Berkovec 1985, Train 1986), in marketing for pricing and production decisions (Ben-Akiva and Lerman 1985), and in the operations management area (Zhang and Cooper 2007, Dong et al. 2008).

We can formulate the $H$ model by substituting the probabilities from (35) in the optimality equation (4). Then $G_t(x, p_t)$, from (5), can be expressed as:

$$G_t(x, p_t) = \sum_{j=1}^{n} \frac{\lambda_t e^{(\theta q_j - p_{jt})/\mu}}{1 + \sum_{k=1}^{n} e^{(\theta q_k - p_{kt})/\mu}} (p_{jt} - \Delta x_j V_{t-1}(x)), \quad p_t \in P_x. \quad (36)$$

To ensure $p_t \in P_x$, that is, $\alpha_j(p) = 0$ when $x_j = 0$, we will set $p_{jt} = \infty$ if $x_j = 0$.

While $G_t(x, p_t)$ in (36) is not a concave function of $p_t$, we can show that it is strictly quasiconcave in $p_t$ using the determinants of its bordered Hessian matrix. Therefore, by Theorem 1, the optimal
prices $p_t(x)$ must satisfy (8). To derive $h(p_t) = -\alpha(p_t) \left( \frac{\partial \alpha(p_t)}{\partial p_t} \right)^{-1}$, we first note that, when the distribution $F$ of $\theta$ is degenerate, the elements of Jacobian matrix of $\alpha(p_t)$ in (13) reduce to

$$
\frac{\partial \alpha_i(p_t)}{\partial p_{jt}} = \begin{cases} 
\frac{e^{(\theta q_i - p_{jt})/\mu}}{\mu (1 + \sum_k e^{(\theta q_k - p_{kt})/\mu})}, & i = j, \\
\frac{e^{(\theta q_i - p_{jt})/\mu} \sum_k e^{(\theta q_k - p_{kt})/\mu}}{\mu (1 + \sum_k e^{(\theta q_k - p_{kt})/\mu})^2}, & i \neq j.
\end{cases}
$$

The inverse of this Jacobian matrix is:

$$(\frac{\partial \alpha(p_t)}{\partial p_t})^{-1} = -\mu \left( 1 + \sum_k e^{\theta q_k - p_{kt}/\mu} \right) \begin{bmatrix} 1 & \cdots & 1 \\
\frac{1 + e(\theta q_1 - p_{1t})/\mu}{e^{\theta q_1 - p_{1t}/\mu}} & \cdots & \frac{1}{e^{\theta q_1 - p_{1t}/\mu}} \\
\vdots & \ddots & \vdots \\
\frac{1}{e^{\theta q_{nt} - p_{nt}/\mu}} & \cdots & \frac{1 + e(\theta q_{nt} - p_{nt})/\mu}{e^{\theta q_{nt} - p_{nt}/\mu}} \\
\end{bmatrix}.$$  

Substituting the choice probabilities and the inverse of the Jacobian matrix into $h_j(p_t)$ yields

$$h_j(p_t) = \mu \left( 1 + \sum_k e^{\theta q_k - p_{kt}/\mu} \right) \left( 1 + \sum_k e^{\theta q_k - p_{kt}/\mu} \right) \left( \frac{1}{e^{\theta q_j - p_{jt}/\mu}} \right) = \mu \left( 1 + \sum_k e^{\theta q_k - p_{kt}/\mu} \right), \quad j = 1, 2, \ldots, n. \quad (37)$$

In (37), $h_j(p_t)$ is identical for all $j$, meaning the current marginal values of all products are the same. This result is consistent with earlier research (Anderson et al. 1992) that reports the optimality of the uniform pricing scheme for a horizontal product assortment for a single period model. Using (37) and applying Theorem 1, we obtain:

$$p_{jt}(x) = \mu \left( 1 + \sum_k e^{\theta q_k - p_{kt}(x)/\mu} \right) + \Delta x_j V_{t-1}(x). \quad (38)$$

When the firm has surplus inventory of product $j$ ($x_j \geq t$), the future value of a surplus unit is zero, that is, $\Delta x_j V_{t-1}(x) = 0$, if $x_j - t \geq 0$. Consequently, from (38), all products that have surplus inventories should be priced the same. However, such a uniform pricing scheme is not optimal for products that have inventory shortfalls ($x_j < t$). From (38), we see that any product with an inventory shortfall, regardless of its attribute rating $q_j$, commands a higher price than the uniform price set for the products with inventory surplus. In other words, inventory shortfall of a product
translates into a premium charged over the uniform price of products with surplus inventories in the assortment, regardless of their respective attribute ratings. This result helps us gain a better understanding of what drives price differences among products in an assortment. In practice, we observe that variants in a product category are offered at different prices. The static, single period pricing models in the literature have failed to explain these differences, concluding instead that the products within a category should be offered at a similar price, despite differences in attribute ratings. Our result shows that the key driver of price differentiation in a horizontal assortment is individual inventory availability, rather than the attribute rating, of the product. We formally state this result in the following theorem.

**Theorem 6** For the H model,

(a) prices of all products with surplus inventories are the same, i.e., \( p_{jt}(x) = p_{kt}(x) \) if \( x_j, x_k \geq t \);

(b) price of a product with inventory shortfall is always higher than that of a product with inventory surplus, i.e., \( p_{i}(x) > p_{jt}(x) \) if \( x_i < t \) and \( x_j \geq t \).

The properties stated above are in sharp contrast with the pricing structure in the V model, where due to the universal ordering of product quality, a product with a higher rating is always priced higher than a product with a lower rating, regardless of their respective inventories. Moreover, the notion of inventory surplus in the H model must be defined at each individual inventory level; in contrast, due to the universal ordering of quality in the V model, the inventory surplus is defined at the aggregate level. Consequently, the optimal prices in the H model are driven by individual inventory levels of products, rather than the aggregate inventory level as in the V model.

Note that the results stated in Theorem 6 do not hold in general for the mixed logit model, where \( \theta \) follows a general distribution. Berry and Pakes (2007) show that the mixed logit model converges to the pure characteristic demand model as \( \mu \) approaches zero. This implies that the MPDP model with linear random utility and uniformly distributed \( \theta \) converges to the V model as \( \mu \to 0 \). From our result in the V model, we know that differentiated pricing \( q_j/2 \) (as opposed to uniform pricing) is optimal for the product with inventory surplus. More generally, from (14), when a product has inventory surplus, its price should be set at the current marginal value, i.e,
\[ p_{jt}(x) = \frac{-\sum_{i=1}^{n} \alpha_i (p_{it}) c_{ij}}{\sum_{i=1}^{n} \partial \alpha_i (p_{jt}) c_{ij}}. \] Observe that this marginal value is uniform across different products only when it is independent of \( j \).

6. Numerical Results

6.1. Behavior of Optimal Prices of the \( V \) Model

First, we illustrate the three monotonicity properties—inventory, time, and quality—of the optimal prices as discussed in Theorem 3. We then examine the behavior of the price difference between adjacent products \( p_{jt}(x) - p_{j+1,t}(x) \) and demonstrate that it depends only on \( |x|_j \).

We select the parameters in the numerical example as follows. We consider a firm offering customers (\( \lambda_t = \lambda = 0.8 \)) three vertically differentiated products, with corresponding quality ratings of \( q_1 = 10, q_2 = 6, \) and \( q_3 = 2 \). Figure 1 shows the three optimal prices as a function of the inventory level of product 1 in period \( t = 40 \), with the inventory levels of products 2 and 3 fixed as \( x_2 = x_3 = 5 \). Figure 2 depicts the optimal prices as a function of the remaining time, at the fixed inventory levels \( x_1 = x_2 = x_3 = 5 \). Figures 1 and 2 illustrate the quality, time and inventory monotonicity properties that we established in Theorem 3. Further, as \( x_1 \) increases (Figure 1) or \( t \) decreases (Figure 2), the price of product \( j \) converges to \( \frac{q_j}{2}, j = 1, 2, 3 \), as established in Theorem 2(a).

![Figure 1](prices_model_V_x_1.png)  ![Figure 2](prices_model_V_t_40.png)

**Figure 1**  Prices for Model \( V \) as a function of \( x_1 \) when \( x_2 = x_3 = 5 \) and \( t = 40 

**Figure 2**  Prices for Model \( V \) as a function of \( t \) when \( x_1 = x_2 = x_3 = 5 

Next we use the same example to illustrate Theorem 2 (d), which states that the price difference between any two adjacent products \( i \) and \( j \) with positive inventories depends only on \( |x|_i \). In our example, it means that the difference \( p_{1t}(x) - p_{2t}(x) \) depends only on \( x_1 \), and \( p_{2t}(x) - p_{3t}(x) \)
depends only on $x_1 + x_2$. Figures 3 and 4 illustrate that for a fixed $x_1$, $p_{1t}(x) - p_{2t}(x)$ is a constant, and for a fixed $x_1 + x_2$, $p_{2t}(x) - p_{3t}(x)$ is a constant.

We also conducted numerical tests for different distributions of $\theta$, including Triangular and Beta distributions. Results (reported in EC.4) showed that the inventory aggregation and monotonicity (quality, time and inventory) properties of the $V$ model still hold, demonstrating the robustness of our results for the $V$ model.

6.2. Behavior of Optimal Prices of the $H$ Model

Suppose that the firm in the previous example offers a horizontally differentiated assortment. We retain the attribute values $q_j$ that we specified earlier (noting that the attribute may refer to popularity rather than quality), to ensure an appropriate comparison with the model $V$ example. We let $\theta = 0.5$, corresponding to the mean customer sensitivity to the attribute value. In this example, we vary the factor $\mu$, which reflects the degree of horizontal differentiation among products, to understand its impact. Figures 5 and 6, the counterparts of Figures 1 and 2 for the $V$ model, respectively, illustrate how the optimal prices vary with inventory, attribute value and time. These figures demonstrate that the optimal prices for the $H$ model do not necessarily possess attribute rating monotonicity (e.g., in Figure 5, for $\mu = 1.5$, $p_{2t}(x)$ starts to dominate $p_{1t}(x)$ when $x_1$ becomes larger), inventory monotonicity (e.g., in Figure 5, $p_{2t}(x)$ first decreases and then increases in $x_1$) and time monotonicity (e.g., in Figure 6, the prices for both products 2 and 3 first decrease and then increase in $t$). Our computational experience suggests that the non-monotonic behavior of prices
with respect to time and inventory is due to the different effects that variable changes have on the
two components of the optimal price—the current value \( h_j(p_t) \) and the future value \( \Delta x_j V_{t-1}(x) \).
Specifically, when time or inventory varies, one of the two terms may increase, while the other
decreases. As a result, the sum of the two terms can have non-monotone behavior depending on
the relative magnitude of each term. In understanding the non-monotonic behavior with respect
to attribute rating, first observe that the current value \( h_j(p_t) \) is uniform across products and the
differences in prices arise from the future marginal value \( \Delta x_j V_{t-1}(x) \) of product inventories. Now,
suppose the higher quality product has a surplus and the lower quality product a shortfall. Then
the future marginal value of the higher quality product is zero whereas the lower quality prod-
uct has a positive future marginal value. Such scenarios can lead to non-monotonic behavior in
attribute rating.

Furthermore, the optimal prices of all three products, at each setting of \( \mu \), converge to the same
value, meaning that the uniform pricing policy starts to take effect when product inventory is
abundant relative to demand. Figures 5 and 6 show sharp contrasts with Figures 1 and 2, in which
the pattern of price differentiation persists regardless of the inventory levels and time remaining,
and the optimal prices always mirror their attribute values at all times. In both Figures 5 and 6,
we observe that a higher value of \( \mu \), which signifies a greater degree of horizontal differentation,
allows the firm to charge higher prices. Higher values of \( \mu \) result in a larger dispersion in consumer
valuations, leading to an increased willingness to pay by consumers.

![Figure 5](image1.png)  
**Figure 5** Prices for Model H as a function of \( x_1 \) when \( x_2 = x_3 = 5, \ t = 40, \ \mu = 1.0 \) or 1.5

![Figure 6](image2.png)  
**Figure 6** Prices for Model H as a function of \( t \) when \( x_1 = x_2 = x_3 = 5, \ \mu = 1.0 \) or 1.5
Next, we illustrate Theorem 6 using another three-product $H$ model with $q_1 = 3$, $q_2 = 2$, $q_3 = 1$, $\mu = 1$, $\theta = 0.5$, and $\lambda_t = 0.8$. Figure 7 depicts the change in product prices over time when $x_1 = 8$, $x_2 = 5$, and $x_3 = 2$. Note that, in region I ($t = 1, 2$), since all three products have surplus inventories, $p_{1t}(x) = p_{2t}(x) = p_{3t}(x)$. In region II ($t = 3, 4, 5$), product 1 and 2 have surplus inventories, whereas product 3 has an inventory shortfall. Hence, $p_{1t}(x) = p_{2t}(x) < p_{3t}(x)$, although product 3 has the lowest attribute rating. In region III ($t = 6, 7, 8$), product 1 has surplus and products 2 and 3 have shortfalls and we have $p_{1t}(x) < p_{2t}(x)$ and $p_{1t}(x) < p_{3t}(x)$. Finally, in region IV ($t > 9$), all three products have shortfall inventories, and hence the firm charges different prices.

7. Conclusions

Motivated by applications in industry we study the dynamic pricing problem of a firm that sells given initial inventories of multiple perishable products over a finite selling season. Using an integrative linear random utility framework, we present a detailed analysis of the structural properties of the MPDP models for both vertically and horizontally differentiated products.

The results in this paper have the following important managerial implications.

(1) Vertically and horizontally differentiated products have fundamentally different pricing policy structures. These differences suggest that managers need to select a consumer choice model that is compatible with the specific nature of product differentiation in their applications. The profitability of a firm may be significantly compromised if an inappropriate consumer
choice model is used in making pricing decisions.

(2) When products can be universally ordered based on their attributes, managers can charge premiums for products ranked higher by consumers, for products with scarce inventories, and for all products as the end of the season approaches. The optimal price of a product depends on higher quality product inventories only through the aggregate inventory level rather than their individual availabilities. Additionally, this aggregate inventory solely determines the product’s markup over an adjacent lower quality product in the assortment. Our exact algorithm allows managers to readily apply the insights and analysis in this paper to practical-sized problems.

(3) If consumer preferences for products are dispersed, the optimal pricing policy is driven by individual product availability. Regardless of product attribute ratings, managers should select a uniform price for products with surplus inventories and charge premiums for those with inventory shortfalls. However, in general, lower inventory availability and longer time-to-sell do not necessarily permit the firm to charge a higher price.

By establishing the importance of incorporating product differentiation in dynamic pricing decisions, our work explores an interesting and novel dimension of the revenue management problem and offers opportunities for interesting future work. For instance, developing models to incorporate firm’s initial inventory choices, alternative distributions of customer sensitivities, multi-product budget and resource constraints, and strategic consumers (who may anticipate the firm’s pricing strategy), in this framework would greatly facilitate the deployment of these models in practice. Exploring industry-specific considerations such as multi-day stays (hospitality) and group purchases (travel) would also greatly enrich the model.

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References

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Proofs of Statements

This appendix contains the proofs of the theorems and lemmas given in the paper.

**EC.1. Properties of the optimal single product price**

When the firm offers only a single product, consumers must choose between purchasing this product and the outside option. In this case, optimality equation (4) simplifies to

\[ \Delta_t V_t(x) = \max_{p_t} \{ G_t(x, p_t) \} , \]

where

\[ G_t(x, p_t) = \lambda_t \alpha(p_t) (p_t - \Delta_x V_{t-1}(x)) , x > 0. \]

This single product dynamic pricing problem has been studied by Zhang and Cooper (2007) and Zhao and Zheng (2000) in a continuous-time framework. For this special case of the MPDP-G problem, we can show the results in the following theorem.

**Theorem EC1** For any inventory \( x > 0 \), let \( G_t(x, p_t) \) be a strictly quasiconcave function of \( p_t \), and let \( p_t(x) \) be the optimal price in period \( t \). Then

(a) \( \Delta_x V_t(x) \) and \( p_t(x) \) are nondecreasing functions of \( t \).

(b) \( \Delta_x V_t(x) \) and \( p_t(x) \) are nonincreasing functions of \( x \).

(c) If \( \lambda_t \geq \lambda_{t-1} \), then \( \Delta_x V_t(x) \) is a non-increasing function of \( t \)

Theorem EC1 implies that \( V_t(x) \) is a supermodular function of \( t \) and \( x \), a concave function of \( x \), and a concave function of \( t \) if \( \lambda_t \) is non-increasing in \( t \). Zhang and Cooper (2007) report the inventory monotonicity property and for a continuous-time single product model, Zhao and Zheng (2000) show that both part (a) (time monotonicity) and part (b) (inventory monotonicity) of Theorem EC1 hold under certain conditions. Since Theorems 1 and Theorem EC1 both require quasiconcavity of \( G_t(x, p_t) \), we next identify conditions on the choice probabilities that are sufficient to ensure this property in our discrete-time model.

**Lemma EC2** If \( \alpha(p_t) \) is decreasing and differentiable in \( p \), and \( h(p) = -\frac{\alpha'(p)}{\alpha(p)} \) is a non-increasing function of \( p \), then \( G_t(x, p) \) is a strictly quasiconcave function of \( p \) for any \( t \geq 1 \) and \( x > 0 \).
Proof. The first derivative of $G_t(x, p_t)$ with respect to $p_t$ is

$$G'_t(x, p_t) = \lambda_t \alpha'(p_t)(p_t - \Delta_x V_{t-1}(x)) + \lambda \alpha(p_t)$$

$$= \lambda_t \alpha'(p_t) \left( p_t + \frac{\alpha(p_t)}{\alpha'(p_t)} - \Delta_x V_{t-1}(x) \right) = \lambda_t \alpha'(p_t) (p_t - h(p_t) - \Delta_x V_{t-1}(x)) \right). \quad (EC.1)$$

We know that $p_t - h(p_t)$ is a strictly increasing function of $p_t$. Therefore, there exists a unique value, $p_t(x)$, such that

$$p_t(x) = h(p_t(x)) + \Delta_x V_{t-1}(x). \quad (EC.2)$$

In addition, $\alpha'(p_t) < 0$ since $\alpha(p_t)$ is strictly decreasing in $p_t$. Therefore, $G'_t(x, p_t) > 0$ for $p_t > p_t(x)$ and $G'_t(x, p_t) < 0$ for $p_t > p_t(x)$. In other words, $G_t(x, p_t)$ strictly increases in $p_t$ for $p_t < p_t(x)$ and strictly decreases in $p_t$ for $p_t > p_t(x)$. Therefore, $G_t(x, p_t)$ is strictly quasi-concave and $p_t$ given in (EC.2) maximizes $G_t(x, p_t)$.

Remark. The conditions in Lemma EC2 are satisfied by many common discrete choice models, e.g. probit and binary logit models.

EC.2. Proof of Theorem 2

We prove parts (a)–(d) simultaneously, using induction on $t$. We first establish the base induction arguments of parts (a)–(d) with $t = 1$. If $|x|_j \geq t = 1$, we know from (24) that

$$p_{k,1}(x) = \frac{1}{2} (q_k + \Delta_{x_k} V_0(x)), \quad k = 1, 2, \ldots, n. \quad (EC.3)$$

Since $\Delta_{x_k} V_0(x) = 0$, $p_{k,1}(x) = \frac{q_k}{2}$, as long as $x_k > 0$ (otherwise the product is not offered). This proves part (a) with $t = 1$. For $t = 1$, the condition for part (b) is automatically satisfied as long as $|x|_j \geq 1$, so part (b) holds trivially, in viewing of (EC.3). Finally, parts (c) and (d) hold trivially since, with one period remaining, the optimal price of any product is held at a constant.

We now show, in turn, parts (a)–(d) for period $t + 1$, based on the hypotheses that all four parts are true for $t$.

Proof of (a). Suppose we have sufficient inventory of the first $j$ types of products to meet the demand during the remaining $t + 1$ periods, i.e., $|x|_j \geq t + 1$. Then, since $|x|_j > |x|_j - 1 \geq t$, our
hypothesis for part (b) states that, for any \( k \geq j \), \( V_t(x - e_k) = V_t(x) \). This further implies \( \Delta_{x_k} V_t(x) = 0 \), \( k \geq j \). Therefore \( p_{k,t+1}(x) = \frac{1}{2}(q_k + \Delta_{x_k} V_t(x)) = \frac{q_k}{2} \), \( k \geq j \). The choice probability of product \( k \), \( k > j \), is given by

\[
\alpha_k(p_t(x)) = \frac{p_{k-1,t}(x) - p_{k,t}(x)}{q_k - q_k} = \frac{q_k - \frac{q_k}{2}}{q_k - q_k} = 0.
\]

This means that as long as we have sufficient inventory from the higher quality products to meet demand, we should set the prices of the lower quality products to the levels so that they will not be chosen by any customer.

**Proof of (b).** We first show that the pricing decisions with initial states \( x \) and \( x + e_k \) are the same in period \( t + 1 \), if \( |x|_j \geq t + 1 \). By our hypothesis for part (b),

\[
\Delta_{x_i} V_t(x) = V_t(x) - V_t(x - e_i) = V_t(x + e_k) - V_t(x - e_i + e_k) = \Delta_{x_i} V_t(x + e_k), \quad i \leq j \leq k.
\]

Therefore,

\[
p_{i,t+1}(x) = \frac{1}{2}(q_i + \Delta_{x_i} V_t(x)) = \frac{1}{2}(q_i + \Delta_{x_i} V_t(x + e_k)) = p_{i,t+1}(x + e_k), \quad i \leq j \leq k.
\]

Thus, we set the same prices for the first \( j \) products in states \( x \) and \( x + e_k \) in period \( t + 1 \). From part (a), we also set the same prices for the next \( n - j \) products in states \( x \) and \( x + e_k \) in period \( t + 1 \), i.e., \( p_{i,t+1}(x) = p_{i,t+1}(x + e_k) = \frac{q_i}{2} \), \( j \leq i \). Next, we argue that the pricing policies for the systems with initial states \( x \) and \( x + e_k \) should be the same from the next period onwards. Note that once the condition \( |x|_j \geq t + 1 \) is satisfied, then, since there is at most one request in each period, we always have sufficient inventory from the first \( j \) products to meet remaining demand from period \( t \) onwards. Therefore, by our hypotheses for part (b), each system will set the same price for each product in each of the remaining \( t + 1 \) periods. This, of course, implies \( V_{t+1}(x) = V_{t+1}(x + e_k) \).

**Proof of (c).** We need to show, for any \( i < i' \leq j \),

\[
p_{j,t+1}(x) = \frac{1}{2}(q_j + \Delta_{x_j} V_t(x)) = \frac{1}{2}(q_j + \Delta_{x_j} V_t(x - e_i + e_{i'})) = p_{j,t+1}(x - e_i + e_{i'}).
\]
Recursively using the above expression leads to (27). Without loss of generality, let \(|x|_j < t + 1\), otherwise the result holds trivially, due to part (b). It is sufficient to show, based on our hypothesis for part (c), for \(i < i' \leq j\),

\[
\Delta_{x_j} V_i(x) = V_i(x) - V_i(x - e_j)
\]

\[
= V_i(x - e_i + e_{i'}) - V_i(x - e_i + e_{i'} - e_j) = \Delta_{x_j} V_i(x - e_i + e_{i'}) ,
\]  

(EC.4)

or, equivalently,

\[
V_i(x) - V_i(x - e_i + e_{i'}) = V_i(x - e_i) - V_i(x - e_i + e_{i'} - e_j) ,
\]  

(EC.5)

The above equation, in essence, states that function \(V_i(x) - V_i(x - e_i + e_{i'})\) is independent of \((x_j, \ldots, x_n)\) for any \(j \geq i' > i\). For notation simplicity, denote the four states, \(x, x - e_i, x - e_j,\) and \(x - e_i + e_{i'} - e_j\) by \(x^1, x^2, x^3\) and \(x^4\), respectively. Note that \(x^1 - e_j = x^3\) and \(x^2 - e_j = x^4\).

Then we can write (EC.5) as

\[
V_i(x^1) - V_i(x^2) = V_i(x^3) - V_i(x^4).
\]  

(EC.6)

To prove (EC.6), we use the expression derived in (25). Given \(x^i, i = 1, 2, 3\) and 4, we have

\[
V_i(x^i) = \lambda_i \left( q_1 - 2p_{1i}(x^i) + \sum_{k=1}^{n-1} \frac{(p_{k-1i}(x^i) - p_{k+1i}(x^i))^2}{q_k - q_{k+1}} + \frac{(p_{n+1i}(x^i))^2}{q_n} \right) + V_{i-1}(x^i),
\]

Consider the difference of \(V_i(x^1)\) and \(V_i(x^2)\). By our hypothesis for (EC.5), for \(k \geq j \geq i' > i\),

\[
\Delta_{x_k} V_{i-1}(x^1) = V_{i-1}(x) - V_{i-1}(x - e_k)
\]

\[
= V_{i-1}(x - e_i + e_{i'}) - V_{i-1}(x - e_i + e_{i'} - e_k) = \Delta_{x_k} V_{i-1}(x^2).
\]

This further implies that

\[
p_{k1t}(x^1) = p_{k1t}(x) = p_{k1t}(x - e_i + e_{i'}) = p_{k1t}(x^2), \quad k \geq j \geq i' > i.
\]

Therefore, the difference of \(V_i(x^1)\) and \(V_i(x^2)\) is reduced to

\[
V_i(x^1) - V_i(x^2) = -2\lambda_i (p_{1i}(x^1) - p_{1i}(x^2)) + (V_{i-1}(x^1) - V_{i-1}(x^2))
\]

By our hypothesis for (EC.5), for \(k \geq j \geq i' > i\),

\[
\Delta_{x_k} V_{i-1}(x^1) = V_{i-1}(x) - V_{i-1}(x - e_k)
\]

\[
= V_{i-1}(x - e_i + e_{i'}) - V_{i-1}(x - e_i + e_{i'} - e_k) = \Delta_{x_k} V_{i-1}(x^2).
\]

This further implies that

\[
p_{k1t}(x^1) = p_{k1t}(x) = p_{k1t}(x - e_i + e_{i'}) = p_{k1t}(x^2), \quad k \geq j \geq i' > i.
\]

Therefore, the difference of \(V_i(x^1)\) and \(V_i(x^2)\) is reduced to

\[
V_i(x^1) - V_i(x^2) = -2\lambda_i (p_{1i}(x^1) - p_{1i}(x^2)) + (V_{i-1}(x^1) - V_{i-1}(x^2))
\]

By our hypothesis for (EC.5), for \(k \geq j \geq i' > i\),

\[
\Delta_{x_k} V_{i-1}(x^1) = V_{i-1}(x) - V_{i-1}(x - e_k)
\]

\[
= V_{i-1}(x - e_i + e_{i'}) - V_{i-1}(x - e_i + e_{i'} - e_k) = \Delta_{x_k} V_{i-1}(x^2).
\]

This further implies that

\[
p_{k1t}(x^1) = p_{k1t}(x) = p_{k1t}(x - e_i + e_{i'}) = p_{k1t}(x^2), \quad k \geq j \geq i' > i.
\]

Therefore, the difference of \(V_i(x^1)\) and \(V_i(x^2)\) is reduced to

\[
V_i(x^1) - V_i(x^2) = -2\lambda_i (p_{1i}(x^1) - p_{1i}(x^2)) + (V_{i-1}(x^1) - V_{i-1}(x^2))
\]
\[ V_t(x^1) - V_t(x^2) = \lambda_t \sum_{k=1}^{j-1} (p_{kt}(x^1) - p_{k+1,t}(x^1))^2 - (p_{kt}(x^2) - p_{k+1,t}(x^2))^2 \]

(EC.7)

We now apply our hypothesis for part (c) to each term in (EC.7). The first term becomes

\[ p_{1t}(x^1) - p_{1t}(x^2) = \frac{1}{2} (\Delta x_1 V_{t-1}(x^1) - \Delta x_1 V_{t-1}(x^2)) \]

\[ = \frac{1}{2} (V_{t-1}(x^1) - V_{t-1}(x^1 - e_1) - V_{t-1}(x^2) + V_{t-1}(x^2 - e_1)) \]

\[ = \frac{1}{2} (V_{t-1}(x^1) - V_{t-1}(x^1 - e_1 - e_j) - V_{t-1}(x^2 - e_1) + V_{t-1}(x^2 - e_1 - e_j)) \]

\[ = \frac{1}{2} (V_{t-1}(x^1) - V_{t-1}(x^3 - e_1) - V_{t-1}(x^4) + V_{t-1}(x^4 - e_1)) \]

\[ = \frac{1}{2} (\Delta x_1 V_{t-1}(x^2) - \Delta x_1 V_{t-1}(x^3)) = p_{1t}(x^3) - p_{1t}(x^4), \]

where the third equality uses the hypothesis that functions \( V_{t-1}(x^1) - V_{t-1}(x^2) \) and \( V_{t-1}(x^1 - e_1) - V_{t-1}(x^2 - e_1) \) are independent of \( x_j \), and the fourth equality uses the fact that \( x^1 - e_j = x^3 \) and \( x^2 - e_j = x^4 \). Now, applying the hypothesis for (EC.6) to the second term in (EC.7) yields

\[ V_{t-1}(x^1) - V_{t-1}(x^2) = V_{t-1}(x^3) - V_{t-1}(x^4). \]

Next, consider each expression in the third summation term of (EC.7). For any \( k < j \), we have

\[ p_{kt}(x^1) - p_{k+1,t}(x^1) = \frac{1}{2} (q_k - q_{k+1} + V_{t-1}(x^1 - e_{k+1}) - V_{t-1}(x^1 - e_k)) \]

\[ = \frac{1}{2} (q_k - q_{k+1} + V_{t-1}(x^1 - e_{k+1} - e_j) - V_{t-1}(x^1 - e_k - e_j)) \]

\[ = \frac{1}{2} (q_k - q_{k+1} + V_{t-1}(x^3 - e_{k+1}) - V_{t-1}(x^3 - e_k)) \]

\[ = p_{kt}(x^3) - p_{k+1,t}(x^3), \]

where the second equality uses the hypothesis, for \( k < k+1 \leq j \), the term \( V_{t-1}(x^1 - e_{k+1}) - V_{t-1}(x^1 - e_k) \) is independent of \( x_j \), and the third equality uses the fact \( x^1 - e_j = x^3 \). Similarly,

\[ p_{kt}(x^2) - p_{k+1,t}(x^2) = p_{kt}(x^4) - p_{k+1,t}(x^4). \]

Substituting the above identities to the terms in (EC.7), we obtain

\[ V_t(x^1) - V_t(x^2) = -2\lambda_t (p_{1t}(x^3) - p_{1t}(x^4)) + (V_{t-1}(x^3) - V_{t-1}(x^4)) \]
\[ + \lambda_t \sum_{k=1}^{t-1} \frac{(p_{kt}(x^k) - p_{k,t+1}(x^k))^2 - (p_{kt}(x^t) - p_{k,t+1}(x^t))^2}{q_k - q_{k+1}} \]
\[ = V_t(x^t) - V_t(x^t). \]

This establishes (EC.6) and also completes the proof of part (c).

**Proof of (d).** This part, in fact, is a direct consequence of (EC.4), which states that the difference function \( V_{t-1}(x - e_i) - V_{t-1}(x - e_j) \) depends only on \((x_1, x_2, \ldots, x_i)\). Since

\[ p_{it}(x) - p_{jt}(x) = \frac{1}{2}(q_i - q_j + V_{t-1}(x - e_j) - V_{t-1}(x - e_i)), \]

the price difference \( p_{it}(x) - p_{jt}(x) \) also depends only on \((x_1, \ldots, x_i)\). However, by part (c), price \( p_{it}(x) \) depends on \((x_1, \ldots, x_i)\) only through their sum \(|x|_i\); similarly, \( p_{jt}(x) \) depends on \((x_1, \ldots, x_j)\) only through their sum \(|x|_j\). This implies that the price difference of two adjacent products, \( i \) and \( j \), depends only on \(|x|_i\). \( \blacksquare \)

**EC.3. Proof of Theorem 3**

**Proof of (a)–(c).** The proof is by induction on \( t + |x|_n = \ell \). The initial step of induction for \( \ell = 1 \) is trivially true. Next we prove parts (a)–(c) simultaneously, assuming \( t + |x|_n = \ell + 1 \). To facilitate further analysis, we state the following preliminary results first, which follow from the general chain rule of derivatives for the composition of multi-variable functions.

**Properties of composition of multivariate functions.** Let \( g : R^n \rightarrow R \) and \( f_j : R^n \rightarrow R \), for \( j = 1, 2, \ldots, n \). Define \( g \circ f(x) = g(f_1(x), f_2(x), \ldots, f_n(x)) \), with \( x = (x_1, x_2, \ldots, x_n) \).

1. If \( g \) is non-increasing in each of its arguments and \( f_j \) is non-decreasing in each of its arguments, then the composite function \( g \circ f \) is non-increasing in each of its arguments.

2. If \( g \) is non-increasing in each of its arguments and \( f_j \) is non-decreasing in each of its arguments, then the composite function \( g \circ f \) is non-decreasing in each of its arguments.

**Proof of (a).** The statement holds for \( t = 0 \) for any \( x \), so assume \( t > 1 \). Let \( t + 1 + |x|_n = \ell + 1 \).

We now need to show \( \Delta_t V_{t+1}(x + e_j) \geq \Delta_t V_{t+1}(x), \) for \( j = 1, 2, \ldots, n \). Let

\[ g(p_t(x)) = \lambda_t \left( q_t - 2p_{it}(x) + \frac{\sum_{k=1}^{t-1} (p_{kt}(x) - p_{k,t+1}(x))^2}{q_k - q_{k+1}} + \frac{p_{it}^2(x)}{q_t} \right). \]  

(EC.8)
Then $\Delta_t V_{t+1}(x_j) = V_{t+1}(x) - V_t(x) = g(p_{t+1}(x))$. It can be seen from the following derivations that

$\frac{\partial g(p_t(x))}{\partial p_{tt}(x)} = \lambda_t \left( -2 + \frac{2(p_{tt}(x) - p_{2t}(x))}{q_1 - q_2} \right) = -2\lambda_t \left( 1 - \frac{(p_{tt}(x) - p_{2t}(x))}{q_1 - q_2} \right) \leq 0$, \hspace{1cm} (EC.9)

$\frac{\partial g(p_t(x))}{\partial p_{kt}(x)} = \lambda_t \left( -2(p_{k-1,t}(x) - p_{kt}(x)) \frac{q_{k-1} - q_k}{q_k - q_{k+1}} + \frac{2(p_{kt}(x) - p_{k+1,t}(x))}{q_k - q_{k+1}} \right) \leq 0, \hspace{1cm} k = 2, \ldots, n$, \hspace{1cm} (EC.10)

$\frac{\partial g(p_t(x))}{\partial p_{nt}(x)} = \lambda_t \left( -2(p_{n-1,t}(x) - p_{nt}(x)) \frac{q_{n-1} - q_n}{q_n(q_{n-1} - q_n)} \right) \leq 0$, \hspace{1cm} (EC.11)

where the nonnegativity of each of the above expressions is the result of $p_t(x) \in \tilde{P}_x$, which is defined in (18). Therefore, we need to show that the composite function $g(p_{t+1}(x))$ is increasing in $x_j, j = 1, 2, \ldots, n$. Based on (EC.9)–(EC.11), we know that $g(p_t(x))$ is a decreasing function of each of its arguments. By the properties of the composition of the multivariate functions stated at the beginning of the proof, $g(p_{t+1}(x))$ will be increasing in $x_j$ if $p_{k,t+1}(x)$ is decreasing in $x_j$ for all $k$, i.e., $p_{k,t+1}(x + e_j) \leq p_{k,t+1}(x)$ for $k, j = 1, 2, \ldots, n$. Also, $p_{k,t+1}(x) = \frac{1}{2} (q_k + \Delta_{x_k} V_t(x))$ for $k, j = 1, 2, \ldots, n$. Now, by our hypotheses for parts (b) and (c) of the theorem, $\Delta_{x_k} V_t(x)$ is decreasing in $x_j$. It follows that $p_{k,t}(x)$, a linear transformation of $\Delta_{x_k} V_t(x)$, is decreasing in $x_j$ for all $k$.

**Proof of (b).** We need to show, for $t + |x|_n = t + 1$,

$$\Delta_{x_j} V_t(x) \geq \Delta_{x_j} V_t(x + e_i), \hspace{1cm} i \neq j, \hspace{1cm} i, j = 1, \ldots, n.$$ \hspace{1cm} (EC.12)

Because a submodular function is symmetric with respect to its two arguments, we assume, without loss of generality, that $i < j$. From Theorem 2 (c), $\Delta_{x_j} V_t(x)$ depends on the values of the first $j$ elements only through their aggregate value. This means

$$\Delta_{x_j} V_t(x) = \Delta_{x_j} V_t(0, \ldots, 0, |x|_j, x_{j+1}, \ldots, x_n),$$ \hspace{1cm} (EC.13)

$$\Delta_{x_j} V_t(x + e_i) = \Delta_{x_j} V_t(0, \ldots, 0, |x|_j + 1, x_{j+1}, \ldots, x_n).$$ \hspace{1cm} (EC.14)

Therefore, (EC.12) is equivalent to stating that $V_t(0, \ldots, 0, |x|_j, x_{j+1}, \ldots, x_n)$ is a concave function of its $j^{\text{th}}$ element. This result will be established in the next part, part (c), of the theorem.
Proof of (c). We prove this result by induction on $n$, the number of products in the system. If $n = 1$, then by Theorem EC1 (b), $V_t(x)$ is a concave function of $x$ in a single-product system.

Suppose that in an $m$-product system, $1 \leq m < n$, the value function $V_t(x_1, \ldots, x_m)$ is concave in $x_j, j = 1, 2, \ldots, m$. Now consider the value function $V_t(x_1, \ldots, x_n)$ in the $n$-product system. We first show that

$$V_t(x) = V_t(x_1, \ldots, x_n)$$

is a concave function of $x_j$ for $j > 1$, i.e., for $t + |x|_n = l + 1$,

$$\Delta x_j V_t(x) \geq \Delta x_j V_t(x + e_j), \quad j > 1. \quad (EC.15)$$

Since, by Theorem 2 (c), $\Delta x_j V_t(x)$ depends on the inventories of the first $j$ products only through $|x|_j$, we obtain

$$\Delta x_j V_t(x) = \Delta x_j V_t(0, \ldots, 0, |x|_j, x_{j+1}, \ldots, x_n), \quad j > 1.$$

Similarly,

$$\Delta x_j V_t(x + e_j) = \Delta x_j V_t(0, \ldots, 0, |x|_j + 1, x_{j+1}, \ldots, x_n), \quad j > 1.$$

Since $j > 1$, this reduces the problem to the system with $n - j + 1 < n$ products. By our hypothesis, the value function for the system with less than $n$ products is concave in $x_j$. Therefore, (EC.15) holds with $j > 1$.

Next, we show (EC.15) with $j = 1$. If $x_1 \geq t$, i.e., the inventory of the highest quality product alone is sufficient to meet demand for the remaining periods, then (EC.15) holds trivially, due to part (a) of Theorem 2. On the other hand, if $x_2 = 0$, then the problem reduces to the $n - 1$ product model, and by our hypothesis, $V_t(x_1, 0, x_2, \ldots, x_n)$ is a concave function of $x_1$. Therefore, without loss of generality, we let $x_1 < t$ and $x_2 > 0$. We consider the following two cases.

Case 1. In this case, the total inventory of the first two products is sufficient to meet all remaining demand in states $x - e_1, x$ and $x + e_1$. By part (b) of Theorem 2, the price of product $k$ in all those three states should be set at a constant, $\frac{q_k}{2}$, for $k = 2, \ldots, n$. Note that (EC.15) can be rewritten as

$$V_t(x) - V_t(x + e_1) \geq V_t(x - e_1) - V_t(x). \quad (EC.16)$$
Since, by part (a) of the theorem, $V_{t-1}(x-e_1) - V_{t-1}(x) \geq V_t(x-e_1) - V_t(x)$, equation (EC.16) will follow if we can show

$$V_t(x) - V_t(x+e_1) \geq V_{t-1}(x-e_1) - V_{t-1}(x),$$

or, equivalently,

$$V_t(x) - V_{t-1}(x-e_1) \geq V_t(x+e_1) - V_{t-1}(x). \quad \text{(EC.17)}$$

We can write the LHS of (EC.17) as

$$V_t(x) - V_{t-1}(x-e_1) = \sum_{k=1}^{n} \lambda_k g_k(p_t(x)) + V_{t-1}(x) - V_{t-1}(x-e_1)$$

$$= g(p_t(x)) + \Delta_{x_1} V_{t-1}(x)$$

$$= g(p_t(x)) + 2p_{1t}(x) - q_1, \quad \text{(EC.18)}$$

where $g(p_t(x))$ is defined in (EC.8). By our hypothesis for part (c),

$$p_{1t}(x) \geq p_{1t}(x+e_1). \quad \text{(EC.19)}$$

Also, when $x_1 + x_2 > t$, we know from Theorem 2(b) that the prices of all other products, except product 1, remain constants when the inventory level changes from $x$ to $x+e_1$, i.e.,

$$p_{kt}(x) = p_{kt}(x+e_1) = \frac{q_k}{2}, \quad k = 2, \ldots, n. \quad \text{Now, from (EC.9), it is easily seen that}$$

$$\frac{\partial (g(p_t(x)) + 2p_{1t}(x) - q_1)}{\partial p_{1t}(x)} = \lambda_t \left( -2 + \frac{2(p_{1t}(x) - p_{2t}(x))}{q_1 - q_2} \right) \geq 2 \frac{\lambda_t (p_{1t}(x) - p_{2t}(x))}{q_1 - q_2} \geq 0,$$

that is, $g(p_t(x)) + 2p_{1t}(x) - q_1$ is increasing in $p_{1t}(x)$. When $x_1$ increases, $p_{1t}(x)$ decreases whereas $p_{kt}(x)$ remains a constant for $k = 2, \ldots, n$, because of the properties of the composition of the multivariate functions. Therefore, the composite function $g(p_t(x)) + 2p_{1t}(x) - q_1$ is decreasing in $x_1$.

**Case 2.** $x_1 + x_2 \leq t$: We can write

$$\Delta_{x_1} V_t(x) = V_t(x) - V_t(x-e_1 + e_2) + V_t(x-e_1 + e_2) - V_t(x-e_1)$$

$$= V_t(x) - V_t(x-e_1 + e_2) + \Delta_{x_2} V_t(x-e_1 + e_2).$$
By part (c) of Theorem 2, $\Delta_{x_2} V_i(x - e_1 + e_2) = \Delta_{x_2} V_i(0, x_1 - 1 + x_2 + 1, x_3, \ldots, x_n)$, which is a decreasing function of $x_1$, due to our hypothesis that the value function for the $n - 1$ product system is concave with each of its arguments. Therefore, it is sufficient to show $V_i(x) - V_i(x - e_1 + e_2)$ is decreasing in $x_1$. But from Theorem 2 (c), $V_i(x) - V_i(x - e_1 + e_2)$ is independent of $x_2$. This means that, for $x' = (x_1, x_2', x_3, \ldots, x_n)$, with $x_1 + x_2' > t$,

$$V_i(x) - V_i(x - e_1 + e_2) = V_i(x') - V_i(x' - e_1 + e_2)$$

$$= V_i(x') - V_i(x' - e_1 + e_2) + \Delta_{x_2} V_i(x' - e_1 + e_2)$$

$$= \Delta_{x_1} V_i(x'),$$

where the second equality uses the fact that, when there is sufficient inventory from the first two products to meet all demand in state $x' - e_1$, an extra unit of product 2 will not bring any more revenue, that is, $\Delta_{x_2} V_i(x' - e_1 + e_2) = 0$. However, we have shown, in Case 1, that $\Delta_{x_1} V_i(x')$ is decreasing in $x_1$ if $x_1 + x_2' > t$. Therefore, $\Delta_{x_1} V_i(x) = \Delta_{x_1} V_i(x') + \Delta_{x_2} V_i(x - e_1 + e_2)$ is decreasing in $x_1$. This completes the proof of Case 2.

**Proof of (d).** It can be shown easily that $\Delta_i V_i(x) \geq \Delta_i V_2(x)$, for any $|x|_n = 1$. Therefore, we assume part (d) is true when $t + |x|_n \leq \ell$. Next we need to show, for $t + |x|_n = \ell + 1$, that $\Delta_i V_i(x) \geq \Delta_i V_{t+1}(x)$. We have established in (25) that

$$\Delta_i V_i(x) = \lambda_i \left( q_1 - 2p_{i1}(x) + \sum_{k=1}^{n-1} \frac{(p_{ik}(x) - p_{ik+1}(x))^2}{q_k - q_{k+1}} + \frac{(p_{i\ell}(x))^2}{q_n} \right)$$

Similar to what we did in part (a), let $\Delta_i V_i(x) = g(p_i(x))$. Since we already assume that $\lambda_i$ is nonincreasing in $t$, $p_i(x)$ increasing in $t$ in each of its arguments, i.e., $p_{kt}(x) \leq p_{k,t+1}(x)$, $k = 1, 2, \ldots, n$, would guarantee that $\Delta_i V_i(x) = g(p_i(x))$ is decreasing in $t$ due the properties of the composition of the multivariate functions. We know that $p_{kt}(x) = \frac{1}{2}(q_k + \Delta_{x_k} V_{t-1}(x))$ for $k = 1, 2, \ldots, n$. Since $t - 1 + \sum_{k=1}^{n} x_k = \ell < \ell + 1$, by our hypothesis for part (a), $\Delta_{x_k} V_{t-1}(x)$ is increasing in $t$ for $k = 1, \ldots, n$. Therefore, $p_{kt}(x)$, a linear transformation of $\Delta_{x_k} V_{t-1}(x)$, must be increasing in $t$ for $k = 1, 2, \ldots, n$. This establishes part (d).
EC.4. Numerical Test for Robustness of the V Model

In this appendix, we illustrate that our results in Theorems 2 and 3 still hold even when \( \theta \) follows a distribution other than the uniform distribution. For this purpose, we consider a three product system and first assume that the scalar \( \theta \) follows a *Triangular distribution* between 0 and 1, with the following density function

\[
f(\theta) = \begin{cases} 
2\theta, & 0 \leq \theta \leq 1, \\
0, & \text{otherwise}
\end{cases}
\]

and then assume that \( \theta \) follows a *Beta distribution* between 0 and 1 (with parameters \( \alpha = 2 \) and \( \beta = 2 \)), with the following density function.

\[
f(\theta) = \begin{cases} 
6\theta(1-\theta), & 0 \leq \theta \leq 1, \\
0, & \text{otherwise}
\end{cases}
\]

For all other problem parameters, we select the same values chosen in our computations in Section 6.1. Figures EC.1 and EC.3 show the optimal prices of the three products as a function of the inventory level of product 1 in period \( t = 40 \), with the inventory levels of products 2 and 3 fixed as \( x_2 = x_3 = 5 \), when \( \theta \) follows a triangular and a Beta distribution, respectively. Note that the prices of all three products are non-increasing in \( x_1 \) in both cases. Similarly, we can also show that the prices of the three products are non-increasing in \( x_2 \) and \( x_3 \) (inventory monotonicity). In addition, the optimal prices are rank ordered according to their qualities (quality monotonicity).

Figures EC.2 and EC.4 show the optimal dynamic prices as a function of the remaining time for both distributions. Fixing the inventory levels at \( x_1 = x_2 = x_3 = 5 \), we observe that, as the sales deadline approaches, the firm has to reduce the prices of all products in order to expedite sales (time monotonicity).

We next illustrate Theorem 5. In our examples, this theorem implies that the price difference between products 1 and 2, \( p_{1t}(x) - p_{2t}(x) \), depends only on \( x_1 \), and the price difference of products 2 and 3, \( p_{2t}(x) - p_{3t}(x) \), depends only on \( x_1 + x_2 \). Figures EC.5 and EC.6 illustrate that for a given value of \( x_1 \), \( p_{1t}(x) - p_{2t}(x) \) is constant, and for a given value of \( x_1 + x_2 \), \( p_{2t}(x) - p_{3t}(x) \) is constant, when \( \theta \) follows a triangular distribution. Figures EC.5 and EC.6 also show that the price difference...
is non-increasing as inventory levels $x_1$ and $x_2$ increase. We make similar observations from Figures EC.7 and EC.8 when $\theta$ follows a Beta distribution. As a result, we conclude that our analytical results, which we derived assuming $\theta$ follows a uniform distribution, are robust.
Figure EC.5  Price difference of products 1 and 2 as a function of $x_1$ and $x_2$ when $x_3 = 5$ and $t = 40$ with $\theta$ following a triangular distribution

Figure EC.6  Price difference of products 2 and 3 as a function of $x_1$ and $x_2$ when $x_3 = 5$ and $t = 40$ and with $\theta$ following a triangular distribution

Figure EC.7  Price difference of products 1 and 2 as a function of $x_1$ and $x_2$ when $x_3 = 5$ and $t = 40$ with $\theta$ following a Beta distribution

Figure EC.8  Price difference of products 2 and 3 as a function of $x_1$ and $x_2$ when $x_3 = 5$ and $t = 40$ with $\theta$ following a Beta distribution