SECOND ORDER SYSTEMS

SECOND ORDER SYSTEM TRANSFER FUNCTION

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

POLES OF THE SYSTEM

\[ p_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]
POSSIBLE STEP RESPONSES OF SECOND ORDER SYSTEMS

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>Poles</th>
<th>Step response</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$j\omega$ &lt;br&gt; $j\omega_n$ &lt;br&gt; $-j\omega_n$</td>
<td>Undamped</td>
</tr>
<tr>
<td>$0 &lt; \zeta &lt; 1$</td>
<td>$j\omega$ &lt;br&gt; $j\omega_n \sqrt{1 - \zeta^2}$ &lt;br&gt; $-\zeta\omega_n$ &lt;br&gt; $-j\omega_n \sqrt{1 - \zeta^2}$</td>
<td>Underdamped</td>
</tr>
<tr>
<td>$\zeta = 1$</td>
<td>$j\omega$ &lt;br&gt; $-\zeta\omega_n$</td>
<td>Critically damped</td>
</tr>
<tr>
<td>$\zeta &gt; 1$</td>
<td>$j\omega$ &lt;br&gt; $-\zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1}$ &lt;br&gt; $-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}$</td>
<td>Overdamped</td>
</tr>
</tbody>
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POSSIBLE STEP RESPONSES OF SECOND ORDER SYSTEMS
UNDERDAMPED RESPONSE

$\gamma(t)$

Exponential decay generated by real part of complex pole pair

Sinusoidal oscillation generated by imaginary part of complex pole pair
\[ %\text{OS} = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100 \]

\[ T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \]

\[ T_s = \frac{4}{\zeta \omega} \]
UNDERDAMPED CASE

\[ s_{1,2} = -\zeta \omega_n \pm i \omega_n \sqrt{1 - \zeta^2} \]

\[ \sigma_d \]

\[ \omega_d \]

\[ T_p = \frac{\pi}{\omega_d} \]

\[ T_s = \frac{4}{\sigma_d} \]

\[ \%OS = e^{-\frac{\sigma_d}{\omega_d} \pi} = e^{-\cot(\theta) \pi} \]
MORAL: If the third pole is far away from the origin, the third order system can be approximated with a second order system. If not...
Effect of a Zero

Normalized $c(t)$ vs. Time (seconds)

- zero at $-3$
- zero at $-5$
- zero at $-10$
- no zero
STEP RESPONSE OF SECOND ORDER SYSTEMS

**System**

- **(a)** \( R(s) = \frac{1}{s} \)

  \[ \frac{b}{s^2 + as + b} \]
  
  General

- **(b)** \( R(s) = \frac{1}{s} \)

  \[ \frac{9}{s^2 + 9s + 9} \]
  
  Overdamped

- **(c)** \( R(s) = \frac{1}{s} \)

  \[ \frac{9}{s^2 + 2s + 9} \]
  
  Underdamped

- **(d)** \( R(s) = \frac{1}{s} \)

  \[ \frac{9}{s^2 + 9} \]
  
  Undamped

- **(e)** \( R(s) = \frac{1}{s} \)

  \[ \frac{9}{s^2 + 6s + 9} \]
  
  Critically damped

**Pole-zero Plot**

**Response**

- **(a)**

  \[ G(s) \]
  
  \[ s^2 + as + b \]
  
  \[ x \text{-plane} \]
  
  \[ -7.854 \text{ - 1.146} \]

  \[ c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t} \]

- **(b)**

  \[ G(s) \]
  
  \[ s^2 + 9s + 9 \]
  
  \[ s \text{-plane} \]
  
  \[ j \omega \]
  
  \[ -1.146 \]
  
  \[ c(t) = 1 - e^{-t} \cos(\sqrt{8}t - 19.4) \]

- **(c)**

  \[ G(s) \]
  
  \[ s^2 + 2s + 9 \]
  
  \[ s \text{-plane} \]
  
  \[ j \sqrt{8} \]
  
  \[ c(t) = 1 - 1.06e^{-t} \cos(\sqrt{8}t - 19.4) \]

- **(d)**

  \[ G(s) \]
  
  \[ s^2 + 9 \]
  
  \[ s \text{-plane} \]
  
  \[ j \sqrt{3} \]
  
  \[ c(t) = 1 - \cos 3t \]

- **(e)**

  \[ G(s) \]
  
  \[ s^2 + 6s + 9 \]
  
  \[ s \text{-plane} \]
  
  \[ -3 \]
  
  \[ c(t) = 1 - 3e^{-3t} - e^{-3t} \]
Overdamped

$20 \log_{10} |H(s)|$ for $\zeta=2$, $\omega_n=2$
Critically Damped

$20\log_{10}|H(s)|$ for $\zeta = 1$, $\omega_n = 2$
Underdamped

$20 \log_{10} |H(s)|$ for $\zeta = 0.3, \omega_n = 2$
Undamped
What if we include some zeros?

$20 \log_{10}|H(s)|$ for $\zeta = 0.3, \ \omega_n = 2$ with zeros at $-0.4 \pm 0.75j$