Homework Problems: 1, 6, 14, 15, 23, 24, 27, 30, 34, 35.

In the following exercises all sequences are assumed to be in \( \mathbb{R} \), \((a_n)\) denote a sequence of real numbers, \( A \) denotes a subset of \( \mathbb{R} \). As usual, you should justify your answers.

1. Prove that neither \((x, y]\) nor \([x, y)\) is closed in \( \mathbb{R} \) for \( x, y \in \mathbb{R} \) with \( x < y \). Are they open?

2. Prove that the infimum of a nonempty, closed and bounded subset \( A \) of \( \mathbb{R} \) belongs to \( A \).

3. Give an example of a subset \( A \) of \( \mathbb{R} \) which is not closed even though both \( \sup(A) \) and \( \inf(A) \) belong to \( A \).

4. Let \( x, y \in \mathbb{R} \) with \( x < y \). Find the closure, interior, accumulation and isolated points of \((x, y), [x, y], (x, y], [x, y), (-\infty, x), (x, \infty), \emptyset, \mathbb{R}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q} \) and \( \{ \frac{1}{n+1} : n \in \mathbb{N} \} \). Which one of these sets are open? Which of them are closed? Which ones are discrete? Which ones are perfect?

5. Suppose that \( A \) is open and bounded. Prove that neither the supremum nor the infimum of \( A \) belongs to \( A \).

6. Are there any subsets of \( \mathbb{R} \) which are both open and closed? If yes, find all of them.

7. Prove that the union of finite number of closed subsets of \( \mathbb{R} \) is closed in \( \mathbb{R} \).

8. Prove that the intersection of any family of closed subsets of \( \mathbb{R} \) is closed in \( \mathbb{R} \).

9. Prove that the union of any family of open subsets of \( \mathbb{R} \) is open in \( \mathbb{R} \).

10. Prove that the intersection of finite number of open sets in \( \mathbb{R} \) is open in \( \mathbb{R} \).

11. Prove that \( A = \overline{A} \) iff \( A \) is closed.

12. Prove that \( \overline{A} = A \cup A' \).

13. Prove that \( A \) is open iff it is the union of a family of open intervals.

14. Find the closure, interior, derived set and the isolated points of \( A = \{0\} \cup (1, 2) \cup \{4\} \). Is this set open? Is it closed? Is it perfect? Is it discrete?

15. Prove that \( A' \) is closed.

16. Prove that \( A \subseteq \overline{A} \).

17. Prove that the following two definitions of an accumulation point are equivalent.
(a) \( x \in \mathbb{R} \) is an \textit{accumulation point of} \( A \) if for every \( \epsilon > 0 \), \((x - \epsilon, x + \epsilon) \cap (A \setminus \{x\}) \neq \emptyset \).
(b) \( x \in \mathbb{R} \) is an \textit{accumulation point of} \( A \) if for every \( \epsilon > 0 \), \((x - \epsilon, x + \epsilon) \cap A \) contains infinitely many elements.

18. Prove that if \( x \) is an isolated point of \( A \), then \( x \) is an accumulation point of \( \mathbb{R} \setminus A \).

19. Suppose that \( x \in \bar{A} \). Prove that \( x \in A' \) iff there is a sequence \((a_n)\) in \( A \) with \( a_n \neq a_m \) for \( n \neq m \) that converges to \( x \).

20. Let \( A = \{a_n : n \in \mathbb{N}\} \) and \( C \) be the set of cluster points of \((a_n)\). Prove that \( \bar{A} = A \cup C \).

21. Give an example of a sequence \((a_n)\) which has a cluster point that is not an accumulation point of \( \{a_n : n \in \mathbb{N}\} \).

22. Let \( a_n \neq a_m \) for \( n \neq m \), \( A = \{a_n : n \in \mathbb{N}\} \) and \((a_n)\) converges to \( L \). Prove that \( A' = \{L\} \) and each element of \( A \) is an isolated point.

23. Let \((a_n)\) be a strictly increasing sequence and \( A = \{a_n : n \in \mathbb{N}\} \). Prove that \( A \) is discrete. Also prove that \( A \) is closed iff \((a_n)\) is unbounded.

24. Prove the following.
   (a) \( A^o \) is open.
   (b) \( A^o \subseteq A \).
   (c) \( A^o \) is the largest open set contained in \( A \).
   (d) \( A^o = A \) iff \( A \) is open.

25. Let \( d \) be a real-valued function on \( X \times X \) for a nonempty set \( X \). Prove that \( d \) is a metric on \( X \) iff \( d \) satisfies the following conditions.
   (a) \( d(x, y) = 0 \) iff \( x = y \)
   (b) \( d(x, z) \leq d(y, x) + d(y, z) \) for every \( x, y \) and \( z \in X \)

26. Let \( X \) be a nonempty set. Prove that the function \( d \) on \( X \times X \) defined by
   \[
   d(x, y) = \begin{cases} 
   0, & \text{if } x = y \\
   1, & \text{if } x \neq y 
   \end{cases}
   \]
   is a metric on \( X \).

27. Let \((X, d)\) be a metric space and \( Y \) be a nonempty subset of \( X \). Prove that the restriction of \( d \) to \( Y \times Y \) is a metric on \( Y \).
28. Let \((X, d)\) be a metric space and \(\rho\) be defined by
\[
\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}
\] for every \(x, y \in X\).
Prove that \(\rho\) is a metric on \(X\).

29. Let \((X, d)\) be a metric space and \(\rho\) be defined by
\[
\rho(x, y) = \min\{1, d(x, y)\}
\] for every \(x, y \in X\).
Prove that \(\rho\) is a metric on \(X\).

30. Let \(f : X \to \mathbb{R}\) be a one-to-one function on a nonempty set \(X\). Prove that \(d\) defined by
\[
d(x, y) = |f(x) - f(y)|
\] for every \(x, y \in X\) is a metric on \(X\).

31. Prove that the function \(d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) defined by
\[
d(x, y) = |\arctan x - \arctan y|
\] is a metric on \(\mathbb{R}\).

32. Let \((X, d)\) be a metric space and \(x, y, z, w \in X\). Prove the following.
   \[(a) \quad |d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)\]
   \[(b) \quad |d(x, z) - d(y, z)| \leq d(x, y)\]

33. Let \(X\) be a nonempty set and \(\sim\) be the relation on the set of all metrics on \(X\) defined by
\(d \sim \rho\) iff \(d\) and \(\rho\) are equivalent metrics on \(X\). Prove that \(\sim\) is an equivalence relation.

34. Let \(m\) be a positive integer and the functions \(d_1, d_2\) and \(d_\infty\) on \(\mathbb{R}^m \times \mathbb{R}^m\) be defined by
\[
d_1(x, y) = \sum_{i=1}^{m} |x_i - y_i|
\]
\[
d_2(x, y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2}
\]
\[
d_\infty(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq m\}
\] for all vectors \(x = (x_1, x_2, \ldots, x_m)\) and \(y = (y_1, y_2, \ldots, y_m)\) in \(\mathbb{R}^m\). Prove that \(d_1, d_2\) and \(d_\infty\) are metrics on \(\mathbb{R}^m\).
35. Let $S$ be any nonempty set. A function $f : S \to \mathbb{R}$ is called bounded if $f(S)$ is a bounded subset of $\mathbb{R}$. Let $B(S)$ be the set of all bounded real-valued functions on $S$. Prove that the function $\rho : B(S) \times B(S) \to \mathbb{R}$ defined by

$$\rho(f, g) = \sup \{|f(s) - g(s)| : s \in S\}$$

is a metric on $B(S)$ ($\rho$ is called the uniform convergence metric or the sup metric).

36. Let $C^1[0, 1]$ be the set of all differentiable functions $f : [0, 1] \to \mathbb{R}$ which have continuous derivatives. Prove that the function $d$ defined by

$$d(f, g) = |f(0) - g(0)| + \sup \{|f'(t) - g'(t)| : 0 \leq t \leq 1\}$$

is a metric on $C^1[0, 1]$. Also prove that $\rho(f, g) \leq d(f, g)$ for every $f, g \in C^1[0, 1]$, where $\rho$ is the uniform convergence metric on the set of all bounded functions on $[0, 1]$.

37. Let $C[0, 1]$ be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Prove that the function $d$ defined by

$$d(f, g) = \int_0^1 |f(t) - g(t)| \, dt$$

is a metric on $C[0, 1]$. 

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4