Maturity and interest-rate risk

Suppose you buy one of these three bonds, originally selling at a yield to maturity of 8 percent.

<table>
<thead>
<tr>
<th>Yield to maturity</th>
<th>One-year 8% coupon %</th>
<th>30-year 8% coupon %</th>
<th>30-year zero-coupon %</th>
</tr>
</thead>
<tbody>
<tr>
<td>8%</td>
<td>$1000</td>
<td>$1000</td>
<td>$99.37</td>
</tr>
<tr>
<td>9%</td>
<td>991 -.9%</td>
<td>897 -10.3%</td>
<td>75.37 -24.1%</td>
</tr>
<tr>
<td>7%</td>
<td>1009 +.9%</td>
<td>1124 +12.4%</td>
<td>131.37 +32.2%</td>
</tr>
</tbody>
</table>

Clearly, the long-term bonds are more sensitive to changes in the interest rate than the one-year bond. But why is the zero-coupon bond more sensitive than the coupon bond?

Think of the coupon bond as a "portfolio" of 30 separate payments.

<table>
<thead>
<tr>
<th>time</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>80</td>
</tr>
<tr>
<td>29</td>
<td>80</td>
</tr>
<tr>
<td>30</td>
<td>1080</td>
</tr>
</tbody>
</table>

The average maturity of this "portfolio of zeros" must be less than 30 years. As a result, it is not surprising that the interest-rate sensitivity of the coupon bond is less than that of the zero with the same maturity.
Duration measures the effective maturity of a coupon bond.

Duration equals the weighted average of the time to receipt of each payment:

\[ D = \sum_{t=1}^{30} w_t \times t \]

where each weight, \( w_t = \frac{\text{Present value of cash flow at time } t}{\text{Total present value of all cash flows}} \)

Example: 3-year, 8% coupon bond selling at yield to maturity of 10%:

<table>
<thead>
<tr>
<th>time</th>
<th>Cash flow</th>
<th>Present value</th>
<th>Weight</th>
<th>( w_t \times t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>72.727 (=80/1.10)</td>
<td>.0765</td>
<td>.0765</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>66.116 (=80/1.10^2)</td>
<td>.0696</td>
<td>.1392</td>
</tr>
<tr>
<td>3</td>
<td>1080</td>
<td>811.420</td>
<td>.8539</td>
<td>2.5617</td>
</tr>
<tr>
<td>SUM:</td>
<td></td>
<td>950.263</td>
<td>1.0000</td>
<td>2.7774</td>
</tr>
</tbody>
</table>

Notice that duration is 2.7774 years, which is less than time to maturity (3 years).

This must be true of bonds paying coupons, because the weighted average maturity is less than the final maturity. For zero-coupon bonds only, duration = time to maturity.

In fact, for a perpetuity, which has an infinite maturity, duration = \( \frac{1+y}{y} \).

For example, if \( y = 10\% \), duration = \( \frac{1.10}{0.1} = 11 \) years, far less than infinity.
Bonds with equal duration have equal interest rate sensitivity.

Example: Compare a perpetuity to a zero-coupon bond. Suppose the interest rate is 8%.

The duration of the perpetuity is \( \frac{1.08}{0.08} = 13.5 \) years.

If the duration pays $1000 a year, its value is \( \frac{1,000}{0.08} = $12,500 \).

We wish to compare the perpetuity to a zero coupon bond with the same duration, so we consider a zero with a maturity of 13.5 years. Its face value is $35,329, so its market value is:

\[ \frac{35,329}{(1.08)^{13.5}} = $12,500, \] the same as the perpetuity.

Now suppose that market yields fall to 7.9%.

The perpetuity increases in value to \( \frac{1,000}{0.079} = $12,658 \).

The zero increases in value to \( \frac{35,329}{(1.079)^{13.5}} = $12,657 \).

Almost identical increases of $157 or $158.

Conclusion: Bonds with equal duration have equal interest rate sensitivity.
The duration approximation to interest rate sensitivity.

\[
\frac{\Delta P}{P} \approx - \frac{\text{Duration}}{1 + y} \times \Delta y
\]

\[
= - \text{Modified duration} \times \Delta y
\]

Modified duration is a "multiplier" that translates a change in yield into a percentage change in price. A natural measure of interest rate sensitivity. Therefore, bonds with equal duration will have equal sensitivity to a given change in the yield to maturity.

For the bonds in our example, \( \Delta y = -0.10\% \) (\( y \) falls from 8\% to 7.9\%)

\[
\frac{-13.5}{1.08} \times (-0.10\%) = 1.25\%
\]

1.25\% of $12,500 = $156

Again, almost identical to the actual increase in the prices of the two bonds.

**Technical point:**

For bonds paying intra-year coupons, the formula for modified duration is slightly different:

\[
\text{Modified duration} = \frac{\text{Duration}}{1 + y/k}
\]

where \( k \) is the number of coupons paid each year (e.g., \( k = 2 \) if the bond pays interest semiannually) and \( y \) is the bond equivalent YTM.

In effect, to obtain modified duration, you divide duration by the periodic (e.g., semi-annual) YTM rather than the annualized YTM.
Price value of a basis point (PVBP)

The duration formula implies that

\[ \Delta P = - \frac{\text{Duration}}{1 + y} \times \Delta y \times P \]

\[ = - \text{Modified duration} \times \Delta y \times P \]

So if we allow \( \Delta y = 1 \text{ bp} = .01\% \) (i.e., \( \Delta y = .0001 \)) then we can obtain the change in the dollar value of the portfolio for each basis point change in the yield on that portfolio.

Since our bond portfolio is worth $12,500,

\[ \text{PVBP} = \frac{13.50}{1.08} \times .0001 \times $12,500 = $15.60 \]

If the portfolio were worth $125,000, the PVBP would be $156

PVBP measures the sensitivity of the dollar value of the fixed income portfolio to changes in yields as opposed to the percentage sensitivity.
PVBP is the slope of the bond pricing curve.
Factors that affect duration

1. *Higher maturity increases duration*

Example: coupon = 8% and yield to maturity = 8%

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>Infinity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration</td>
<td>1</td>
<td>7.3</td>
<td>10.7</td>
<td>13.5</td>
</tr>
</tbody>
</table>

2. *Higher coupon decreases duration*

Example: maturity = 20 years and yield to maturity = 8%

<table>
<thead>
<tr>
<th>Coupon</th>
<th>0</th>
<th>6</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration</td>
<td>20</td>
<td>11</td>
<td>9.6</td>
</tr>
</tbody>
</table>

**Digression:** why are yields on long-term zeros generally greater than yields on equal-maturity coupon bonds?

Probable answer: The zeros have far greater durations. So if the yield curve is upward sloping, the zeros are viewed as longer-term bonds and are priced at a higher YTM.

Another interpretation: the zeros are viewed as having higher price risk so receive a higher term premium.

3. *Higher yield to maturity decreases duration*

Example: maturity = 20 years and coupon = 15%

<table>
<thead>
<tr>
<th>Yield to maturity</th>
<th>6%</th>
<th>15%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration</td>
<td>9.5</td>
<td>7</td>
</tr>
</tbody>
</table>
Immunization requires duration – not maturity – matching.

One reason to immunize is *active* bond management. The bonds you find attractively priced cannot be made to cash flow match your obligation.

Consider a company that has issued a GIC. Based on analysis of various securities in the fixed income market, it chooses to fund the obligation with 3-year zeros and perpetuities, which it believes are attractively priced.

*Question*: How can it immunize itself against interest rate risk?

To simplify, suppose that

i. The GICs promise a 10% return over 7 years on an investment of $10,000, for a single payment obligation of $19,487. The GIC is to be "fully funded."

ii. The obligation is funded with 3-year zeros and perpetuities, each currently yielding 10%.

Therefore to immunize, choose weights to match the durations of assets and liabilities.

\[
\begin{align*}
  w_z D_z + w_p D_p &= 7 \\
  w_z 3 + (1 - w_z)11 &= 7 \\
  w_z &= .5 \\
  w_p &= 1 - w_z = .5 
\end{align*}
\]

You buy $5,000 market value of 3-year zeros (face value $6,655) and $5,000 of perpetuities (coupon rate = 10% \(\Rightarrow\) coupon = $500)

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero-coupon</td>
<td>GIC</td>
</tr>
<tr>
<td>Perpetuity</td>
<td>PV = 10,000</td>
</tr>
<tr>
<td>5,000</td>
<td>5,000</td>
</tr>
</tbody>
</table>
The immunized portfolio is protected against interest rate changes.

Suppose rates fall to 9.5%; you are still obligated to provide a rate of return of 10% on the GIC (meaning that you still have to provide a payoff of $19,487), so the market value (i.e., the present value) of your liability rises. But your assets also are worth more.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zeros ( \frac{6,655}{1.095^3} = \frac{5,069}{1.095^7} = $5,069 )</td>
<td>GIC ( \frac{19,487}{1.095^7} = $10,324 )</td>
</tr>
<tr>
<td>Perpetuity ( \frac{500}{0.095} = \frac{5,263}{10,332} )</td>
<td>Surplus ( \frac{8}{10,332} )</td>
</tr>
</tbody>
</table>

Surplus is still near zero, so you are still fully funded. Reflects the fact that you were immunized. Duration matching works.

**Notice:** It doesn't matter that the cash flows of the asset portfolio do not match the cash flows of the perpetual obligation. The focus is on maintaining the match between the *market values* of assets versus liabilities, *not* on matching cash flows.

If the assets generate less cash than the payments to beneficiaries, you can sell some assets; if the assets generate excess cash, you can reinvest it. The important point is that at all times you have enough assets to fund the present value of the obligation.
To remain immunized, we require dynamic hedging.

Duration matching is not a once and for all strategy. It requires dynamic hedging, meaning that the optimal hedge changes over time.

At this new interest rate, what are the immunizing weights?

- The duration of the zero is still 3 years.
- The duration of the obligation is still 7 years (since we simplify by assuming that no time has passed.)
- The duration of the perpetuity is now higher, however:

\[
\frac{1 + y}{y} = \frac{1.095}{0.095} = 11.53 \text{ years.}
\]

Therefore, the new immunizing weights are determined by

\[
w_z D_z + w_p D_p = 7
\]

\[
w_z 3 + (1-w_z)11.53 = 7
\]

\[
w_z = .53
\]

So we should hold:

\[
.53 \times 10,324 = 5,472 \text{ in the 3-year zero coupon bond}
\]

\[
.47 \times 10,324 = 4,852 \text{ in the perpetuities.}
\]

Compare to the balance sheet, which shows that current mixture is $5,069 in zeros and $5,263 in perpetuities.

We would need to sell off some of our perpetuities and place the funds into zeros. One must update the hedged position as interest rates change and as time passes.
Rebalancing is required even if rates do not change.

Suppose one year passes and rates stay at 10%.

Obligation is now a 6-year "bullet" worth $11,000.

Perpetuity is still worth $5,000 but you've also received the first $500 coupon.

The zero is worth $5500.

So you're still fully funded.

But you are no longer immunized. While the duration of the perpetuity is still 11 years, the duration of the zero has fallen to 2 years and the duration of the obligation fallen to 6 years.

\[ w_z D_z + w_p D_p = 6 \]

\[ w_z 2 + (1 - w_z) 11 = 6 \]

\[ w_z = 5/9 \]

\[ 5/9 \times 11,000 = 6,111.11 \text{ in the zero} \]

\[ 4/9 \times 11,000 = 4,888.89 \text{ in the perpetuity.} \]

You must invest the entire coupon into the zero, as well as sell off $111.11 of the perpetuity and invest the proceeds in the zero.

Practice problem: What if time passes and the interest rate changes? Try Concept Check #3 in the chapter.
Duration matching strategies assume that shifts in the yield curve are parallel.

Recall the PVBP approach:

$$\Delta P = - \frac{\text{Duration}}{1+y} \times \Delta y \times P$$

We matched P to fully fund the obligation, and we matched durations to immunize.

Therefore, as long as $\Delta y$ is equal for the two sides of the balance sheet, we have achieved a perfect hedge. This is a big "if" however: it generally will not be true that yields on all assets change equally. This would require parallel shifts in the yield curve.
Convexity implies that the duration rule is only approximate.

Example:

A zero-coupon 20-year maturity bond with face value $67,275 is selling for $10,000 at a YTM of 10%, and has duration of 20 years. If interest rates rise so that the bond's YTM increases by 1% to 11%, the percentage change in the bond's price will be approximately

\[ \text{Modified duration} \times \Delta y = -\frac{\text{Duration}}{1 + y} \times \Delta y \]

\[ = -\frac{20}{1.10} \times 1\% = -18.18 \times 1\% = -18.18\% \]

which implies a change in price of \(-18.18 \times 10,000 = -1,818\), to a new price of $8,182.

Similarly a yield decrease to 9% would imply a price gain of approximately 18.18% to a new price of $11,818

What are the exact prices at these interest rates?

<table>
<thead>
<tr>
<th>YTM:</th>
<th>9%</th>
<th>10%</th>
<th>11%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual price:</td>
<td>$12,004</td>
<td>$10,000</td>
<td>$8,344</td>
</tr>
<tr>
<td>Approximate price:</td>
<td>$11,818</td>
<td>$10,000</td>
<td>$8,182</td>
</tr>
<tr>
<td>Error in duration rule:</td>
<td>$186</td>
<td>$0</td>
<td>$ 162</td>
</tr>
</tbody>
</table>

In both cases, actual price is higher than predicted price. Why? Predicted price versus \(\Delta y\) plots as a straight line, but actual price curve is convex.
Convexity means that immunization also is only an approximate hedge.

In our immunization example, when we immunized the bullet obligation (the GIC) using the perpetuity and the zero, our assets had more convexity than our liability. So when yields changed, we ended up with a small surplus.
The convexity correction to the duration rule

An improvement in the duration approximation can be obtained by accounting for the curvature of the pricing function, which is measured by the convexity of the bond. The more accurate formula is

\[
\text{% change in bond price} = - D^{\text{mod}} \times \Delta y + \frac{1}{2} \times \text{Convexity} \times (\Delta y)^2
\]

Example: The zero-coupon bond we just considered (page 13) has convexity of 347.1. Therefore, a better approximation of its price change when its YTM increases from 10% to 11% is:

<table>
<thead>
<tr>
<th>YTM</th>
<th>%ΔP</th>
<th>Duration rule</th>
<th>Duration w/Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>+20.04</td>
<td>+18.18</td>
<td>+19.19</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>−16.56</td>
<td>−18.18</td>
<td>−16.44</td>
</tr>
</tbody>
</table>

[Remember that the duration rule predicted a price decline of 18.18%. Notice that we need to use decimals not percentages when using convexity formula.]

The true change in the bond price is 16.56% (from $10,000 to $8,344), so the convexity term increases the accuracy of the approximation.

Notice also that the part of the price change due to convexity is positive regardless of whether YTM rises or falls. Therefore, all else equal, investors prefer convexity, and will be willing to accept lower yields on more convex bonds. Convexity is most valuable in high volatility environments, when the potential change in yields [and therefore the expected value of $(\Delta y)^2$] is greater.
Formula for convexity

The formula for the convexity of a bond with a maturity of \( n \) years making annual coupon payments is:

\[
\text{Convexity} = \frac{1}{P \times (1 + y)^2} \sum_{t=1}^{n} \left[ \frac{\text{CF}_t}{(1 + y)^t} \left( t^2 + t \right) \right]
\]

where \( \text{CF}_t \) is the cash flow paid to the bondholder at date \( t \); \( \text{CF}_t \) represents either a coupon payment before maturity or final coupon plus par value at the maturity date.
This analysis suggests that convexity is desirable.

Here are strategies with equal durations, but increasing amounts of convexity as you go down the list:

- **Bullet**: _____________ ↑
- **Ladder**: ↑ ↑ ↑ ↑ ↑ ↑ ↑
- **Barbell**: ↑ ___________________ ↑
Negative convexity can result from call provisions.

In negative convex region, curve lies below tangent, not above as in the positive convexity region.

Same effect in mortgage-backed securities market, where negative convexity is a big issue.