Question 1: (20 Points)
It is reported that two out of five adult smokers in the population acquired (=edinmek) the smoking habit by age 14.

(a) If 15 smokers are randomly selected, find the probability that more than 5 but fewer than 11 acquired the habit by age 14. (Use the Binomial table for this question)

\[ \Pr(5 \leq X \leq 11) = \Pr(X \leq 10) - \Pr(X \leq 5) \]
\[ = 0.991 - 0.403 = 0.588 \]

(b) If 7 smokers are randomly selected, find the probability that at least 3 of them acquired the habit by age 14.

\[ \Pr(X \geq 3) = 1 - \Pr(X \leq 2) \]
\[ = 1 - \left( \binom{7}{0}(0.4)^0(0.6)^7 - \binom{7}{1}(0.4)(0.6)^6 - \binom{7}{2}(0.4)^2(0.6)^5 \right) \]
\[ = 1 - 0.0279336 - 0.1306368 - 0.2612736 \]
\[ = 0.58 \]

(c) If 400 smokers are randomly selected, approximate the probability that 170 or more acquired the habit by age 14.

\[ \Pr(X \geq 170) = \Pr(X \geq 169.5) \]
\[ \approx \Pr(Z \geq \frac{169.5 - 160}{\sqrt{160}}) \]
\[ = \Pr(Z \geq 0.97) \]
\[ = 0.5 - 0.3340 \]
\[ = 0.166 \]
Question 2: (20 Points)

"X is N(μ, σ)" means that the random variable X has a Normal pdf with mean μ and standard deviation σ.

\[ \text{If } X \sim N(10, 2.2), \text{ what is } P(9 < X \leq 12) ? \]

\[ \begin{align*}
\text{a) } & \quad X \sim N(10, 2.2), \text{ what is } P(9 < X \leq 12) ? \\
\text{b) } & \quad \text{Find } P(X + Y > 22), \text{ if } X \text{ has the pdf given in part (a) and } Y \sim N(10, 2) \text{ independent of } X. \\
\text{c) } & \quad \text{For the standard Normal variable } Z, P(Z < z_1) = 0.3 \text{ and } P(Z > z_2) = 0.1. \\
\text{What is } P(Z < z_1 + z_2) ? \\
\text{d) } & \quad X \sim N(\mu, \sigma), P(X < 3) = 0.3 \text{ and } P(X > 8) = 0.1. \text{ Find } \mu \text{ and } \sigma. 
\end{align*} \]

\[ \begin{align*}
a) \quad & \quad P(9 < X \leq 12) = P \left( \frac{9 - 10}{2.2} \leq Z \leq \frac{12 - 10}{2.2} \right) \\
& \quad = P(-0.87 \leq Z \leq 0.87) \\
& \quad = 0.2957 + 0.4525 = 0.7492 \tag{1} \\

b) \quad & \quad X + Y \sim N(\mu_X + \mu_Y, \frac{\sigma_X^2 + \sigma_Y^2}{10 + 10}) \\
& \quad \Rightarrow X + Y \sim N(20, 2.33) \tag{2} \\
& \quad \Rightarrow P(X + Y > 22) = P \left( Z > \frac{22 - 20}{2.33} \right) \\
& \quad = P(Z > 0.86) = 0.5 - 0.3051 = 0.1949 \tag{3} \\

c) \quad & \quad P(Z < z_1) = 0.3 \Rightarrow P(z_1 < Z \leq 0) = 0.5 - 0.3 = 0.2 \\
& \quad \Rightarrow \begin{align*}
z_1 & \approx -0.52 \\
\text{Similarly, } P(Z > z_2) = 0.1 & \Rightarrow 0.5 - 0.1 = 0.4 \\
& \Rightarrow z_2 = 1.28 \\

P(Z \leq 1.28 - 0.52) = P(Z \leq 0.76) & = 0.5 + 0.2764 = 0.7764 \tag{4} \\

d) \quad & \quad P(X < 3) = 0.3 \Rightarrow P \left( Z \leq \frac{3 - \mu}{\sigma} \right) = 0.3 \\
& \quad \Rightarrow \frac{3 - \mu}{\sigma} = z_1 \text{ of (c)} \Rightarrow 3 - \mu = -0.52 \sigma \tag{5} \\
& \quad \text{Similarly, } \frac{8 - \mu}{\sigma} = z_2 = 1.28 \Rightarrow 8 - \mu = 1.28 \sigma \tag{6} \\
& \quad \text{Solving (5) and (6) gives } \mu = 4.45 \quad \sigma \approx 2.78
\end{align*} \]
Question 3: (20 Points)
The heartbeat rate of a person in a stress situation has a mean of 120 beats and a standard deviation of 5 beats per minute.

(10 points) a) If the distribution of the heartbeat rate is normal, what is the probability that the average heartbeat rate of randomly selected 10 people subjected to a stress situation is at least 119?

\[ X \sim N(120, 5) \Rightarrow \bar{X} \sim N(120, \frac{5}{\sqrt{10}}) \]

\[ P(\bar{X} > 119) = P(\frac{Z}{\frac{119 - 120}{\frac{5}{\sqrt{10}}}}) = P(Z > -0.63) = 0.2357 + 0.5 = 0.7357 \]

(10 points) b) If the distribution of the heartbeat rate is normal, what is the probability that the average heartbeat rate of randomly selected 100 people subjected to a stress situation is at least 119?

CLT \Rightarrow \bar{X}^{off} \sim N\left(120, \frac{5}{\sqrt{100}}\right) \Rightarrow \bar{X} \sim N(120, 0.5)

\[ P(\bar{X} > 119) \approx P(Z > \frac{119 - 120}{0.5}) = P(Z > -2) = 0.4772 + 0.5 = 0.9772 \]
Question 4: (20 Points)
In the following bivariate distribution, the joint probabilities are unknown.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>p(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>3</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

\[ p(x) = a+c \quad b+d \]

It is known that \( E(X) = 2.8 \), \( E(Y) = 1.6 \), \( \text{Cov}(X,Y) = -0.08 \). To calculate \( a, b, c, d \), we need four equations. Given the above information, write these four equations (DON'T SOLVE THEM).

1. \[ E(X) = 2(a+c) + 4(b+d) = 2.8 \]
2. \[ E(Y) = 1(a+b) + 3(c+d) = 1.6 \]
3. \[-0.08 = \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 2a + 4b + 6c + 12d - (2.8)(1.6) \]
4. All probabilities must add up to 1
   \[ a+b+c+d = 1 \]
Question 5: (20 Points)
Vanessa firmly believes in reincarnation. She is convinced that in each of her lifetimes, the number of times she marries has the following probability distribution function.

<table>
<thead>
<tr>
<th>x</th>
<th>( p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

(6 points) a) According to her, what is the expected number of times she will marry within her next 3 lifetimes?

\[
E(X_1 + X_2 + X_3) = 3E(X)
\]

\[
E(X) = 0(0.2) + 1(0.6) + 2(0.2) = 1
\]

\[
\Rightarrow E(X_1 + X_2 + X_3) = (3)(1) = 3
\]

(14 points) b) What is the probability that she will marry more than 50 times in her next 40 lifetimes?

\[
b) \quad P(X_1 + \cdots + X_{40} > 50) = ?
\]

\[
E(X_1 + \cdots + X_{40}) = 40 \quad \Rightarrow E(X) = (40)(1) = 40
\]

\[
\sqrt{V(X_1 + \cdots + X_{40})} = 40 \quad \Rightarrow V(X) = 40(0.4) = 16
\]

Since

\[
V(X) = E(X^2) - (E(X))^2 = 1^2(0.6) + 2^2(0.2) - 1^2
\]

\[
= 0.6 + 0.8 - 1 = 0.4
\]

and by assuming independence of Vanessa's "lives",

\[
X_1 + \cdots + X_{40} \approx N(40, 16) \quad \text{by CLT}
\]

\[
\Rightarrow P(X_1 + \cdots + X_{40} > 50) \approx P(Z > \frac{50 - 40}{\sqrt{16}})
\]

\[
\approx P(Z > 2.63)
\]

\[
= 0.5 - 0.4957
\]

\[
= 0.0043
\]