Math 301 - Problem Set # 3 - Fall 2010

**Homework Problems:** 2, 8, 10, 12, 17, 21

In the following exercises all sequences are assumed to be in $\mathbb{R}$ and $(a_n)$ denote a sequence of real numbers. As usual, you should justify your answers.

1. Prove that any subsequence of a convergent sequence is convergent and has the same limit as the sequence.

2. Find all the cluster points of the following sequences if they exist.
   
   (a) $a_n = 5$ for every natural number $n$.
   
   (b) $a_n = (-1)^n$ for every natural number $n$.
   
   (c) For every natural number $n$, $a_n$ is the remainder when $n$ is divided by 3.
   
   (d) $a_n = n$ for every natural number $n$.
   
   (e) $a_n = f(n)$ for every natural number $n$, where $f$ is any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0,1]$.
   
   *(How do we know that there is such a bijection?)*

3. Prove that the limit of a convergent sequence is a cluster point and in fact the only cluster point of that sequence. If possible, find a sequence which diverges even though it has a unique cluster point.

4. If possible, give an example of a set which has a maximum, i.e. a greatest element, and an example of a bounded set which does not have a maximum.

5. If possible, find a convergent sequence with more than one cluster point.

6. Prove that any bounded sequence has at least one cluster point.

7. Prove that $(a_n)$ converges iff both $(a_{2n})$ and $(a_{2n+1})$ converges to the same limit.

8. If possible, give an example of a Cauchy sequence which isn’t convergent and an example of a sequence which isn’t Cauchy. What would change if we had considered sequences in $\mathbb{Z}$ or $\mathbb{Q}$?

9. Prove that if $(a_n)$ is monotone and unbounded, then $\lim \frac{1}{a_n} = 0$.

   **Definition.** Let $S$ be a subset of $\mathbb{R}$. A real number $a$ is called an accumulation point of $S$ if for every $\epsilon > 0$ the interval $(a - \epsilon, a + \epsilon)$ contains infinitely many points of $S$.

10. Given a sequence $(a_n)$, what can you say about the accumulation points of the set $\{a_n : n \in \mathbb{N}\}$?
11. (a) Prove that \( \mathbb{N} \) has no accumulation points.
(b) Find the accumulation points of \( S = (0, 1) \cup \{2\} \).
(c) Prove that any real number is an accumulation point of \( \mathbb{Q} \).

12. Prove that \( a \) is an accumulation point of \( S \) iff there is a sequence \((a_n)\) in \( S \) such that \( \lim a_n = a \) and \( a_n \neq a_m \) whenever \( n \neq m \).

13. Prove that if \( S \) is a bounded subset of \( \mathbb{R} \) with infinitely many elements, then \( S \) has at least one accumulation point.

14. Find a sequence \((A_n)\) of non-empty subsets of \( \mathbb{R} \) such that \( A_0 \) is bounded, \( A_{n+1} \subseteq A_n \) for every \( n \in \mathbb{N} \) and the intersection \( \cap_{n \in \mathbb{N}} A_n \) is empty.

15. Let \([a_0, b_0] \supseteq [a_1, b_1] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots\) be a sequence of closed and bounded intervals in \( \mathbb{R} \). Prove that the intersection \( \cap_{n \in \mathbb{N}} [a_n, b_n] \) contains only one element iff for every \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( b_n - a_n < \epsilon \).

16. If possible, find an unbounded sequence whose limit inferior and limit superior both exist.

17. Find limit superior and limit inferior of each of the following sequences, if they exist.
   (a) \( a_n = (-1)^n \) for every natural number \( n \).
   (b) For every natural number \( n \), \( a_n \) is the remainder when \( n \) is divided by 4.
   (c) \( a_n = \frac{1}{n+1} \) for every natural number \( n \).
   (d) \( a_n = (-1)^n \frac{2n}{n+1} \) for every natural number \( n \).
   (e) \( a_n = \frac{\sin((n+1)\pi)}{n+1} \) for every natural number \( n \).
   (f) \( a_n = f(n) \) for every natural number \( n \), where \( f \) is any bijection from \( \mathbb{N} \) to \( \mathbb{Q} \cap [0, 1] \).
   (g) \( a_n = n \) for every natural number \( n \).
   (h) \( a_0 = 0, a_{2n+1} = \frac{1}{3} + a_{2n} \) and \( a_{2n+2} = \frac{1}{3}a_{2n+1} \) for every natural number \( n \).

18. Prove that if \( \lim \inf a_n = \lim \sup a_n \), then \((a_n)\) is convergent.

19. Prove that if \((a_n)\) is convergent, then \( \lim \sup a_n = \lim \inf a_n = \lim a_n \).

20. In each of the following, either give an example of a sequence \((a_n)\) in \( \mathbb{R} \) satisfying the given condition or prove that there is no such sequence.
   (a) \( \lim \sup a_n \leq a_m \) for every \( m \in \mathbb{N} \)
   (b) \( (\lim \sup a_n)^2 < (\lim \inf a_n)^2 \)
   (c) \( \lim \inf a_n < \lim \sup a_n \) and \( \lim a_n = 2 \).

21. Let \((a_n)\) and \((b_n)\) be two bounded sequences. Prove that
   (a) \( \lim \inf a_n + \lim \inf b_n \leq \lim \inf(a_n + b_n) \)
   (b) \( \lim \sup(a_n + b_n) \leq \lim \sup a_n + \lim \sup b_n. \)