Chapter 6

Magnetostatic Fields in Matter

Problem 6.1

\[ N = m_2 \times B_1; \quad B_1 = \frac{\mu_0}{4\pi r^3} [3(m_1 \cdot \hat{r}) \hat{r} - m_1]; \quad \hat{f} = \hat{y}; \quad m_1 = m_1 \hat{z}; \quad m_2 = m_2 \hat{y}. \quad B_1 = -\frac{\mu_0 m_1}{4\pi r^3} \hat{z}. \]

\[ N = -\frac{\mu_0 m_1 m_2}{4\pi r^3} (\hat{y} \times \hat{z}) = -\frac{\mu_0 m_1 m_2}{4\pi r^3} \hat{x}. \]

Here \( m_1 = \pi a^2 I, \quad m_2 = b^2 I. \) So \[ N = -\frac{\mu_0}{4\pi r^3} \frac{(abI)^2}{3} \]

Final orientation:

downward (\(-\hat{z}\)).

Problem 6.2

\[ d\mathbf{F} = \mathbf{I} \times \mathbf{B}; \quad d\mathbf{N} = \mathbf{r} \times d\mathbf{F} = \mathbf{I} \times (dl \times \mathbf{B}). \]

Now (Prob. 1.6): \( \mathbf{r} \times (dl \times \mathbf{B}) + dl \times (\mathbf{B} \times \mathbf{r}) + \mathbf{B} \times (r \times dl) = 0. \)

But \( d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] = dr \times (\mathbf{r} \times \mathbf{B}) + \mathbf{r} \times (dr \times \mathbf{B}) \) (since \( \mathbf{B} \) is constant), and \( dr = dl \), so \( dl \times (\mathbf{B} \times \mathbf{r}) = r \times (dl \times \mathbf{B}) - d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})]. \)

Hence \( 2r \times (dl \times \mathbf{B}) = d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times (\mathbf{r} \times dl). \)

\( d\mathbf{N} = \frac{1}{2} \mathbf{I} \{d[\mathbf{r} \times (\mathbf{r} \times \mathbf{B})] - \mathbf{B} \times (\mathbf{r} \times dl)\} \).

But the first term is zero (\( \oint d\mathbf{\cdots} = 0 \)), and the second integral is \( 2a \) (Eq. 1.107). So \( \mathbf{N} = -\mathbf{I}(\mathbf{B} \times \mathbf{a}) = \mathbf{m} \times \mathbf{B}. \) qed

Problem 6.3

According to Eq. 6.2, \( \mathbf{F} = 2\pi I R B \cos \theta. \)

But \( \mathbf{B} = \frac{\mu_0}{4\pi r^3} [3(m_1 \cdot \hat{r}) \hat{r} - m_1], \) and \( B \cos \theta = B \cdot \hat{y}, \) so \( B \cos \theta = \frac{\mu_0}{4\pi r^3} [3(m_1 \cdot \hat{r}) \hat{r} - m_1]. \)

But \( m_1 \cdot \hat{y} = 0 \) and \( \hat{r} \cdot \hat{y} = \sin \phi, \) while \( m_1 \cdot \hat{r} = m_1 \cos \theta. \)

\[ F = 2\pi I R \frac{\mu_0}{4\pi r^3} 3m_1 \sin \phi \cos \phi. \]

Now \( \sin \phi = \frac{R}{r}, \cos \phi = \sqrt{r^2 - R^2}/r, \) so \( F = \frac{3\mu_0}{2\pi} m_1 IR^2 \frac{\sqrt{r^2 - R^2}}{r^4}. \)

But \( IR^2 = m_2, \) so \( F = \frac{3\mu_0}{2\pi} m_1 m_2 \frac{\sqrt{r^2 - R^2}}{r^5}, \) while for a dipole, \( R \ll r, \) so \( F = \frac{3\mu_0}{2\pi} \frac{m_1 m_2}{r^4}. \)

(b) \( \mathbf{F} = \nabla (m_2 \cdot \mathbf{B}) = (m_2 \cdot \nabla) \mathbf{B} = (m_2 \frac{d}{dz}) \left[ \frac{\mu_0}{4\pi z^3} (3(m_1 \cdot \hat{z}) \hat{z} - m_1) \right] = \frac{\mu_0}{2\pi} m_1 m_2 \hat{z} \frac{d}{dz} \left( \frac{1}{z^3} \right), \)

or, since \( z = r: \]

\[ \mathbf{F} = \frac{3\mu_0}{2\pi} \frac{m_1 m_2}{r^4} \hat{z}. \]

113
CHAPTER 6. MAGNETOSTATIC FIELDS IN MATTER

Problem 6.4

\[ dF = I \left\{ (dy \hat{y}) \times B(0, y, 0) + (dz \hat{z}) \times B(0, z, 0) - (dy \hat{y}) \times B(0, y, 0) - (dz \hat{z}) \times B(0, z, 0) \right\} \]
\[ = I \left\{ \epsilon \frac{\partial B}{\partial y} \right\} \]
\[ \Rightarrow I e^2 \left\{ \hat{y} \times \frac{\partial B}{\partial y} - \hat{z} \times \frac{\partial B}{\partial z} \right\}. \]

\[ F = m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{vmatrix} = m \begin{vmatrix} \hat{y} \frac{\partial B_y}{\partial y} - \hat{z} \frac{\partial B_z}{\partial z} - \hat{x} \frac{\partial B_x}{\partial x} \\ \hat{z} \frac{\partial B_z}{\partial z} + \hat{y} \frac{\partial B_y}{\partial y} + \hat{x} \frac{\partial B_x}{\partial x} \end{vmatrix} \]
\[ \text{using } \nabla \cdot B = 0 \text{ to write } \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial x}. \]

But \( m \cdot B = m B_x \) (since \( m = m \hat{x} \), here), so \( \nabla (m \cdot B) = m \nabla (B_x) = m \left( \frac{\partial B_x}{\partial x} \hat{x} + \frac{\partial B_y}{\partial y} \hat{y} + \frac{\partial B_z}{\partial z} \hat{z} \right). \)

Therefore \( F = \nabla (m \cdot B) \). QED

Problem 6.5

(a) \( B = \mu_0 J_0 x \hat{y} \) (Prob. 5.14).
\[ m \cdot B = 0, \text{ so Eq. 6.3 says } F = 0. \]

(b) \( m \cdot B = m_0 \mu_0 J_0 x, \) so \( F = m_0 \mu_0 J_0 \hat{x}. \)

(c) Use product rule #4: \( \nabla (p \cdot E) = p \times (\nabla \times E) + E \times (\nabla \times p) + (p \cdot \nabla)E + (E \cdot \nabla)p. \)
But \( p \) does not depend on \( (x, y, z) \), so the second and fourth terms vanish, and \( \nabla \times E = 0 \), so the first term is zero. Hence \( \nabla (p \cdot E) = (p \cdot \nabla)E. \) QED

This argument does not apply to the magnetic analog, since \( \nabla \times B \neq 0. \) In fact, \( \nabla (m \cdot B) = (m \cdot \nabla)B + \mu_0 (m \times J). \)
\[ (m \cdot \nabla)B_a = m_0 \frac{\partial}{\partial x} (B) = m_0 \mu_0 J_0 y, \]
\[ (m \cdot \nabla)B_b = m_0 \frac{\partial}{\partial y} (\mu_0 J_0 x \hat{y}) = 0. \]

Problem 6.6

Aluminum, copper, copper chloride, and sodium all have an odd number of electrons, so we expect them to be paramagnetic. The rest (having an even number) should be diamagnetic.

Problem 6.7

\[ J_b = \nabla \times M = 0; \]
\[ K_b = M \times \hat{n} = M \hat{\phi}. \]
The field is that of a surface current \( K_b = M \hat{\phi}, \) but that's just a solenoid, so the field outside is zero, and inside \( B = \mu_0 K_b = \mu_0 M. \) Moreover, it points upward (in the drawing), so \( B = \mu_0 M. \)
Problem 6.8

\[ \nabla \times M = J_b = \frac{1}{s} \frac{\partial}{\partial s} (s k_s^2) \hat{z} = \frac{1}{s} (3k_s^2) \hat{z} = 3k_s \hat{z}, \quad K_b = M \times \hat{n} = k_s^2(\hat{\phi} \times \hat{s}) = -kR^2 \hat{z}. \]

So the bound current flows up the cylinder, and returns down the surface. [Incidentally, the total current should be zero ... is it? Yes, for \( \int J_b \, ds = \int_0^R (3k_s)(2\pi s \, ds) = 2\pi kR^3 \), while \( \int K_b \, dl = (-kR^2)(2\pi R) = -2\pi kR^3 \).] Since these currents have cylindrical symmetry, we can get the field by Ampère's law:

\[ B : 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 \int_0^s J_b \, ds = 2\pi k\mu_0 s^3 \Rightarrow B = \mu_0 k_s^2 \hat{\phi} = \mu_0 M. \]

Outside the cylinder \( I_{\text{enc}} = 0 \), so \( B = 0 \).

Problem 6.9

\[ K_b = M \times \hat{n} = M \hat{\phi}. \]

(Essentially a long solenoid)

(Intermediate case)

[The external fields are the same as in the electrical case; the internal fields (inside the bar) are completely different—in fact, opposite in direction.]

Problem 6.10

\( K_b = M \), so the field inside a complete ring would be \( \mu_0 M \). The field of a square loop, at the center, is given by Prob. 5.8: \( B_{\text{sq}} = \sqrt{2} \mu_0 I/\pi R \). Here \( I = Mw \), and \( R = a/2 \), so

\[ B_{\text{sq}} = \sqrt{2} \mu_0 Mw \frac{1}{\pi(a/2)} = \frac{2\sqrt{2} \mu_0 Mw}{\pi a}; \text{ net field in gap: } B = \mu_0 M \left( 1 - \frac{2\sqrt{2} w}{\pi a} \right). \]
Problem 6.11

As in Sec. 4.2.3, we want the average of \( B = B_{\text{out}} + B_{\text{in}} \), where \( B_{\text{out}} \) is due to molecules outside a small sphere around point \( P \), and \( B_{\text{in}} \) is due to molecules inside the sphere. The average of \( B_{\text{out}} \) is same as field at center (Prob. 5.57b), and for this it is OK to use Eq. 6.10, since the center is “far” from all the molecules in question:

\[
A_{\text{out}} = \frac{\mu_0}{4\pi} \int_{\text{outside}} \frac{M \times \hat{z}}{r^2} \, d\tau
\]

The average of \( B_{\text{in}} \) is \( \frac{\mu_0}{4\pi} \left( \frac{2}{3} \pi R^3 \right) \) —Eq. 5.89—where \( m = \frac{1}{2} \pi R^3 M \). Thus the average \( B_{\text{in}} \) is \( 2\mu_0 M/3 \). But what is left out of the integral \( A_{\text{out}} \) is the contribution of a uniformly magnetized sphere, to wit: \( 2\mu_0 M/3 \) (Eq. 6.16), and this is precisely what \( B_{\text{in}} \) puts back in. So we’ll get the correct macroscopic field using Eq. 6.10. qed

Problem 6.12

(a) \( M = ks\hat{z} \); \( J_b = \nabla \times M = -k\hat{\phi} \); \( K_b = M \times \hat{n} = kR\hat{\phi} \).

\( B \) is in the \( z \) direction (this is essentially a superposition of solenoids). So

\[
\oint B \cdot dl = B l = \mu_0 I_{\text{enc}} = \mu_0 \left[ \int J_b \, da + K_b l \right] = \mu_0 \left[ -k l(R - s) + k R l \right] = \mu_0 kl s.
\]

\[
\therefore B = \mu_0 ks\hat{z} \text{ inside.}
\]

(b) By symmetry, \( H \) points in the \( z \) direction. That same amperian loop gives \( \oint H \cdot dl = H l = \mu_0 I_{\text{enc}} = 0 \), since there is no free current here. So \( H = 0 \), and hence \( B = \mu_0 M \). Outside \( M = 0 \), so \( B = 0 \); inside \( M = ks\hat{z} \), so \( B = \mu_0 ks\hat{z} \).

Problem 6.13

(a) The field of a magnetized sphere is \( \frac{2}{3} \mu_0 M \) (Eq. 6.16), so \( B = B_0 - \frac{2}{3} \mu_0 M \), with the sphere removed.

In the cavity, \( H = \frac{1}{\mu_0} B \), so \( H = \frac{1}{\mu_0} \left( B_0 - \frac{2}{3} \mu_0 M \right) = H_0 + M - \frac{2}{3} M \Rightarrow H = H_0 + \frac{1}{3} M \).

(b) The field inside a long solenoid is \( \mu_0 K \). Here \( K = M \), so the field of the bound current on the inside surface of the cavity is \( \mu_0 M \), pointing down. Therefore

\[
B = B_0 - \mu_0 M; \quad H = \frac{1}{\mu_0} (B_0 - \mu_0 M) = \frac{1}{\mu_0} B_0 - M \Rightarrow H = H_0.
\]

(c) This time the bound currents are small, and far away from the center, so \( B = B_0 \), while \( H = \frac{1}{\mu_0} B_0 = H_0 + M \Rightarrow H = H_0 + M \).

[Comment: In the wafer, \( B \) is the field in the medium; in the needle, \( H \) is the \( H \) in the medium; in the sphere (intermediate case) both \( B \) and \( H \) are modified.]
Problem 6.14

M: ~; B is the same as the field of a short solenoid; $H = \frac{1}{\mu_0} B - M$.

Problem 6.15

"Potentials":

\[
\begin{align*}
W_{\text{in}}(r, \theta) &= \sum A_l r^l P_l(\cos \theta), \quad (r < R); \\
W_{\text{out}}(r, \theta) &= \sum \frac{B_l}{R^{l+1}} P_l(\cos \theta), \quad (r > R).
\end{align*}
\]

Boundary Conditions:

\[
\begin{align*}
(i) & \quad W_{\text{in}}(R, \theta) = W_{\text{out}}(R, \theta), \\
(ii) & \quad -\frac{\partial W_{\text{out}}}{\partial \theta} |_{R} + \frac{\partial W_{\text{in}}}{\partial \theta} |_{R} = M \mathbf{\hat{z}} \cdot \mathbf{\hat{r}} = M \cos \theta.
\end{align*}
\]

(The continuity of $W$ follows from the gradient theorem: $W(b) - W(a) = \int_{a}^{b} \nabla W \cdot dl = -\int_{a}^{b} H \cdot dl$; if the two points are infinitesimally separated, this last integral $\to 0$.)

\[
\begin{align*}
(i) & \quad A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow B_l = R^{2l+1} A_l, \\
(ii) & \quad \sum(l + 1) \frac{B_l}{R^{l+1}} P_l(\cos \theta) + \sum l A_l R^{l-2} P_l(\cos \theta) = M \cos \theta.
\end{align*}
\]

Combining these:

\[
\sum (2l + 1) R^{l-1} A_l P_l(\cos \theta) = M \cos \theta, \quad \text{so} \quad A_l = 0 \quad (l \neq 1), \quad \text{and} \quad 3 A_1 = M \Rightarrow A_1 = \frac{M}{3}.
\]

Thus $W_{\text{in}}(r, \theta) = \frac{M}{3} r \cos \theta = \frac{M}{3} r$, and hence $H_{\text{in}} = -\nabla W_{\text{in}} = -\frac{M}{3} \mathbf{\hat{z}} = -\frac{1}{3} M$, so

\[
B = \mu_0 (H + M) = \mu_0 \left( -\frac{1}{3} M + M \right) = \frac{2}{3} \mu_0 M.
\]
Problem 6.16

\[ \oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = I, \text{ so } \mathbf{H} = \frac{I}{2\pi s} \hat{\phi}. \]

\[ \mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu_0 (1 + \chi_m) \frac{I}{2\pi s} \hat{\phi}. \]

\[ \mathbf{M} = \chi_m \mathbf{H} = \frac{\chi_m I}{2\pi s} \hat{\phi}. \]

\[ \mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{s} \frac{\partial}{\partial s} \left( \frac{\chi_m I}{2\pi s} \right) \hat{z} = 0. \]

\[ \mathbf{K}_b = \mathbf{M} \times \hat{n} = \begin{cases} \frac{\chi_m I}{2\pi s} \hat{z}, & \text{at } s = a; \\ -\frac{\chi_m I}{2\pi s} \hat{z}, & \text{at } r = b. \end{cases} \]

Total enclosed current, for an amperian loop between the cylinders:

\[ I + \frac{\chi_m I}{2\pi a} = (1 + \chi_m) I, \text{ so } \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0 (1 + \chi_m) I \Rightarrow \mathbf{B} = \frac{\mu_0 (1 + \chi_m) I}{2\pi s} \hat{\phi}. \]

Problem 6.17

From Eq. 6.20:

\[ \oint \mathbf{H} \cdot d\mathbf{l} = H(2\pi s) = I_{\text{enc}} = \begin{cases} I(s^2/a^2), & (s < a); \\ I, & (s > a). \end{cases} \]

\[ \mathbf{H} = \left\{ \begin{array}{ll} \frac{I s}{2\pi}, & (s < a) \\ \frac{I}{2\pi s}, & (s > a) \end{array} \right\}, \text{ so } \mathbf{B} = \mu_0 \mathbf{H} = \begin{cases} \frac{\mu_0 (1 + \chi_m) I s}{2\pi s}, & (s < a); \\ \frac{\mu_0 I}{2\pi s}, & (s > a). \end{cases} \]

\[ J_b = \chi_m J_f \text{ (Eq. 6.33), and } J_f = \frac{I}{\pi a^2}, \text{ so } J_b = \frac{\chi_m I}{\pi a^2} \text{ (same direction as } I). \]

\[ \mathbf{K}_b = \mathbf{M} \times \hat{n} = \chi_m \mathbf{H} \times \hat{n} \Rightarrow \mathbf{K}_b = \frac{\chi_m I}{2\pi a} \text{ (opposite direction to } I). \]

\[ I_b = J_b (\pi a^2) + K_b (2\pi a) = \chi_m I - \chi_m I = 0 \text{ (as it should be, of course).} \]

Problem 6.18

By the method of Prob. 6.15:

For large \( r \), we want \( \mathbf{B}(r, \theta) \rightarrow \mathbf{B}_0 = B_0 \hat{z} \), so \( \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} \rightarrow \frac{1}{\mu_0} B_0 \hat{z} \), and hence \( W \rightarrow -\frac{1}{\mu_0} \mathbf{B}_0 z = -\frac{1}{\mu_0} B_0 r \cos \theta \).

"Potentials":

\[ \begin{align*}
W_{\text{in}}(r, \theta) &= \sum A_l r^l \cos \frac{l+1}{2} P_l(\cos \theta), & (r < R); \\
W_{\text{out}}(r, \theta) &= -\frac{1}{\mu_0} B_0 r \cos \theta + \sum \frac{B_{l+1}}{l+1} P_l(\cos \theta), & (r > R).
\end{align*} \]

Boundary Conditions:

\[ \begin{align*}
& \text{(i) } W_{\text{in}}(R, \theta) = W_{\text{out}}(R, \theta), \\
& \text{(ii) } -\mu_0 \frac{\partial W_{\text{out}}}{\partial r} \bigg|_R + \mu_0 \frac{\partial W_{\text{out}}}{\partial r} \bigg|_R = 0.
\end{align*} \]

(The latter follows from Eq. 6.26.)

\[ \begin{align*}
\text{(ii) } & \Rightarrow \mu_0 \left[ \frac{1}{\mu_0} B_0 \cos \theta + \sum (l + 1) \frac{B_{l+1}}{R^{l+2}} P_l(\cos \theta) \right] + \mu \sum A_l R^{l-1} P_l(\cos \theta) = 0.
\end{align*} \]

For \( l \neq 1 \), (i) \( \Rightarrow B_l = R^{2l+1} A_l \), so \( [\mu_0 (l+1) + \mu] A_l R^{l-1} = 0 \), and hence \( A_l = 0 \).

For \( l = 1 \), (i) \( \Rightarrow A_1 = -\frac{1}{\mu_0} B_0 R + B_1 / R^2 \), and (ii) \( \Rightarrow B_0 + 2\mu_0 B_1 / R^3 + \mu A_1 = 0 \), so \( A_1 = -3B_0 / (2\mu_0 + \mu) \).

\[ \begin{align*}
W_{\text{in}}(r, \theta) &= -\frac{3B_0}{(2\mu_0 + \mu)} r \cos \theta = -\frac{3B_0 z}{(2\mu_0 + \mu)} \hat{z}, \\
H_{\text{in}} &= -\nabla W_{\text{in}} = -\frac{3B_0}{(2\mu_0 + \mu)} \hat{z} = \frac{3B_0}{(2\mu_0 + \mu)} \hat{z}.
\end{align*} \]

\[ \mathbf{B} = \mu_0 \mathbf{H} = \frac{3\mu B_0}{(2\mu_0 + \mu)} = \frac{1 + \chi_m}{(1 + \chi_m/3)} B_0. \]
By the method of Prob. 4.23:

**Step 1:** \( \mathbf{B}_0 \) magnetizes the sphere: \( \mathbf{M}_0 = \chi_m \mathbf{H}_0 = \frac{\chi_m}{\mu_0(1+\chi_m)} \mathbf{B}_0 \). This magnetization sets up a field within the sphere given by Eq. 6.16:

\[
\mathbf{B}_1 = \frac{2}{3} \mu_0 \mathbf{M}_0 = \frac{2}{3} \frac{\chi_m}{1+\chi_m} \mathbf{B}_0 = \frac{2}{3} \kappa \mathbf{B}_0 \quad (\text{where } \kappa \equiv \frac{\chi_m}{1+\chi_m}).
\]

**Step 2:** \( \mathbf{B}_1 \) magnetizes the sphere an additional amount \( \mathbf{M}_1 = \frac{\kappa}{\mu_0} \mathbf{B}_1 \). This sets up an additional field in the sphere:

\[
\mathbf{B}_2 = \frac{2}{3} \mu_0 \mathbf{M}_1 = \frac{2}{3} \kappa \mathbf{B}_1 = \left(\frac{2\kappa}{3}\right)^2 \mathbf{B}_0, \quad \text{etc.}
\]

The total field is:

\[
\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2 + \cdots = \mathbf{B}_0 + \frac{2\kappa}{3} \mathbf{B}_0 + \frac{2\kappa}{3} \mathbf{B}_0 + \cdots = \left[1 + \frac{2\kappa}{3} + \left(\frac{2\kappa}{3}\right)^2 + \cdots\right] \mathbf{B}_0 = \frac{\mathbf{B}_0}{1 - \frac{2\kappa}{3}}.
\]

\[
\frac{1}{1 - \frac{2\kappa}{3}} = \frac{3}{3 - 2\chi_m/(1 + \chi_m)} = \frac{3 + 3\chi_m - 2\chi_m}{3 + \chi_m}, \quad \text{so } \mathbf{B} = \left(\frac{1 + \chi_m}{1 + \chi_m/3}\right) \mathbf{B}_0.
\]

**Problem 6.19**

\[
\Delta m = -\frac{e^2 r^2}{4\pi m_e} \mathbf{B}; \quad \mathbf{M} = \frac{\Delta m}{V} = -\frac{e^2 r^2}{4\pi m_e V} \mathbf{B}, \quad \text{where } V \text{ is the volume per electron. } \mathbf{M} = \chi_m \mathbf{H} \text{ (Eq. 6.29)}
\]

\[
= \frac{\chi_m}{\mu_0(1 + \chi_m)} \mathbf{B} \text{ (Eq. 6.30). So } \chi_m = -\frac{e^2 r^2}{4\pi m_e V} \mu_0. \quad \text{[Note: } \chi_m \ll 1, \text{ so I won’t worry about the } (1 + \chi_m) \text{ term; for the same reason we need not distinguish } \mathbf{B} \text{ from } \mathbf{B}_\text{else}, \text{ as we did in deriving the Clausius-Mossotti equation in Prob. 4.38.]}
\]

Let’s say \( V = \frac{4}{3}\pi r^3 \). Then \( \chi_m = -\frac{\mu_0}{4\pi} \left(\frac{3e^2}{4\pi m_e r}\right). \text{ I’ll use } 1 \text{ Å} = 10^{-10} \text{ m for } r. \)

Then \( \chi_m = -(10^{-7}) \left(\frac{2(1.6 \times 10^{-19})^2}{4(9.1 \times 10^{-31})(10^{-10})}\right) = -2 \times 10^{-5} \), which is not bad—Table 6.1 says \( \chi_m = -1 \times 10^{-5}. \)

However, I used only one electron per atom (copper has 29) and a very crude value for \( r. \) Since the orbital radius is smaller for the inner electrons, they count for less (\( \Delta m \sim r^2 \)). I have also neglected competing paramagnetic effects. But never mind … this is in the right ball park.

**Problem 6.20**

Place the object in a region of zero magnetic field, and heat it above the Curie point—or simply drop it on a hard surface. If it’s delicate (a watch, say), place it between the poles of an electromagnet, and magnetize it back and forth many times; each time you reverse the direction, reduce the field slightly.

**Problem 6.21**

(a) Identical to Prob. 4.7, only starting with Eqs. 6.1 and 6.3 instead of Eqs. 4.4 and 4.5.

(b) Identical to Prob. 4.8, but starting with Eq. 5.87 instead of 3.104.

(c) \( U = -\frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3 \cos \theta_1 \cos \theta_2 - \cos(\theta_2 - \theta_1)\right] m_1 m_2. \) Or, using \( \cos(\theta_2 - \theta_1) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2, \)

\[
U = \frac{\mu_0 m_1 m_2}{4\pi} \left(\sin \theta_1 \sin \theta_2 - 2 \cos \theta_1 \cos \theta_2\right).
\]

Stable position occurs at minimum energy: \( \frac{\partial U}{\partial \theta_1} = \frac{\partial U}{\partial \theta_2} = 0 \)

\[
\begin{cases}
\frac{\partial U}{\partial \theta_1} = \frac{\mu_0 m_1 m_2}{4\pi r^3} (\cos \theta_1 \sin \theta_2 + 2 \sin \theta_1 \cos \theta_2) = 0 \Rightarrow 2 \sin \theta_1 \cos \theta_2 = -\cos \theta_1 \sin \theta_2; \\
\frac{\partial U}{\partial \theta_2} = \frac{\mu_0 m_1 m_2}{4\pi r^3} (\sin \theta_1 \cos \theta_2 + 2 \cos \theta_1 \sin \theta_2) = 0 \Rightarrow 2 \sin \theta_1 \cos \theta_2 = -4 \cos \theta_1 \sin \theta_2.
\end{cases}
\]
Thus \( \sin \theta_1 \cos \theta_2 = \sin \theta_2 \cos \theta_1 = 0 \). \[
\begin{align*}
\text{Either } & \quad \sin \theta_1 = \sin \theta_2 = 0 : & \quad \rightarrow \rightarrow & \quad \text{or } \rightarrow \leftarrow \\
\text{or } & \quad \cos \theta_1 = \cos \theta_2 = 0 : & \quad \uparrow \uparrow & \quad \text{or } \uparrow \downarrow \end{align*}
\]
Which of these is the stable minimum? Certainly not \( \rightarrow \rightarrow \) or \( \rightarrow \leftarrow \) for these \( \mathbf{m}_2 \) is not parallel to \( \mathbf{B}_1 \), whereas we know \( \mathbf{m}_2 \) will line up along \( \mathbf{B}_1 \). It remains to compare \( \rightarrow \rightarrow \) (with \( \theta_1 = \theta_2 = 0 \)) and \( \uparrow \downarrow \) (with \( \theta_1 = \pi/2, \theta_2 = -\pi/2 \)).
\[
U_1 = \frac{\mu_0 m_1^2 m_2}{4\pi} (-2); \\
U_2 = \frac{\mu_0 m_1^2 m_2}{4\pi} (-1).
\]
\( U_1 \) is the lower energy, hence the more stable configuration.

**Conclusion:** They line up parallel, along the line joining them: \( \rightarrow \rightarrow \)

(d) They'd line up the same way: \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \)

**Problem 6.22**

\[
\mathbf{F} = \int \mathbf{F} \times \mathbf{B} = \int \left( \oint dl \times \mathbf{B} \right) + \int \oint \mathbf{F} \times \left( \mathbf{r} \cdot \nabla_0 \right) \mathbf{B}_0 - \int \oint \mathbf{F} \times \left( \mathbf{r} \cdot \nabla_0 \right) \mathbf{B}_0 = \int \oint \mathbf{F} \times \left( \mathbf{r} \cdot \nabla_0 \right) \mathbf{B}_0
\]

(because \( \oint dl = 0 \)). Now
\[
(\mathbf{d} \times \mathbf{B}_0)_i = \sum_{j,k} \epsilon_{ijk} \text{d}l_j (\mathbf{B}_0)_k, \quad \text{and } (\mathbf{r} \cdot \nabla_0) = \sum_i \mathbf{r}_i (\nabla_0)_i, \quad \text{so}
\]

\[
F_i = \int \sum_{j,k,l} \epsilon_{ijk} \left( \oint \mathbf{r}_l \mathbf{d}l_j \right) [(\nabla_0)_l (\mathbf{B}_0)_k] \quad \text{Lemma 1: } \oint \mathbf{r}_l \mathbf{d}l_j = \sum_m \epsilon_{ljm} a_m \quad (\text{proof below}).
\]

\[
= \int \sum_{j,k,l,m} \epsilon_{ijk} \epsilon_{ljm} a_m (\nabla_0)_l (\mathbf{B}_0)_k \quad \text{Lemma 2: } \sum_j \epsilon_{ijk} \epsilon_{ljm} = \delta_{il} \delta_{km} - \delta_{im} \delta_{kl} \quad (\text{proof below}).
\]

\[
= \int \sum_{k,l,m} (\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}) a_m (\nabla_0)_l (\mathbf{B}_0)_k = \int \sum_k [a_k (\nabla_0)_l (\mathbf{B}_0)_k - a_i (\nabla_0)_l (\mathbf{B}_0)_k]
\]

\[
= \int [(\nabla_0)_i (\mathbf{a} \cdot \mathbf{B}_0) - a_i (\nabla_0) \cdot \mathbf{B}_0].
\]

But \( \nabla_0 \cdot \mathbf{B}_0 = 0 \) (Eq. 5.48), and \( \mathbf{m} = \mathbf{I} \mathbf{a} \) (Eq. 5.84), so \( \mathbf{F} = \nabla_0 (\mathbf{m} \cdot \mathbf{B}_0) \) (the subscript just reminds us to take the derivatives at the point where \( \mathbf{m} \) is located). qed

**Proof of Lemma 1:**

Eq. 1.108 says \( \oint (\mathbf{c} \cdot \mathbf{r}) \mathbf{d}l = \mathbf{a} \times \mathbf{c} = -\mathbf{c} \times \mathbf{a} \). The \( j \)-th component is \( \sum_p \oint c_p r_p \mathbf{d}l_j = -\sum_{p,m} \epsilon_{pcm} c_p a_m \). Pick \( c_p = \delta_{pl} \) (i.e. 1 for the \( l \)-th component, zero for the others). Then \( \oint \mathbf{r}_l \mathbf{d}l_j = -\sum_m \epsilon_{ljm} a_m = \sum_m \epsilon_{ljm} a_m \). qed

**Proof of Lemma 2:**

\( \epsilon_{ijk} \epsilon_{ljm} = 0 \) unless \( ijk \) and \( ljm \) are both permutations of 123. In particular, \( i \) must either be \( l \) or \( m \), and \( k \) must be the other, so

\[
\sum_{j} \epsilon_{ijk} \epsilon_{ljm} = A \delta_{il} \delta_{km} + B \delta_{im} \delta_{kl}.
\]

To determine the constant \( A \), pick \( i = l = 1, k = m = 3 \); the only contribution comes from \( j = 2 \):

\[
\epsilon_{123} \epsilon_{123} = 1 = A \delta_{11} \delta_{33} + B \delta_{13} \delta_{31} = A \Rightarrow A = 1.
\]

To determine \( B \), pick \( i = m = 1, k = l = 3 \):

\[
\epsilon_{123} \epsilon_{321} = -1 = A \delta_{13} \delta_{31} + B \delta_{11} \delta_{33} = B \Rightarrow B = -1.
\]

So

\[
\sum_{j} \epsilon_{ijk} \epsilon_{ljm} = \delta_{il} \delta_{km} - \delta_{im} \delta_{kl}. \quad \text{qed}
\]
Problem 6.23

(a) The electric field inside a uniformly polarized sphere, \( E = -\frac{1}{3\epsilon_0} \mathbf{P} \) (Eq. 4.14) translates to \( \mathbf{H} = -\frac{1}{3\mu_0} (\mu_0 \mathbf{M}) = -\frac{1}{3} \mathbf{M} \). But \( \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \). So the magnetic field inside a uniformly magnetized sphere is \( \mathbf{B} = \mu_0 (\frac{2}{3} \mathbf{M} + \mathbf{M}) = \frac{2}{3} \mu_0 \mathbf{M} \) (same as Eq. 6.16).

(b) The electric field inside a sphere of linear dielectric in an otherwise uniform electric field is \( E = \frac{1}{1 + \chi_m/3} E_0 \) (Eq. 4.49). Now \( \chi_e \) translates to \( \chi_m \), for then Eq. 4.30 (\( \mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \)) goes to \( \mu_0 \mathbf{M} = \mu_0 \chi_m \mathbf{H} \), or \( \mathbf{M} = \chi_m \mathbf{H} \) (Eq. 6.29). So Eq. 4.49 \( \Rightarrow \mathbf{H} = \frac{1}{1 + \chi_m/3} \mathbf{H}_0 \). But \( \mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} \), and \( \mathbf{B}_0 = \mu_0 \mathbf{H}_0 \) (Eqs. 6.31 and 6.32), so the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field is \( \mathbf{B} = \frac{1}{\mu_0 (1 + \chi_m)} \mathbf{B}_0 \), or \( \mathbf{B} = \left( \frac{1 + \chi_m}{1 + \chi_m/3} \right) \mathbf{B}_0 \) (as in Prob. 6.18).

(c) The average electric field over a sphere, due to charges within, is \( E_{\text{ave}} = -\frac{1}{4\pi \epsilon_0} \frac{1}{R^3} \int \mathbf{P} d\mathbf{r} \) (for uniform charge density). Let's pretend the charges are all due to the frozen-in polarization of some medium (whatever \( \rho \) might be, we can solve \( \nabla \cdot \mathbf{P} = -\rho \) to find the appropriate \( \mathbf{P} \)). In this case there are no free charges, and \( \mathbf{p} = \int \mathbf{P} d\mathbf{r} \), so \( E_{\text{ave}} = -\frac{1}{4\pi \epsilon_0} \frac{1}{R^3} \int \mathbf{P} d\mathbf{r} \), which translates to

\[
H_{\text{ave}} = -\frac{1}{4\pi \mu_0} \frac{1}{R^3} \int \mu_0 \mathbf{M} d\mathbf{r} = -\frac{1}{4\pi} R^3 \mathbf{m}.
\]

But \( \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \), so \( \mathbf{B}_{\text{ave}} = -\frac{\mu_0}{4\pi} \frac{\mathbf{m}}{R^3} + \mu_0 \mathbf{M}_{\text{ave}} \), and \( \mathbf{M}_{\text{ave}} = \frac{\mathbf{m}}{\frac{3}{2} \pi R^2} \), so \( B_{\text{ave}} = \frac{\mu_0}{4\pi} \frac{2\mathbf{m}}{R^3} \), in agreement with Eq. 5.89. (We must assume for this argument that all the currents are bound, but again it doesn't really matter, since we can model any current configuration by an appropriate frozen-in magnetization. See G. H. Goedcke, Am. J. Phys. 66, 1010 (1998).)

Problem 6.24

Eq. 2.15: \( \mathbf{E} = \rho \left\{ \frac{1}{4\pi \epsilon_0} \int \nabla \cdot \frac{\mathbf{P}}{\epsilon_0} d\tau \right\} \) (for uniform charge density);

Eq. 4.9: \( V = \mathbf{P} \cdot \left\{ \frac{1}{4\pi \epsilon_0} \int \nabla \cdot \frac{\mathbf{P}}{\epsilon_0} d\tau \right\} \) (for uniform polarization);

Eq. 6.11: \( \mathbf{A} = \mu_0 \epsilon_0 \mathbf{M} \times \left\{ \frac{1}{4\pi \epsilon_0} \int \nabla \cdot \frac{\mathbf{P}}{\epsilon_0} d\tau \right\} \) (for uniform magnetization).

For a uniformly charged sphere (radius \( R \)): \[
\left\{ \begin{array}{l}
E_{\text{in}} = \rho \left( \frac{1}{3\epsilon_0} r \right) \\
E_{\text{out}} = \rho \left( \frac{1}{3\epsilon_0} R^3 \mathbf{r} \right)
\end{array} \right. \] (Prob. 2.12),

So the scalar potential of a uniformly polarized sphere is:
\[
\left\{ \begin{array}{l}
V_{\text{in}} = \frac{1}{3\epsilon_0} (\mathbf{P} \cdot \mathbf{r}) \\
V_{\text{out}} = \frac{1}{3\epsilon_0} R^3 (\mathbf{P} \cdot \mathbf{r})
\end{array} \right.
\] (Ex. 2.2).

And the vector potential of a uniformly magnetized sphere is:
\[
\left\{ \begin{array}{l}
A_{\text{in}} = \frac{\mu_0}{3} (\mathbf{M} \times \mathbf{r}) \\
A_{\text{out}} = \frac{\mu_0}{3} R^3 (\mathbf{M} \times \mathbf{r})
\end{array} \right.
\]
(Confirming the results of Ex. 4.2 and of Exs. 6.1 and 5.11).

Problem 6.25

(a) \( \mathbf{B}_1 = \frac{\mu_0}{4\pi} \frac{2m^2}{z^3} \hat{z} \) (Eq. 5.86, with \( \theta = 0 \)). So \( m^2 \mathbf{B}_1 = -\frac{\mu_0 m^2}{2\pi z^3} \). \( \mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \) (Eq. 6.3) \( \Rightarrow \mathbf{F} = \frac{\partial}{\partial z} \left[ -\frac{\mu_0 m^2}{2\pi z^3} \right] \hat{z} = \frac{3\mu_0 m^2}{2\pi z^4} \hat{z} \). This is the magnetic force upward (on the upper magnet); it balances the gravitational force downward \(-m_d \hat{z}\):

\[
\frac{3\mu_0 m^2}{2\pi z^4} - m_d = 0 \Rightarrow z = \left[ \frac{3\mu_0 m^2}{2\pi m_d} \right]^{1/4}.
\]
(b) The middle magnet is repelled upward by lower magnet and downward by upper magnet:

\[
\frac{3\mu_0 m^2}{2\pi x^4} - \frac{3\mu_0 m^2}{2\pi y^4} - m dg = 0.
\]

The top magnet is repelled upward by middle magnet, and attracted downward by lower magnet:

\[
\frac{3\mu_0 m^2}{2\pi y^4} - \frac{3\mu_0 m^2}{2\pi(x + y)^4} - m dg = 0.
\]

Subtracting: \[
\frac{3\mu_0 m^2}{2\pi} \left[ \frac{1}{x^4} - \frac{1}{y^4} - \frac{1}{(x+y)^4} \right] - m dg + m dg = 0, \text{ or } \frac{1}{x^4} - \frac{2}{y^4} + \frac{1}{(x+y)^4} = 0, \text{ so: } 2 = \frac{1}{(x/y)^4} + \frac{1}{(x/y+1)^4}.
\]

Let \( \alpha \equiv x/y; \) then \( 2 = \frac{1}{\alpha^4} + \frac{1}{(\alpha+1)^4}. \) Mathematica gives the numerical solution \( \alpha = \frac{x}{y} = 0.850115 \ldots \)

**Problem 6.26**

At the interface, the perpendicular component of \( \mathbf{B} \) is continuous (Eq. 6.26), and the parallel component of \( \mathbf{H} \) is continuous (Eq. 6.25 with \( K_f = 0 \)). So \( B_1^\parallel = B_2^\parallel, H_1^\parallel = H_2^\parallel. \) But \( \mathbf{B} = \mu \mathbf{H} \) (Eq. 6.31), so \( \frac{1}{\mu_1} B_1^\parallel = \frac{1}{\mu_2} B_2^\parallel. \)

Now \( \tan \theta_1 = B_1^\parallel / B_1^\perp, \) and \( \tan \theta_2 = B_2^\parallel / B_2^\perp, \) so

\[
\frac{\tan \theta_2}{\tan \theta_1} = \frac{B_2^\parallel / B_1^\parallel}{B_2^\perp / B_1^\perp} = \frac{\mu_2/\mu_1}{1}
\]

(the same form, though for different reasons, as Eq. 4.68).

**Problem 6.27**

In view of Eq. 6.33, there is a bound dipole at the center: \( m_b = \chi_m m. \) So the net dipole moment at the center is \( m_{\text{center}} = m + m_b = (1 + \chi_m)m = \frac{\mu}{\mu_0} m. \) This produces a field given by Eq. 5.87:

\[
B_{\text{center}} = \frac{\mu}{4\pi r^3} \left[ 3(m \cdot \hat{r})\hat{r} - m \right].
\]

This accounts for the first term in the field. The remainder must be due to the bound surface current \( K_b \) at \( r = R \) (since there can be no volume bound current, according to Eq. 6.33). Let us make an educated guess (based either on the answer provided or on the analogous electrical Prob. 4.34) that the field due to the surface bound current is (for interior points) of the form \( B_{\text{surface}} = \lambda m \) (i.e. a constant, proportional to \( m \)). In that case the magnetization will be:

\[
M = \chi_m H = \frac{\chi_m}{\mu} B = \frac{\chi_m}{4\pi r^3} \left[ 3(m \cdot \hat{r})\hat{r} - m \right] + \frac{\chi_m}{\mu} \lambda m. 
\]

This will produce bound currents \( J_b = \nabla \times M = 0, \) as it should, for \( 0 < r < R \) (no need to calculate this curl—the second term is constant, and the first is essentially the field of a dipole, which we know is curl-less, except at \( r = 0 \)), and

\[
K_b = M(R) \times \hat{r} = \frac{\chi_m}{4\pi R^3} (-m \times \hat{r}) + \frac{\chi_m A}{\mu} (m \times \hat{r}) = \chi_m m \left( -\frac{1}{4\pi R^3} + \frac{A}{\mu} \right) \sin \theta \hat{\phi}.
\]

But this is exactly the surface current produced by a spinning sphere: \( K = \sigma \mathbf{v} = \sigma \omega R \sin \theta \hat{\phi}, \) with \( \sigma \omega R \leftrightarrow \chi_m m \left( \frac{A}{\mu} - \frac{1}{4\pi R^3} \right). \) So the field it produces (for points inside) is (Eq. 5.68):

\[
B_{\text{surface}} = \frac{2}{3} \mu_0 (\sigma \omega R) = \frac{2}{3} \mu_0 \chi_m m \left( \frac{A}{\mu} - \frac{1}{4\pi R^3} \right).
\]
Everything is consistent, therefore, provided \( A = \frac{2}{3} \mu_0 \chi_m \left\{ \frac{A}{\mu} - \frac{1}{4 \pi R^3} \right\} \), or \( A \left(1 - \frac{2 \mu_0}{3 \mu} \right) = -\frac{2 \mu_0 \chi_m}{3} \). But 
\[ \chi_m = \left( \frac{\mu}{\mu_0} \right) - 1, \] 
so \( A \left(1 - \frac{2}{3} + \frac{2 \mu_0}{3 \mu} \right) = -\frac{2}{3} \left( \frac{\mu - \mu_0}{4 \pi R^3} \right) \), or \( A \left(1 + \frac{2 \mu_0}{\mu} \right) = 2 \frac{(\mu_0 - \mu)}{4 \pi R^3} \); \( A = \frac{\mu}{4 \pi R^3} \frac{2(\mu_0 - \mu)}{2(\mu_0 + \mu)} \), and hence

\[
B = \frac{\mu}{4 \pi} \left\{ \frac{1}{r^3} \left[ 3(m \cdot \hat{r})\hat{r} - m \right] + \frac{2(\mu_0 - \mu)m}{R^3} \right\}.
\]

**Problem 6.28**

The problem is that the field inside a cavity is not the same as the field in the material itself.

(a) **Ampère type.** The field deep inside the magnet is that of a long solenoid, \( B_0 \approx \mu_0 M \). From Prob. 6.13:

\[
\begin{align*}
\text{Sphere} & \colon B = B_0 - \frac{\mu}{3} \mu_0 M = \frac{1}{3} \mu_0 M; \\
\text{Needle} & \colon B = B_0 - \mu_0 M = 0; \\
\text{Wafer} & \colon B = \mu_0 M.
\end{align*}
\]

(b) **Gilbert type.** This is analogous to the electric case. The field at the center is approximately that midway between two distant point charges, \( B_0 \approx 0 \). From Prob. 4.16 (with \( E \to B, 1/\varepsilon_0 \to \mu_0, \mathbf{P} \to M \)):

\[
\begin{align*}
\text{Sphere} & \colon B = B_0 + \frac{\mu_0}{3} M = \frac{1}{3} \mu_0 M; \\
\text{Needle} & \colon B = B_0 = 0; \\
\text{Wafer} & \colon B = B_0 + \mu_0 M = \mu_0 M.
\end{align*}
\]

In the cavities, then, the fields are the same for the two models, and this will be no test at all. [Yes] Fund it with \$1 M from the Office of Alternative Medicine.