List of formulas

\[ I = P \cdot r \cdot t \quad A = P \cdot (1 + r \cdot t) \quad A = P \left(1 + \frac{r}{m}\right)^{m \cdot t} \quad APY = \left(1 + \frac{r}{m}\right)^m - 1 \]

\[ FV = PMT \cdot \frac{(1+i)^n-1}{i} \quad PV = PMT \cdot \frac{1-(1+i)^{-n}}{i} \quad i = \frac{r}{m} \quad n = m \cdot t \]

Problem 1 a) (7 pts) A group of 100 people includes 35 who play only tennis, 45 who play only golf and 7 who play neither sport. How many people in the group play both tennis and golf?

\[ 35 + 45 + 7 = 87 \]

\[ 100 - 87 = 13 \]

b) (8 pts) From a committee of 10 people, in how many ways can we choose a chairperson (baskan), a vice-chairperson (baskan yardimcisi) and 3 members?

\[ \binom{10}{1} \cdot \binom{9}{1} \cdot \binom{8}{3} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{3! \cdot 5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{3!} = 5040 \]
Problem 2 a) (10 pts) A family wants to buy a house whose price is $100,000. How much downpayment (pesinat) should they pay to buy the house with $600 monthly payment for 20 year mortgage at 6% compounded monthly.

$$PV = PMT \frac{1 - (1+i)^{-n}}{i}$$

$$= 600 \frac{1 - (1,005)^{-240}}{0.005}$$

$$= 83,748.46$$

Downpayment = 100,000 - 83,748.46

$$= 16,251.53$$

b) (8 pts) In the above question, calculate how much interest is paid at the end of 10 years.

$$X = PMT \frac{1 - (1+i)^{-120}}{i}$$

$$= 600 \frac{1 - (1,005)^{-120}}{0.005}$$

$$= 54,049.07$$

$$83,748.46 - 54,049.07 = 29,709.39 \rightarrow \text{money gone to have}$$

$$120 \times 600 = 72,000 \rightarrow \text{money paid}$$

$$72,000 - 29,709.39 = 42,290.61$$
Problem 3 (15 pts) Solve the following system by using Gauss-Jordan elimination.

\[
\begin{align*}
3x - 2y + z &= -7 \\
2x + y - 4z &= 0 \\
x + y - 3z &= 1
\end{align*}
\]

\[
\begin{bmatrix}
3 & -2 & 1 \\
2 & 1 & -4 \\
1 & 1 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -3 \\
2 & 1 & -4 \\
3 & 2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -3 \\
2 & 1 & -4 \\
3 & 2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -3 \\
0 & 1 & -2 \\
0 & 5 & 10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -3 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
x - z = -1 \\
y - 2z = 2
\]

\[
(x, y, z) = (z-1, 2z+2, z)
\]

\(\Rightarrow\) \(x = z-1\), \(y = 2z + 2\)
Problem 4 (10 pts) Find two negative numbers whose product is 30 and their sum is maximum.

\[ x \cdot y = 30 \]
\[ y = \frac{30}{x} \]
\[ f(x) = x + \frac{30}{x} \]
\[ f'(x) = 1 - \frac{30}{x^2} \]
\[ f'(x) = 0 \quad \Rightarrow \quad \frac{x^2 - 30}{x^2} = 0 \]
\[ x^2 = 30 \]
\[ x = \sqrt{30}, \quad x = -\sqrt{30} \]

xy negative:
\[ x = -\sqrt{30} \]
\[ y = -\sqrt{30} \]
Problem 5 a) (8 pts) If \( f \) is continuous everywhere, find \( a \) and \( b \).

\[
f(x) = \begin{cases} 
x^2 - ax + b & x > 2 \\
 bx - 3a & -1 \leq x \leq 2 \\
 8x - a + b & x < -1 
\end{cases}
\]

\[
\begin{align*}
x = 2 & \quad 4 - 2a + b = 2b - 3a \implies a - b = -4 \\
x = -1 & \quad -b - 3a = -8 - a + b \implies a + b = 4
\end{align*}
\]

\[
\begin{array}{c}
a = 0 \\
b = 4
\end{array}
\]

b) (7 pts) In the question above, is \( f \) differentiable everywhere?

Points to check: \( x = 2 \) \quad \( x = -1 \)

\[
\begin{align*}
x = 2 & \quad \text{Right } f'(x) = 2x \implies f'(2)^+ = 4 & \checkmark \\
& \quad \text{Left } f'(x) = 4 \implies f'(2)^- = 4 \\

x = -1 & \quad \text{Right } f'(x) = 4 \implies f'(-1)^+ = 4 & \text{corner!} \\
& \quad \text{Left } f'(x) = 8 \implies f'(-1)^- = 8
\end{align*}
\]

\( f \) is not differentiable at \( -1 \).
Problem 6 (15 pts) Let \( f(x) = \frac{x-2}{x^2} \)

a) Find all local extrema and intervals on which \( f \) is increasing & decreasing.

\[
f'(x) = \frac{1 \cdot x^2 - (x-2) \cdot 2x}{x^4} = \frac{x^2 - 2x + 4x}{x^4} = \frac{x^2 + 4x}{x^4} = \frac{-x + 4}{x^3}
\]

\[f'\]

\[
\frac{-}{\downarrow} \quad 0 \quad \nearrow \quad \downarrow \quad 6
\]

\[
\begin{align*}
\text{local max} & \quad \text{not defined!} \\
4 & \quad (4, \frac{1}{8})
\end{align*}
\]

b) Find inflection points, and intervals on which \( f \) is concave up & concave down.

\[
f''(x) = \frac{-1 \cdot x^3 - (-x+4) \cdot 2x^2}{x^6} = \frac{-x^3 + 2x^2 - 12x^2}{x^6} = \frac{2x^2 - 12x^4}{x^6} = \frac{2(x-6)}{x^4}
\]

\[f''\]

\[
\frac{+}{\downarrow} \quad 0 \quad \text{con.} \quad 6 \quad \text{con.}
\]

\[
\text{concave} \quad \text{down} \quad \text{up}
\]

\[
\begin{align*}
\text{inflection pt.} & \quad (6, \frac{1}{9})
\end{align*}
\]

c) Find the asymptotes, if exist.

\[
\lim_{x \to \infty} \frac{x-2}{x^2} = 0
\]

\[
\lim_{x \to -\infty} \frac{x-2}{x^2} = 0
\]

\[
\lim_{x \to 0^+} \frac{x-2}{x^2} = -\infty
\]

\[
\lim_{x \to 0^-} \frac{x-2}{x^2} = -\infty
\]

\[
\text{vertical asymptote: } x = 0
\]

\[
\text{horizontal asymptote: } y = 0
\]

e) Sketch the graph of \( f \).
List of formulas

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \int e^x \, dx = e^x + C \quad \int \frac{1}{x} \, dx = \ln x + C
\]

Problem 7 a) (10 pts) Find the area of the region between \( y = x^2 - 1 \) and \( y = 3 \).

\[
\begin{align*}
D &= \int_{-1}^{1} (x^2 - 1) \, dx = \left[ \frac{x^3}{3} - x \right]_{-1}^{1} = \left( \frac{1}{3} - 1 \right) - \left( -\frac{1}{3} + 1 \right) = -\frac{4}{3} \\
\implies \quad B &= \frac{4}{3} \\
A &= 12 - \left( \int_{1}^{2} (x^2 - 1) \, dx + \int_{1}^{2} (x^2 - 1) \, dx \right) = 12 - \frac{8}{3} = \frac{28}{3} \\
\text{Area} &= \frac{4}{3} + \frac{28}{3} = \frac{32}{3}
\end{align*}
\]

b) (7 pts) \( \int_{2}^{5} \frac{x^2 - 3}{\sqrt{x}} \, dx \)

\[
\begin{align*}
\int_{2}^{5} x^{\frac{3}{2}} - x^{\frac{1}{2}} \, dx &= \left[ \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_{2}^{5} = 2 \left( \frac{5 \sqrt{5} - 6 \sqrt{2}}{\sqrt{2}} \right) \\
&= \left( \frac{2}{\sqrt{5}} \right) \left( 5 \sqrt{5} - 6 \sqrt{2} \right) = \left( 10 \sqrt{5} - 6 \sqrt{2} \right) - \left( \frac{2}{\sqrt{5}} \right) \left( 6 \sqrt{5} - 6 \sqrt{2} \right) \\
&= 4 \sqrt{5} + 2 \sqrt{2}
\end{align*}
\]