Homework Set #2
Due: Friday, October 25, 2013.

1. **Projection Matrices** A matrix $P \in \mathbb{R}^{n \times n}$ is called an orthogonal projection matrix if $P = P^T$ and $P^2 = P$.
   
   (a) Show that if $P$ is an orthogonal projection matrix then so is $I - P$.
   
   (b) Suppose that the columns of $U \in \mathbb{R}^{n \times k}$ are orthonormal. Show that $UU^T$ is an orthogonal projection matrix.
   
   (c) Suppose $A \in \mathbb{R}^{n \times k}$ is full rank with $k \leq n$. Show that $A(A^T A)^{-1}A^T$ is an orthogonal projection matrix.
   
   (d) if $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the point $y$ in $S$ closest to $x$ is called the orthogonal projection of $x$ onto $S$. Show that if $P$ is a orthogonal projection matrix, then $y = Px$ is the orthogonal projection of $x$ on $\mathcal{R}(P)$. (Which is why such matrices are called projection matrices...)

2. **Orthogonal matrices**
   
   (a) Show that if $U$ and $V$ are orthogonal then so is $UV$.
   
   (b) Suppose that $U \in \mathbb{R}^{2 \times 2}$ is orthogonal. Show that $U$ is either a rotation or a reflection. Make clear how you decide whether a given orthogonal $U$ is a rotation or reflection.

3. Let $A \in \mathbb{R}^{n \times n}$ and if $\lambda, \mu$ are to distinct eigenvalues of $A$, i.e. $\mu \neq \lambda$, then show that any left eigenvector of $A$ corresponding to $\mu$ is orthogonal to any right eigenvector of $A$ corresponding to $\lambda$.

4. Suppose $a, b \in \mathbb{R}^{n \times n}$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to $a$ than $b$ is a half space, i.e.:

   $$\{x \mid \|x - a\| \leq \|x - b\|\} = \{x \mid c^T x \leq d\}$$

   for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Give $c$ and $d$ explicitly, and draw a picture showing $a, b, c$ and the halfspace (for $n=2$). Note that the set $\{x \mid c^T x = d\}$ is called an hyperplane.

5. **Reflection through a hyperplane**. Find the matrix $Q \in \mathbb{R}^{n \times n}$ such that the reflection of $x$ through the hyperplane $\{z \mid a^T z = 0\}$ is given by $Qx$. Verify that matrix $Q$ is orthogonal. (To reflect $x$ through the hyperplane means the following: find the point $z$ on the hyperplane closest to $x$. Starting from $x$ go in the direction $z - x$ through the hyperplane to a point on the opposite side, which has the same distance to $z$ as $x$ does.)
6. (Bonus) *Communication Channel*

(a) Suppose a single user is transmitting information signal \( \{x_n\} \) to a receiver with a single antenna. \( Z \) transform of the received signal can be written as

\[
Y(z) = H(z)X(z)
\]

where \( H(z) \) corresponds to the stable transfer function corresponding to the channel from the transmitter to the receiver. What is the condition on \( H(z) \) such that we can invert it perfectly with a receiver filter \( K(z) \) so that we can recover the original transmitted sequence error free. Note that the user is using all of \([-\pi, \pi)\) bandwidth.

(b) Now a single user is transmitting to a receiver using an antenna array with \( N \) elements. If we label the \( Z \) transform of the sequence received at antenna \( i \) as \( Y_i(z) \), then we can write

\[
\begin{bmatrix}
Y_1(z) \\
Y_2(z) \\
\vdots \\
Y_N(z) \\
\end{bmatrix} = \begin{bmatrix}
H_{11}(z) & H_{12}(z) & \cdots & H_{1M}(z) \\
H_{21}(z) & H_{22}(z) & \cdots & H_{2M}(z) \\
\vdots & \vdots & \ddots & \vdots \\
H_{N1}(z) & H_{N2}(z) & \cdots & H_{NM}(z) \\
\end{bmatrix} \begin{bmatrix}
X_1(z) \\
X_2(z) \\
\vdots \\
X_M(z) \\
\end{bmatrix}
\]

The receiver first filters the signals from different antennas and then adds them to recover the original transmitted sequence, i.e., it forms the signal

\[
\hat{X}(z) = K_1(z)Y_1(z) + K_2(z)Y_2(z) + \ldots + K_N(z)Y_N(z).
\]

What is the condition on \( H(z) \) for the error free recovery of the transmitted sequence \( \{x_n\} \). Note that the user is using all of \([-\pi, \pi)\) bandwidth.

(c) In a typical wireless reverse link scenario, multiple users transmit to the same base station. Suppose there are \( M \) users transmitting to a base station and the base station employs an antenna array with \( N \) elements. If we label the \( Z \) transform of the sequence received at antenna \( i \) as \( Y_i(z) \), then we can write

\[
\begin{bmatrix}
Y_1(z) \\
Y_2(z) \\
\vdots \\
Y_N(z) \\
\end{bmatrix} = \begin{bmatrix}
H_{11}(z) & H_{12}(z) & \cdots & H_{1M}(z) \\
H_{21}(z) & H_{22}(z) & \cdots & H_{2M}(z) \\
\vdots & \vdots & \ddots & \vdots \\
H_{N1}(z) & H_{N2}(z) & \cdots & H_{NM}(z) \\
\end{bmatrix} \begin{bmatrix}
X_1(z) \\
X_2(z) \\
\vdots \\
X_M(z) \\
\end{bmatrix}
\]

For each user, the receiver first filters the signals from different antennas and then adds them to recover the original transmitted sequence, i.e., for user \( i \) it forms the signal

\[
\hat{X}_i(z) = K_{i1}(z)Y_1(z) + K_{i2}(z)Y_2(z) + \ldots + K_{iN}(z)Y_N(z)
\]

for \( i = 1, \ldots, M \). What is the condition on \( H(z) \) for the error free recovery of all users. Note that each user is using all of \([-\pi, \pi)\) bandwidth.