Math 301 - Problem Set # 5 - Fall 2010

Homework Problems: 5, 6, 13, 15, 20, 22.

In the following problems \((X, d)\) is assumed to be a metric space, \(A \subseteq X\) and \(B \subseteq X\).

1. Prove that \(A\) is bounded iff the subset \(\{d(x, y) : x, y \in A\}\) of \(\mathbb{R}\) is bounded from above.

2. Prove that the subset of a bounded set is bounded.

3. Prove that the closure of a bounded set is bounded.

4. Prove that the union of a finite number of bounded sets is bounded. Find an infinite family of bounded sets whose union is not bounded.

5. Give a counterexample to this statement: for every \(x \in X\) and \(r > 0\), the closure of the open ball \(B_r(x)\) is the closed ball \(\overline{B}_r(x)\).

6. Prove that if \(A\) is open and \(A \cap B = \emptyset\), then \(A \cap \overline{B} = \emptyset\).

7. Prove that \(A\) is closed iff for every \(x \in X \setminus A\) there exists \(\epsilon > 0\) such that \(B_\epsilon(x) \cap A = \emptyset\).

8. Let \(\tau_d\), also called the topology on \(X\) induced by the metric \(d\), denote the collection of all the open sets in \(X\). Prove the following.

   (a) \(X\) and \(\emptyset\) are both in \(\tau_d\).
   (b) If \(U\) and \(V\) are in \(\tau_d\), then so is \(U \cap V\).
   (c) If \(\{U_\alpha\}_{\alpha \in I}\) is a family of sets in \(\tau_d\), then \(\bigcup_{\alpha \in I} U_\alpha \in \tau_d\).

9. If \(A\) is open, then it is the union of some open balls.

10. Give an example of a metric space with a family of open sets such that the intersection of these open sets is not open. For the same metric space find a family of closed sets such that the union of these closed sets is not closed.

11. Give an example of a metric space which contains a subset that is neither open nor closed.

12. Give an example of a metric space which contains a proper nonempty subset that is both open and closed.

13. Prove that \(x \in \overline{A}\) iff \(\inf\{d(x, a) : a \in A\} = 0\).

14. Consider a subset \(S\) of \(\mathbb{R}\). We have defined the interior, closure, openness and closedness of \(S\) in Section 1.7. Verify that these definitions are consistent with those for \(S\) when it is considered as a subset of the metric space \((\mathbb{R}, d)\), where \(d\) is the absolute value metric.
15. Consider $\mathbb{R}$ as a metric space with the absolute value metric. What are the interior, closure, exterior and boundary of $\mathbb{Q}$? What are the interior, closure, exterior and boundary of $\mathbb{N}$? What are the interior, closure, exterior and boundary of a finite interval? Does it matter what kind of an interval we consider?

16. Prove the following.
   (a) $A^o$ is open.
   (b) $A^o \subseteq A$.
   (c) $A^o$ is the largest open set contained in $A$.
   (d) $A^o = A$ iff $A$ is open.
   (e) $(A^o)^0 = A^0$

17. Prove the following.
   (a) $\overline{A}$ is closed.
   (b) $A \subseteq \overline{A}$.
   (c) $\overline{A}$ is the smallest closed set containing $A$.
   (d) $\overline{A} = A$ iff $A$ is closed.
   (e) $\overline{\overline{A}} = \overline{A}$.

18. Prove the following.
   (a) $A^0 \cup B^0 \subseteq (A \cup B)^0$
   (b) $A^0 \cap B^0 = (A \cap B)^0$
   (c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
   (d) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

19. Prove that $\partial A = \{x \in X : \text{for every } r > 0, B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap (X \setminus A) \neq \emptyset \}.$

20. Prove that $\partial A \subseteq \overline{A},$ in fact $\overline{A} = A \cup \partial A.$

21. Prove that $\partial A = \overline{A} \cap X \setminus A = \partial (X \setminus A).$

22. Prove that $\partial A$ is closed.

23. Prove that $A$ is open iff $A \cap \partial A = \emptyset.$

24. Prove that $A$ is closed iff $\partial A \subseteq A.$

25. Prove that $A^0$ is the same as the exterior of $X \setminus A.$

26. Prove or disprove: for every $x \in X$ and $r > 0$ we have $\partial B_r(x) = \{y \in X : d(x, y) = r\}.$