Simple Zeros

**Theorem 1** If $p$ is a polynomial and $\lambda$ is a zero of $p$, then one solution of the difference equation $p(E)x = 0$ is $[\lambda, \lambda^2, \lambda^3, \ldots]$. If all the zeros of $p$ are simple and nonzero, then each solution of the difference equation is a linear combination of such special solutions.

**Proof** First, if $\lambda$ is any complex number and $u = [\lambda, \lambda^2, \lambda^3, \ldots]$, then $Eu = \lambda u$ because $(Eu)_n = u_{n+1} = \lambda^{n+1} = \lambda(\lambda^n) = \lambda u_n$. By reapplying the operator $E$, one obtains in general $E^iu = \lambda^i u$. Since $E^0$ has been defined as the identity map, we have $E^0 u = \lambda^0 u$. Thus, if $p$ is a polynomial defined by $p(\lambda) = \sum_{i=0}^{m} c_i \lambda^i$, then

$$p(E)u = \left(\sum_{i=0}^{m} c_i E^i\right)u = \sum_{i=0}^{m} c_i (E^i u) = \sum_{i=0}^{m} c_i \lambda^i u = p(\lambda)u$$

If $p(\lambda) = 0$, then $p(E)u = 0$, as asserted.

Let $p$ be a polynomial all of whose zeros, $\lambda_1, \lambda_2, \ldots, \lambda_m$, are simple and nonzero. Corresponding to any zero $\lambda_k$ there is a solution of the difference equation $p(E)x = 0$; namely, we have the solution $u^{(k)} = [\lambda_k, \lambda_k^2, \lambda_k^3, \ldots]$. Let $x$ denote an arbitrary solution of the difference equation. We seek to express $x$ in the form $x = \sum_{k=1}^{m} a_k u^{(k)}$. Taking the first $m$ components of the sequences in this equation, we obtain

$$x_i = \sum_{k=1}^{m} a_k \lambda_k^i \quad (i = 1, 2, \ldots, m) \quad (4)$$

The $m \times m$ matrix having elements $\lambda_k^i$ is nonsingular because its singularity would imply a nontrivial equation

$$\sum_{i=1}^{m} b_i \lambda_k^i = 0, \quad \text{or} \quad \sum_{i=1}^{m} b_i \lambda_k^{i-1} = 0$$

(This last equation exhibits a polynomial of degree $m - 1$ having $m$ zeros.) Equation (4) thus determines $a_1, a_2, \ldots, a_m$ uniquely. It remains to be proven that Equation (4) is valid for all values of $i$. Put $x = x - \sum_{k=1}^{m} a_k u^{(k)}$. Then $p(E)x = 0$, or equivalently $\sum_{i=0}^{m} c_i x_{n+i} = 0$ for all $n$. In other words

$$z_{n+m} = -c_m^{-1}(c_0 z_n + c_1 z_{n+1} + \cdots + c_{m-1} z_{n+m-1}) \quad (n = 1, 2, \ldots) \quad (5)$$

Note that $c_m \neq 0$ because the polynomial $p$ has $m$ distinct zeros and is therefore of degree $m$. Since $z_i = 0$ for $i = 1, 2, \ldots, m$, Equation (5) used repeatedly shows that

$$z_{m+1} = z_{m+2} = \cdots = 0$$
Multiple Zeros

There remains the problem of solving a difference equation \( p(E) x = 0 \) when \( p \) has multiple zeros. Define \( x(\lambda) = [\lambda, \lambda^2, \lambda^3, \ldots] \). If \( p \) is any polynomial, we have seen that

\[
p(E)x(\lambda) = p(\lambda)x(\lambda)
\]

A differentiation with respect to \( \lambda \) yields

\[
p(E)x'(\lambda) = p'(\lambda)x(\lambda) + p(\lambda)x''(\lambda)
\]

If \( \lambda \) is a multiple zero of \( p \), then \( p(\lambda) = p'(\lambda) = 0 \), and Equations (6) and (7) show that \( x(\lambda) \) and \( x'(\lambda) \) are solutions of the difference equation. Thus, a solution is the sequence \( x'(\lambda) = [1, 2\lambda, 3\lambda^2, \ldots] \). If \( \lambda \neq 0 \), it is independent of the solution \( x(\lambda) \) because

\[
\det \begin{bmatrix} \lambda & \lambda^2 \\ 1 & 2\lambda \end{bmatrix} \neq 0
\]

and thus, if the sequences are truncated at the second term, the resulting pair of vectors in \( \mathbb{R}^2 \) is linearly independent.

By extending this reasoning, one can prove that if \( \lambda \) is a zero of \( p \) having multiplicity \( k \), then the following sequences are solutions of the difference equation \( p(E)x = 0 \):

\[
x(\lambda) = [\lambda, \lambda^2, \lambda^3, \ldots] \\
x'(\lambda) = [1, 2\lambda, 3\lambda^2, \ldots] \\
x''(\lambda) = [0, 2, 6\lambda, \ldots] \\
\vdots
\]

\[
x^{(k-1)}(\lambda) = \frac{d^{(k-1)}}{d\lambda^{k-1}} [\lambda, \lambda^2, \lambda^3, \ldots]
\]

**THEOREM 2**  Let \( p \) be a polynomial satisfying \( p(0) \neq 0 \). Then a basis for the null space of \( p(E) \) is obtained as follows. With each zero \( \lambda \) of \( p \) having multiplicity \( k \), associate the \( k \) basic solutions \( x(\lambda), x'(\lambda), \ldots, x^{(k-1)}(\lambda) \), where \( x(\lambda) = [\lambda, \lambda^2, \lambda^3, \ldots] \).

**Example 1** Determine the general solution of this difference equation:

\[
4x_n + 7x_{n-1} + 2x_{n-2} - x_{n-3} = 0
\]

**Solution** The given equation is of the form \( p(E)x = 0 \), where \( p(\lambda) = 4\lambda^3 + 7\lambda^2 + 2\lambda - 1 \). The factors of \( p \) are \( (\lambda + 1)^2 \) and \( (4\lambda - 1) \). Thus, \( p \) has a double zero at \(-1\) and a simple zero at \(1/4\). The basic solutions are

\[
x(-1) = [-1, 1, -1, 1, \ldots] \\
x'(-1) = [1, -2, 3, -4, \ldots] \\
x(\frac{1}{4}) = [\frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \ldots]
\]

The general solution is

\[
x = \alpha x(-1) + \beta x'(-1) + \gamma x(\frac{1}{4})
\]

or

\[
x_n = \alpha(-1)^n + \beta n(-1)^{n-1} + \gamma(\frac{1}{4})^n
\]
1.3 Difference Equations

Stable Difference Equations

An element \( x = [x_1, x_2, \ldots] \) of \( V \) is said to be bounded if there is a constant \( c \) such that \( |x_n| \leq c \) for all \( n \). In other words, \( \sup_n |x_n| < \infty \). A difference equation of the form \( p(E)x = 0 \) is said to be stable if all of its solutions are bounded. The difference equation (2) is not stable since one of its solutions is given by \( x_n = 2^n \). We now ask whether there is an easy method of identifying a stable difference equation.

Theorem 3
For a polynomial \( p \) satisfying \( p(0) \neq 0 \), these properties are equivalent:

(i) \( \) The difference equation \( p(E)x = 0 \) is stable.

(ii) \( \) All zeros of \( p \) satisfy \( |x| \leq 1 \), and all multiple zeros satisfy \( |x| < 1 \).

Proof
Assume that (ii) is true of \( p \). Let \( \lambda \) be a zero of \( p \). Then one solution of the corresponding difference equation is \( x(\lambda) = [\lambda, \lambda^2, \lambda^3, \ldots] \). Since \( |\lambda| \leq 1 \), this sequence is bounded. If \( \lambda \) is a multiple zero, then one or more of \( x'(\lambda), x''(\lambda), \ldots \) will also be a solution of the difference equation. In this case, \( |\lambda| < 1 \), by (ii). Since, by elementary calculus (L'Hôpital's Rule)

\[
\lim_{n \to \infty} n^k \lambda^n = 0 \quad (k = 0, 1, \ldots)
\]

we see that each sequence \( x'(\lambda), x''(\lambda), \ldots \) is bounded. (See Problems 22 and 23.)

For the converse, suppose that (i) is false. If \( p \) has a zero \( \lambda \) satisfying \( |\lambda| > 1 \), then the sequence \( x(\lambda) \) is unbounded. If \( p \) has a multiple zero \( \lambda \) satisfying \( |\lambda| \geq 1 \), then \( x'(\lambda) \) is unbounded since its general term satisfies the inequality

\[
|n_x_n| = |n \lambda^{n-1}| = n|\lambda|^{n-1} \geq n
\]

Example 2
Determine whether this difference equation is stable:

\[4x_n + 7x_{n-1} + 2x_{n-2} - x_{n-3} = 0\]

Solution
The given equation is of the form \( p(E)x = 0 \), where \( p(\lambda) = 4\lambda^3 + 7\lambda^2 + 2\lambda - 1 \). By the preceding example, \( p \) has a double zero at \(-1\) and a simple zero at \( 1/4 \). The equation is therefore unstable.

An example of a difference equation having nonconstant coefficients arises in the theory of Bessel functions. The Bessel functions \( J_n \) are defined by the formula

\[J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta - n\theta) \, d\theta\]

It is obvious from the definition that \( |J_n(x)| \leq 1 \). Not so obvious, but true, is the recurrence formula

\[J_n(x) = 2(n - 1)x^{-1}J_{n-1}(x) - J_{n-2}(x)\]

If (for a certain \( x \)) we know \( J_0(x) \) and \( J_1(x) \), then the recurrence relation can be used to compute \( J_2(x), J_3(x), \ldots, J_n(x) \). This procedure becomes unstable and useless
when \(2n > |x|\) because roundoff errors that inevitably occur will be multiplied by the factor \(2nx^{-1}\). This factor eventually becomes very large. (See Problem 25.)

For further information on computing functions by means of recurrence relations, see Abramowitz and Stegun [1964, p. xiii], Cash [1979], Gautschi [1961, 1967, 1975], and Wimp [1984].

**PROBLEM SET 1.3**

1. For the sequences following Equation (2), express the first as a linear combination of the second and third.

2. Let \(p\) be a polynomial of degree \(m\). Is the solution space of \(p(E)x = 0\) necessarily of dimension \(m\)?

3. Let \(p\) be a polynomial of degree \(m\), with \(p(0) \neq 0\). If a sequence \(x\) contains \(m\) consecutive zeros and \(p(E)x = 0\), then \(x = 0\).

4. Is the operator \(E\) injective (one-to-one)? Does it have a right or left inverse? Is it surjective (onto)? Define an operator \(F\) by \((Fx)_n = x_{n-1}\), with \((Fx)_1 = 0\), and answer the same questions for \(F\). Explore the relationship between \(E\) and \(F\). Suppose that \(V\) were redefined as the space of all functions on the set \(\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}\), and suppose that \(F\) were defined simply by \((Fx)_n = x_{n-1}\). How are the answers to the previous questions affected?

5. What are the eigenvalues and eigenvectors of the operator \(E\)?

6. What can you say about the question of convergence in the following infinite series?

\[
\sum_{n=1}^{\infty} x_n v^{(n)}(n)
\]

Prove that \(x = \sum_{n=1}^{\infty} x_n v^{(n)}\), in the pointwise sense.

7. If \(\{v^{(1)}, v^{(2)}, \ldots\}\) is adopted as a basis for \(V\), show that \(\sum_{i=0}^{m} c_i E^i\) can be represented by an infinite matrix.

8. Prove that any two operators of the form described in Problem 7 commute with each other.

9. Prove that if \(L_1\) and \(L_2\) are linear combinations of powers of \(E\), and if \(L_1 x = 0\), then \(L_1 L_2 x = 0\).

10. Develop a complete theory for the difference equation \(E^n x = 0\).

11. Give bases consisting of real sequences for the solution space of

(a) \((4E^0 - 3E^2 + E^3)x = 0\) \hspace{1cm} (c) \((2E^0 - 9E^8 + 12E^4 - 4E^3)x = 0\)

(b) \((3E^0 - 2E + E^2)x = 0\) \hspace{1cm} (d) \((\pi E^0 - \sqrt{2}E + \log 2 \cdot E^0)x = 0\)

12. Prove that if \(p\) is a polynomial with real coefficients, and if \(x \equiv [x_1, x_2, \ldots]\) is a (complex) solution of \(p(E)x = 0\), then the conjugate of \(x\), the real part of \(x\), and the imaginary part of \(x\) are also solutions.

13. Solve

(a) \(x_{n+1} - nx_n = 0\) \hspace{1cm} (b) \(x_{n+1} - x_n = n\) \hspace{1cm} (c) \(x_{n+1} - x_n = 2\)

14. Define an operator \(\Delta\) by putting

\[
\Delta x = [x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots]
\]

Show that \(E = I + \Delta\). Show that if \(p\) is a polynomial,

\[
p(E) = p(I) + p'(I)\Delta + \frac{1}{2!}p''(I)\Delta^2 + \cdots
\]