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1 Risk

- $r$ ratio of the sale price plus dividends (or coupons) to the purchase price.
- *Risk:* we are not sure whether we will have high or low returns. We represent our knowledge about uncertain future returns using the probability of different outcomes, $Pr(s)$.
- Measure of position of outcomes: *mean* or mathematical expectation:
  \[ E(r) = \sum_s Pr(s) \times r(s), \]
  where $Pr(s)$ stands for the probability of state $s$ realizing itself, and $r(s)$ is the rate of return in state $s$. We will see that the expected rate of return in excess of the risk free rate (e.g., the one-month T-bill rate) is the *compensation* for risk.
- Measure of dispersion of outcomes: *variance* of the distribution of returns,
  \[ \text{Var}(r) = \sigma^2 = E[r - E(r)]^2 \]
  \[ = \sum_s Pr(s) \times [r(s) - E(r)]^2. \]
  Standard deviation:
  \[ \sigma = \sqrt{\text{Var}(r)} \]
  Both variance and standard deviation of the distribution of returns measure the risk associated with that asset.

**Example 1** Consider the following percentage rates of return on a risky asset $A$ and on a risk-free asset $f$:

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Probability</th>
<th>$r_A(s)$</th>
<th>$r_f(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>1/2</td>
<td>.23</td>
<td>.03</td>
</tr>
<tr>
<td>Recession</td>
<td>1/2</td>
<td>-.05</td>
<td>.03</td>
</tr>
</tbody>
</table>

We have
\[ E(r_A) = \frac{1}{2} \times .23 + \frac{1}{2} \times -.05 = .09; \]
and
\[ \sigma_A^2 = \frac{1}{2} \times (.23 - .09)^2 + \frac{1}{4} \times (-.05 -.09)^2 = .0196, \]

hence $\sigma_A = .14$. 

• Q: mean and variance of \( r_f \)?
• Variance is zero only if the outcomes are equal to a constant for every state of the world.

2 Properties of Mean and Variance

• sums: \( E(r_A + r_B) = E(r_A) + E(r_B) \)
• \( c = \text{constant} \): \( E(c \times r_A) = c \times E(r_A) \)
• \( c = \text{constant} \): \( \text{Var}(c \times r_A) = c^2 \times \sigma_A^2 \)
• \( c_A, c_B = \text{constants} \): \( \text{Var}(c_A r_A + c_B r_B) = c_A^2 \times \sigma_A^2 + c_B^2 \times \sigma_B^2 + 2 \times c_A c_B \times \sigma_{AB} \),
  where

\[
\sigma_{AB} = E[(r_A - E(r_A) \times (r_B - E(r_B))]
\]
\[
= \sum_s P_r(s) \times [r_A(s) - E(r_A)](r_B(s) - E(r_B))
\]
\[
= \rho_{AB} \sigma_A \sigma_B
\]

Example 2 Consider the following percentage rates of return on the risky asset \( A \) and \( B \):

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Probability</th>
<th>( r_A(s) )</th>
<th>( r_B(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>1/2</td>
<td>.23</td>
<td>.16</td>
</tr>
<tr>
<td>Recession</td>
<td>1/2</td>
<td>-.05</td>
<td>-.04</td>
</tr>
</tbody>
</table>

We have

\[
\sigma_{AB} = \frac{1}{2} \times (.23 - .09)(.16 - .06) + \frac{1}{2} \times (-.05 - .09)(-.04 - .06) = .014.
\]

Note that \( \rho_{AB} = 1 \). In fact

\[
\sigma_{AB} = \sigma_A \sigma_B = .14 \times .10.
\]

Example 3 The expected return on a portfolio with \( w_A = .5 \) invested in asset \( A \), and \( (1 - w_A) = .5 \) invested in the risk-free asset \( f \), where \( r_f = .03 \), is

\[
E(r_p) = E[w_A \times r_A + (1 - w_A) \times r_f]
\]
\[
= E(w_A \times r_A) + E[(1 - w_A) \times r_f]
\]
\[
= w_A \times E(r_A) + (1 - w_A) \times E(r_f)
\]
\[
= .5 \times .09 + .5 \times .03 = .06.
\]
The standard deviation of the portfolio is half the standard deviation of the risky asset $A$ (the risk-free asset does not contribute to risk):

$$\sigma_p = \sqrt{\text{Var}[w_A \times r_A + (1 - w_A) \times r_f]}$$

$$= \sqrt{w_A^2 \times \sigma_A^2}$$

$$= w_A \times \sigma_A = .5 \times .14 = .07.$$  

Intuitively, if you carry half of the portfolio in a riskless security, the risk is reduced by half.

3 Risk Premium

- Investors dislike risk $\implies$ the market usually offers higher returns for risky securities.
- We measure this higher return in terms of the average excess return over the risk-free rate:

$$E(r) - r_f,$$

which we call the risk premium.

(In our Example the risk premium is $E(r_A) - r_f = .09 - .03 = .06$.)

4 Utility Functions

- *Fair game:* risky game that carries a zero risk premium.
  
  *Risk-averse* investors reject fair games and demand a premium for risk.
  
- *Utility function:* describes the trade-off between risk and expected returns

$$U = E(r) - \frac{1}{2} \times A \times \text{Var}(r)$$

The value of the function $U$ is the certainty-equivalent: the return for sure which is equivalent to the risky return, after adjusting for risk.

- $A$ captures the degree of aversion to risk: the higher the $A$ coefficient, the higher the penalty for risk.

  $A = 0 \implies$ risk neutrality

**Example 4** Assume $A = 7$. We can identify the points of an indifference curve, for $U = .03$. These are combinations of risk and expected return
which keep an investor indifferent between a risky asset and a riskless asset with \( r_f = .03 \):

\[
E(r) = 0.03 + \frac{1}{2} \times 7 \times \sigma_r^2.
\]

<table>
<thead>
<tr>
<th>( \sigma_r^2 )</th>
<th>( U )</th>
<th>( E(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.0</td>
<td>.03</td>
<td>.03</td>
</tr>
<tr>
<td>.1</td>
<td>.03</td>
<td>.38</td>
</tr>
<tr>
<td>.2</td>
<td>.03</td>
<td>.73</td>
</tr>
<tr>
<td>.3</td>
<td>.03</td>
<td>1.08</td>
</tr>
<tr>
<td>.4</td>
<td>.03</td>
<td>1.43</td>
</tr>
</tbody>
</table>

5  Risk and Returns: the Complete Portfolio

- In the following we shall consider a simple portfolio composed of:
  one risky asset, \( p \) (which one should think of as a mutual fund of stocks), and
  one risk-free asset, \( f \) (which one should think of as a money market mutual fund or an equivalent investment in money-market securities).

- \( w_p \): weight of the risky portfolio, as a fraction of the complete portfolio value,

- \( w_f = 1 - w_p \): weight of risk-free portfolio.

- Expected return on the complete portfolio \( c \):

  \[
  E(r_c) = w_p \times E(r_p) + (1 - w_p) \times r_f = r_f + w_p \times [E(r_p) - r_f].
  \]

**Example 5** \( E(r_p) = .09, r_f = 0.03 \), then

- if \( w_p = 0.0 \implies E(r_c) = 0.03 \);
- if \( w_p = .5 \implies E(r_c) = .06 \);
- if \( w_p = 1 \implies E(r_c) = .09 \);
- if \( w_p > 1 \implies E(r_c) > .09 \).

- Note that if the risk premium on \( p \) is positive, expected returns increase with the percentage of the portfolio allocated to the risky asset.
• What is the risk of the complete portfolio?

\[
\text{Var}(r_c) = w_p^2 \times \text{Var}(r_p) + (1 - w_p)^2 \times \text{Var}(r_f) \\
+ 2 \times w_p \times (1 - w_p) \times \text{Cov}(r_p, r_f) \\
= w_p^2 \times \text{Var}(r_p).
\]

• The standard deviation of the complete portfolio is therefore:

\[
\sigma_c = w_p \times \sigma_p.
\]

**Example 6** \(\sigma_p = .14,\)

if \(w_p = 0 \implies \sigma_c = 0.0;\)

if \(w_p = .5 \implies \sigma_c = .07;\)

if \(w_p = 1 \implies \sigma_c = .14;\)

if \(w_p > 1 \implies \sigma_c > .14.\)

6 **The Capital Allocation Line (CAL)**

• What happens to expected returns and standard deviation of the portfolio as we change \(w_p?\)

We have

\[
\sigma_c = w_p \times \sigma_p,
\]

hence

\[
w_p = \sigma_c / \sigma_p.
\]

Also, we have

\[
E(r_c) = (1 - w_p) \times r_f + w_p \times E(r_p) \\
= r_f + w_p \times [E(r_p) - r_f] \\
= r_f + \frac{[E(r_p) - r_f]}{\sigma_p} \times \sigma_c.
\]

The relation above, between expected rate of return and risk, is the **Capital Allocation Line (CAL)**.

The slope of the line,

\[
S = \frac{[E(r_p) - r_f]}{\sigma_p},
\]

is the *reward-to-variability* or Sharpe ratio.
Example 7 Assume $E(r_p) = .9$, $r_f = .03$, and $\sigma_p = .14$. The intercept of the CAL is $r_f$, while the slope is given by

$$S = \frac{[E(r_p) - r_f]}{\sigma_p} = \frac{.06}{.14} = .4286.$$ 

- **Leveraged position**: borrowing at the risk-free rate to hold more than 100% invested in the risky portfolio $p$.
- **Short sales**: You can sell the risky portfolio short to invest more than 100% in the riskless asset. (You can think of borrowing as a short sale of bonds yielding the risk-free rate.)

7 Optimal Capital Allocation

- Objective of the investor: achieve maximum utility

  Formally, the optimal portfolio weight is given by choosing $w_p$ such that

  $$\max_{w_p} \left[ E(r_c) - \frac{1}{2} \lambda \sigma_c^2 \right],$$

subject to the opportunity set summarized by the CAL.

The optimal combination of risk and expected return is achieved at the tangency point between the CAL and the highest indifference curve.

- To find the solution analytically we want to set the slopes of the indifference curves and that of the CAL equal.

  We have

  $$\frac{dE(r_c)}{d\sigma_c} \bigg|_U = -\frac{dU/d\sigma_c}{dU/dE(r_c)} = A \times \sigma_c$$

  $$= \frac{[E(r_p) - r_f]}{\sigma_p}.$$ 

Solving the equality above for $\sigma_c$, we have

$$\sigma_c = \frac{[E(r_p) - r_f]}{A \times \sigma_p}.$$ 

In turn, we have $\sigma_c = w_p \times \sigma_p$.

Substituting in the expression above, we obtain

$$w_p^* = \frac{[E(r_p) - r_f]}{A \times \sigma_p^2}.$$
Hence, the optimal weight allocated to the risky portfolio: increases with the risk premium, decreases with risk aversion, and decreases with the riskiness of the return on the risky portfolio.

- Alternative derivation of the optimal portfolio:

\[
\max_{w_p} \left[ w_p E(r_p) + (1 - w_p) r_f - \frac{1}{2} \times A \times w_p^2 \sigma_p^2 \right].
\]

Take the derivative w.r.t. \(w_p\), and set it equal zero:

\[
E(r_p) - r_f - w_p \times A \sigma_p^2 = 0.
\]

**Example 8** \(E(r_p) = 0.09, r_f = 0.03, \sigma_p^2 = 0.0196,\) and \(A = 7 \implies w_p^* = 0.4373.\)
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1 Diversification and Portfolio Risk

- **Diversification** (main intuition): do not put all your eggs in the same basket.
  Statistics point of view: *law of large numbers* $\implies$ positive and negative outcomes tend to compensate each other.

- **Intuition**: By reducing the weight of each individual asset in your portfolio you can substantially reduce the variance of the portfolio.

- In reality there are common factors which affect all returns at the same time $\implies$ limit to the risk reduction that can be achieved through diversification.

- Variance of a portfolio

  \[
  \sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_{ij}.
  \]

- Assume to invest equally in all the $n$ different assets, $(w_i = 1/n)$, then the variance of the portfolio is

  \[
  \begin{align*}
  \sigma_p^2 &= \sum_i \frac{1}{n^2} \sigma_i^2 + \sum_{i \neq j} \frac{1}{n^2} \sigma_{ij} \\
  &= \frac{1}{n} \sum_i \frac{1}{n} \sigma_i^2 + \sum_{i \neq j} \frac{1}{n^2} \sigma_{ij}.
  \end{align*}
  \]

- Define the *average variance* as

  \[
  \bar{\sigma}^2 = \frac{1}{n} \sum_i \sigma_i^2,
  \]

  and the *average covariance* as

  \[
  \bar{\sigma}_{ij} = \frac{1}{n(n-1)} \sum_i \sum_{j \neq i} \sigma_{ij},
  \]

  where $n(n-1)$ is the number of pairs of assets in the portfolio.

- The variance of the portfolio returns can be written as

  \[
  \sigma_p^2 = \frac{1}{n} \bar{\sigma}^2 + \frac{n-1}{n} \bar{\sigma}_{ij}.
  \]

  As the number of assets in the portfolio tends to infinity, the variance of portfolio returns becomes

  \[
  \lim_{n \to \infty} \sigma_p^2 = \bar{\sigma}_{ij}.
  \]
• To reduce risk maintain the size of the portfolio (§1) fixed, and increase the number of securities $n$.
As $n$ grows, nonsystematic or diversifiable risk is eliminated.
Only systematic risk, as measured by the average covariance between any two asset returns, cannot be diversified away.

• Note that when the rates of return on the available assets are uncorrelated, increasing the number of securities in the portfolio allows to reduce risk to zero.

• Systematic risk has to do with fundamental determinants of risk: e.g. the state of the economy, the trade deficit, the business cycle, inflation.

• Only systematic risk receives a premium: non-systematic risk can be eliminated.

2 Portfolios of Two Risky Assets

• Two assets $A$ and $B$, with expected returns
  \[ E(r_A) > E(r_B), \]
  standard deviations
  \[ \sigma_A > \sigma_B, \]
  and correlation coefficient
  \[ \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}. \]

• $w_A$ is invested in $A$, $(1 - w_A)$ is invested in $B$

• The expected return on a portfolio is a weighted average of the expected returns on the individual portfolios
  \[ E(r_p) = w_A E(r_A) + (1 - w_A) E(r_B) = E(r_B) + w_A [E(r_A) - E(r_B)], \]
  and variance
  \[ \sigma_p^2 = w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2 \times w_A (1 - w_A) \rho_{AB} \sigma_A \sigma_B. \]
  where $\rho_{AB} \sigma_A \sigma_B = \sigma_{AB}$.

• Note that by varying the composition of the portfolio one obtains all the combinations of expected rate of return and standard deviation attainable with the two assets.
**Example 1** Assume $E(r_A) = .09$, $E(r_B) = .06$, $\sigma_A = .14$, $\sigma_B = .10$, and $\rho_{AB} = 0$. For $w_A = .1$ we have

\[ \sigma_p = \sqrt{.1^2 \times .14^2 + .9^2 \times .10^2} = .0911 \]

and

\[ E(r_p) = .1 \times .09 + .9 \times .06 = .063. \]

For $w_A = .2$ we have

\[ \sigma_p = \sqrt{.2^2 \times .14^2 + .8^2 \times .10^2} = .0848 \]

and

\[ E(r_p) = .2 \times .09 + .8 \times .06 = .0660. \]

and so on.

- The case of no correlation between the two returns is an intermediate case relative to the two extreme cases: $\rho_{AB} = 1$ and $\rho_{AB} = -1$.

- $\rho_{AB} = 1$: the two assets are perfect substitutes, in the sense that their returns always move in the same direction.

  We have

  \[ \sigma_p^2 = w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2 \times w_A(1 - w_A)\sigma_A \sigma_B = [w_A \sigma_A + (1 - w_A) \sigma_B]^2 \]

  and

  \[ \sigma_p = w_A \sigma_A + (1 - w_A) \sigma_B. \]

  Hence, the standard deviation of the portfolio is simply a weighted average of the standard deviations.

  As a result, the combinations of $E(r_p)$ and $\sigma_p$ are on a straight line connecting the two portfolios.

- $\rho_{AB} = 0$: the two assets perfectly offset each other, in the sense that their returns always move in the opposite direction.

  We have

  \[ \sigma_p^2 = w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 - 2 \times w_A(1 - w_A)\sigma_A \sigma_B = [w_A \sigma_A - (1 - w_A) \sigma_B]^2 \]

  and

  \[ \sigma_p = |w_A \sigma_A - (1 - w_A) \sigma_B|. \]

  Hence, the standard deviation of the portfolio falls to zero for $w_A/(1 - w_A) = \sigma_B/\sigma_A$.

  As a result, the combinations of $E(r_p)$ and $\sigma_p$ are on a ”cone” originating from the vertical axis.
3 Tangency Portfolios

- The point where the CAL and the minimum variance frontier are tangent is the tangency portfolio.

This portfolio is important because it is the one that offers the highest trade-off between return (in excess of the risk-free rate) and risk.

- When the tangency portfolio \( P^* \) has been identified, then we can apply the analysis of the previous session to select the optimal complete portfolio \( C^* \).

- Formally, the selection of the optimal portfolio of risky assets amounts to solving the problem

\[
\max_{w_i} S = \frac{E(r_p) - r_f}{\sigma_p}.
\]

We have

\[
\frac{dS}{dw_i} = \frac{dE(r_p)/dw_i \times \sigma_p - [E(r_p) - r_f] \times d\sigma_p/dw_i}{\sigma_p^2} = 0.
\]

- In the two-risky-assets case, we have

\[
\frac{dE(r_p)}{dw_A} = E(r_A) - E(r_B),
\]

and

\[
\frac{d\sigma_p^2}{dw_A} = 2\sigma_p[2w_A\sigma_A^2 - 2(1 - w_A)\sigma_B^2 + 2(1 - 2w_A)\rho_{AB}\sigma_A\sigma_B].
\]

Substituting and solving for \( w_A \), we obtain

\[
w_A = \frac{[E(r_A) - r_f]\sigma_B^2 - [E(r_B) - r_f]\sigma_{AB}}{[E(r_A) - r_f]\sigma_B^2 + [E(r_B) - r_f]\sigma_A^2 - [E(r_A) + E(r_B) - 2r_f]\sigma_{AB}}.
\]

**Example 2** Assume again \( E(r_A) = .09, E(r_B) = .06, \sigma_A = .14, \sigma_B = .10, \) and \( \rho_{AB} = 0 \). Assume \( r_f = .03 \).

The optimal weight for the tangency portfolio equals

\[
w_A = \frac{.06 \times .001}{.06 \times .001 + .03 \times .0196} = .5051.
\]
Given the composition of the tangency portfolio, we can now construct an optimal complete portfolio. We have

\[ \sigma_p = \sqrt{.5051^2 \times .14^2 + .4949^2 \times .01^2} = .0863 \]

and

\[ E(r_p) = .5051 \times .09 + .4949 \times .06 = .0752. \]

Assume a risk aversion of 7, we have

\[ w_p^* = \frac{.0752 - .03}{7 \times .0863^2} = .8659. \]

Hence, the complete portfolio is invested 13.41% (= 100% – 86.59%) in the risk-free asset, 43.73% (= 86.59% × 50.51%) in asset A, and 42.86% (= 86.59% × 49.49%) in asset B.

The expected rate of return on the complete portfolio is

\[ .8659 \times .0752 + .1341 \times .03 = .0691 \]

and the standard deviation is

\[ .8659 \times .0863 = .0747. \]

4 Several Risky Assets

- The analysis above can be extended to the situation where there are several risky assets available for investment.

- In this case the mean-standard deviation frontier reflects the opportunities from diversification by using all the risky assets available.

- On noteworthy feature is that now an investor may invest in assets which do not lie on the frontier.