Math 207: Quiz # 2B
Fall 2004

- You have 25 minutes.
- You may ask any question about the quiz within the first 5 minutes. After this time for any question you may want to ask 2 points will be deducted from your grade.

1. Let $A$, $B$, $H$ and $U$ be $n \times n$ matrices such that $H$ is Hermitian and $U$ is unitary.

   1.a) Show that $(BA)^T = A^T B^T$, where $T$ stands for the “transpose”.
   Hint: Let $M = BA$ and compute the entries of $M^T$ in terms of the entries of $A$ and $B$. (3 points)
   
   $$(M^T)_{ij} = M_{ji} = \sum_{k=1}^{n} B_{jk} A_{ki}$$
   
   
   
   $$(A^T B^T)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = (BA^T B^T)_{ij}$$
   
   
   

   1.b) Show that $(BA)^* = A^* B^*$, where $*$ stands for the “Hermitian conjugate”. (2 points)
   
   $$(BA)^* = (BA)^T = (\overline{BA})^T = (\overline{B} \overline{A})^T = A^T B^* = A^* B^*$$

   1.c) Show that $UHU^{-1}$ is Hermitian. (5 points)

   $$((UHU^{-1})^*) = (U^*)^{-1} U^{-1} (H^*) = (U^*)^{-1} (UH^{-1})^* (U^{-1})$$
   
   $$= (U^{-1})^{-1} H^* U^{-1}$$
   
   (if $H$ is Hermitian, $U$ is unitary)
   
   $$= U^{-1} H U^{-1}$$
2. Let \( a, b, r, s \) be real numbers and \( U = \begin{pmatrix} 2re^{ia} & se^{-ia} \\ 2re^{ib} & -se^{-ib} \end{pmatrix} \). How should \( a \) and \( b \) be related so that \( U \) is a unitary matrix? Use the condition that \( U \) is unitary to determine \( r \) and \( s \).

10 points

If \( U \) is unitary, then \( \mathbf{v}_1, \mathbf{v}_2 \) must be orthonormal:

\[
1 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 4r^2 + 4r^2 = 8r^2 = r^2 = \frac{1}{\sqrt{8}}
\]

\[
1 = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = s^2 + s^2 = 2s^2 = s^2 = \frac{1}{4}
\]

\[
0 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1^\dagger \mathbf{v}_2 = (2re^{ia} \quad 2re^{-ia}) (\begin{pmatrix} s e^{-ia} \\ -s e^{-ib} \end{pmatrix}) = 2rs (e^{-2ia} - e^{-2ib})
\]

\[
e^{-2ia} = e^{-2ib} = e^{2i(b-a)} = 1
\]

\[
2(b-a) = 2\pi k \quad \text{for some integer } k
\]

\[
b = a + nk
\]

\[
e^{ib} = e^{ia} e^{nk} = e^{ia}
\]