Lecture 1

Root Locus

• What is Root-Locus? : A graphical representation of closed loop poles as a system parameter varied.

• Based on Root-Locus graph we can choose the parameter for stability and the desired transient response.
How does the Root-Locus graph look-like?

For the system

\[
\frac{K}{s^2 + 10s + K}
\]

where \( K = K_1K_2 \)

The location of poles as a function of \( K \) can be calculated as

<table>
<thead>
<tr>
<th>( K )</th>
<th>Pole 1</th>
<th>Pole 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>-9.47</td>
<td>-0.53</td>
</tr>
<tr>
<td>10</td>
<td>-8.87</td>
<td>-1.13</td>
</tr>
<tr>
<td>15</td>
<td>-8.16</td>
<td>-1.84</td>
</tr>
<tr>
<td>20</td>
<td>-7.24</td>
<td>-2.76</td>
</tr>
<tr>
<td>25</td>
<td>-5</td>
<td>-5</td>
</tr>
<tr>
<td>30</td>
<td>-5 + j2.24</td>
<td>-5 - j2.24</td>
</tr>
<tr>
<td>35</td>
<td>-5 + j3.16</td>
<td>-5 - j3.16</td>
</tr>
<tr>
<td>40</td>
<td>-5 + j3.87</td>
<td>-5 - j3.87</td>
</tr>
<tr>
<td>45</td>
<td>-5 + j4.47</td>
<td>-5 - j4.47</td>
</tr>
<tr>
<td>50</td>
<td>-5 + j5</td>
<td>-5 - j5</td>
</tr>
</tbody>
</table>
Our First Root Locus

The corresponding root locus can be drawn
Drawing the Root Locus

• How do we draw root locus
  – for more complex systems,
  – and without calculating poles.

• We exploit the properties of Root-Locus to do a rough sketch.

• Therefore, let’s explore the properties of root locus.
Properties of Root Locus

For the closed loop system

- The transfer function is
  \[ T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \]

- For a given \( K \), \( s^* \) is a pole if
  \[ 1 + KG(s^*)H(s^*) = 0 \]
  which is equivalent to
  \[ -|KG(s^*)H(s^*)| = 1, \]
  \[ -\angle KG(s^*)H(s^*) = (2m + 1)\pi \]

Since \( K \) is real and positive these conditions would be equivalent to

- \( \angle G(s^*)H(s^*) = (2m + 1)\pi \)
- \( K = \frac{1}{|G(s^*)H(s^*)|} \)
Number of Branches and Symmetry

• The number of branches of the root locus equals the number of closed loop poles

• Since the poles appear as complex conjugate pairs, root locus is symmetric about real axis
Which parts of real line will be a part of root locus?

Remember the angle condition
\[ \angle G(\sigma)H(\sigma) = (2m + 1)\pi \]

The angle contribution of off-real axis poles and zeros is zero. (Because they appear in complex pairs).

What matters is the the real axis poles and zeros.

Rule: If the total number of open loop poles and zeros on the right of a point is odd then that point is part of root-locus.
Real Axis Segments: Examples

- **Example 1**

  The real axis segments: \([-2, -1]\) and \([-4, -3]\)

- **Example 2**

  The real axis segments: \([-2, -1]\) and \([3, 5]\)
Start and End Points

- Let's write
  \[ H(s) = \frac{N_H(s)}{D_H(s)} \quad \quad G(s) = \frac{N_G(s)}{D_G(s)} \]

Therefore
\[ T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)} \]

As a result,
- when \( K \) is close to zero
  \[ T(s) \approx \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)} \]
  i.e. the closed loop poles are essentially the the poles of \( G(s)H(s) \).
- when \( K \) is large
  \[ T(s) \approx \frac{KN_G(s)D_H(s)}{KN_G(s)N_H(s)} \]
  i.e. the closed loop poles are essentially the zeros of \( G(s)H(s) \).

**Conclusion:** The root locus begins at the finite and infinite poles of \( G(s)H(s) \) and ends at the finite and infinite zeros of \( G(s)H(s) \).
Behavior at Infinity

• What if the number of (finite) open loop poles are more than (finite) open loop zeros, e.g.,

\[ KG(s)H(s) = \frac{K}{s(s + 1)(s + 2)} \]

  – The poles are at 0, −1, −2
  – The zeros are at \( s \to \infty \).

• Let \( s \) approach to \( \infty \) then

\[ KG(s)H(s) \approx \frac{K}{s^3} \]

• Skipping the details, the asymptotes are calculated using formulas:

  – The real axis intercept: The point where the asymptotes merge on the real axis

\[
\sigma_a = \frac{\sum \text{finitepoles} - \sum \text{finitezeros}}{\# \text{finitepoles} - \# \text{finitezeros}}
\]

  – The angles with real line:

\[
\theta_a = \frac{(2m + 1)\pi}{\# \text{finitepoles} - \# \text{finitezeros}}
\]
Asymptotes: Example

- Consider the unity feedback system

  ![Feedback System Diagram](image)

  \[ R(s) + \frac{K(s+3)}{s(s+1)(s+2)(s+4)} \]

- The real axis intercept for the asymptotes:

  \[
  \sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = \frac{-4}{3}
  \]

- The angles

  \[
  \theta_a = \frac{(2m + 1)\pi}{3}
  \]

  which yields \( \frac{\pi}{3}, \pi \) and \( \frac{5\pi}{3} \)
Break-away and Break-in Points

- **Break-away point**: The point where root-locus leaves the real axis.
- **Break-in point**: The point where root locus enters the real axis.
- **Variation of $K$** as a function of $\sigma$
• Note that the curves have their local maximum and minimum points at break-away and break-in points. So the derivative of

\[ K = -\frac{1}{G(\sigma)H(\sigma)} \]

should be equal to zero at break-away and break-in points.
Break-away and Break-in Points: Example

Find the break-away and break-in point of the following figure

Solution: From the figure

\[ K(s)H(s) = \frac{K(s - 3)(s - 5)}{(s + 1)(s + 2)} = \frac{K(s^2 - 8s + 15)}{s^2 + 3s + 1} \]

On the real axis

\[ K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15} \]

Differentiating \( K \) with respect to \( \sigma \) and equating to
zero
\[
\frac{dK}{d\sigma} = \frac{11\sigma^2 - 26\sigma - 61}{(\sigma^2 - 8\sigma + 15)^2} = 0
\]
which is achieved for \(\sigma_1 = -1.45\) and \(\sigma_2 = 3.82\).

- it can be shown that a break-away or break-in point satisfy
\[
\sum \frac{1}{\sigma + z_i} = \sum \frac{1}{\sigma + p_i}
\]
- Applying this formula to our problem, we obtain
\[
\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2}
\]
which would yield
\[
11\sigma^2 - 26\sigma - 61 = 0
\]
(same as what we obtained before)
**$j\omega$ Axis Crossings**

- Use Routh-Hurwitz to find $j\omega$ axis crossings.
- When we have $j\omega$ axis crossings, the Routh-table has all zeros at a row.
- Find the $K$ value for which a row of zeros is achieved in the Routh-table.

**Example:** Consider

$$T(s) = \frac{K(s + 3)}{s^4 + 7s^3 + 14s^2 + (8 + K)s + 3K}$$

The Routh table

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
<th>14</th>
<th>3K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>7</td>
<td>8 + K</td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td>90 − K</td>
<td>21K</td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td>$\frac{-K^2 - 65K + 72}{90 - K}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td>21K</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The row $s^1$ is zero for $K = 9.65$. For this $K$, the previous row polynomial is

$$(90 - K)s^2 + 21K = 80.35s^2 + 202.7 = 0$$

whose roots are $s = \pm j1.59$. 
Angles of Departure and Arrival

- Angles of Departure from Open Loop Poles

\[ \theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k + 1)180^\circ \]
• Angles of Departure form Open Loop Zeros

\[ \theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k + 1)180^\circ \]
Angle of Departure: Example

Consider the system

\[
\angle \theta_1 - \angle \theta_2 + \angle \theta_3 - \angle \theta_4 = -\theta_1 - 90^\circ + \arctan\left(\frac{1}{1}\right) - \arctan\left(\frac{1}{2}\right) = 180^\circ
\]

from which we obtain \( \theta_1 = -108.4^\circ \).
Root Locus Example

Problem: Sketch the Root-Locus of the system

- The number of branches: 2
- Open Loop Poles: $-2, -4$ (starting points)
- Open Loop Zeros: $2 + j4, 2 - j4$ (ending points)
- Number of finite poles = Number of Finite Zeros $\Rightarrow$ No Asymptotes
- Break-away point: Take the derivative of $K = -\frac{1}{\sigma}$

$$\frac{dK}{d\sigma} = -\frac{d}{d\sigma} \frac{(\sigma + 2)(\sigma + 4)}{\sigma^2 - 4\sigma + 20} = -\frac{-10s^2 + 24s + 152}{(\sigma^2 - 4\sigma + 20)^2}$$

equating to zero we obtain $\sigma_b = -2.87$ and $K = 0.0248$.
- $j\omega$ axis crossing occurs for $K = 1.5$ and at $\pm j3.9$.
- The root locus crosses $\zeta = 0.45$ line for $K = 0.417$ at $3.4\angle 116.7^\circ$
The resulting Root Locus:
Root Locus Example 2

Problem: Sketch the Root-Locus of the third order system

\[
\begin{align*}
\frac{R(s) + E(s)}{K(s + 1.5)} & \to C(s) \\
\frac{K(s + 1.5)}{s(s + 1)(s + 10)} & \to C(s)
\end{align*}
\]

- Number of branches: 3
- Open Loop Poles: 0, \(-1\), \(-10\) (Starting points)
- Open Loop Zero: \(-1.5\) (One of the end points)
- Real Axis Segments: \([-1, 0]\) and \([-10, -1.5]\)
- Asymptotes: \(\sigma_a = \frac{-11 - (-1.5)}{3 - 1} = -4.75\) and \(\theta_a = \frac{\pi}{2}, \frac{3\pi}{2}\).
- Break-in,away points: The derivative of \(K = -\frac{1}{G(\sigma)}\) yields
  \[
  2\sigma^3 + 15.5\sigma^2 + 33\sigma + 15
  \]
equating to zero we obtain
- \(\sigma_1 = -0.62\) with gain \(K = 2.511\) (Break-away point)
- \(\sigma_2 = -4.4\) with gain \(K = 28.89\) (Break-away point)
\[-\sigma_3 = -2.8\] with gain \(K = 27.91\) (Break-in point)

The resulting root locus