PROBLEM 1 (20 points): Let $D$ be a region, i.e., an open connected set in $\mathbb{C}$.

(a) Suppose that $u(x, y) : D \rightarrow \mathbb{R}$ has vanishing partial derivatives $u_x$ and $u_y$ at every point in $D$. Show that $u$ is constant on $D$.

Solution: Note that any two points in a region can be connected by a polygonal line containing only horizontal and vertical line segments. Let $(a, b)$ and $(c, d)$ be any two points in $D$. Then there exists a polygonal line connecting $(a, b)$ and $(c, d)$. Since $u_x$ and $u_y$ vanish at every point, by the mean value theorem, the change in $u$ between the successive vertices is 0.

To be more precise, if $(a, b)$ is connected to $(a + h, b)$ by a horizontal line segment, then $u(a + h, b) - u(a, b) = u_x(a + th, b)$ for some $0 \leq t \leq 1$. But since $u_x = 0$ at every point in $D$, we conclude that $u(a + h, b) = u(a, b)$. Similarly the same holds in the vertical direction. Thus $u(a, b) = u(c, d)$ and hence $u$ is constant on $D$.

(b) Suppose that $f : D \rightarrow \mathbb{C}$ is an analytic function such that $f'(z) = 0$ for every $z \in D$. Show that $f$ is constant on $D$.

Solution: Suppose that $f = u + iv$. Since $f'$ is identically zero in $D$, we obtain that the partial derivatives $u_x$, $u_y$, $v_x$ and $v_y$ are zero throughout $D$. Then by part (a), we conclude that $f$ is constant on $D$. 


(c) Suppose that $f : D \to \mathbb{C}$ is an analytic function such that at every point $z \in D$, either $f(z) = 0$ or $f'(z) = 0$. Show that $f$ is constant on $D$.

**Solution:** Note that $f^2$ is analytic on $D$ and $(f^2(z))' = 2f(z)f'(z)$. Since either $f(z) = 0$ or $f'(z) = 0$, we obtain $(f^2(z))' = 0$ for every $z \in D$. Then by part (b), we conclude that $f^2$ is constant which implies that $|f|^2$, and hence $|f|$ is constant. By part (d), the function $f$ is constant on $D$.

(d) Let $f : D \to \mathbb{C}$ be an analytic function. Prove that if $|f|$ is constant on $D$, then so is $f$.

**Solution:** If $|f| = 0$, then $f = 0$. So assume that $|f| = |u + iv| = c > 0$ for some constant $c$. Then $u^2 + v^2 = c$. Taking the partial derivatives gives us the equations $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Now using the Cauchy-Riemann equations, we obtain $uu_x - vu_y = 0$ and $uu_y + vu_x = 0$. Then $(uu_x - vu_y)^2 = u^2u_x^2 - 2uvu_xu_y + v^2u_y^2 = 0$ and $(uu_y + vu_x)^2 = u^2v_y^2 + 2uvu_xu_y + v^2u_x^2 = 0$. Hence we obtain, $(u^2 + v^2)(u_x^2 + u_y^2) = c(u_x^2 + u_y^2) = 0$ and this implies that $u_x = 0$ and $u_y = 0$. Moreover by the Cauchy-Riemann equations, $v_x = v_y = 0$. Hence $f$ is constant by part (b).

**Problem 2 (10 points):**

Let $f(z) = \frac{1}{z^3 + 1}$ and let $\gamma(t) = Re^{it}$ $(0 \leq t \leq \pi)$. Show that

$$\lim_{R \to \infty} \left| \int_{\gamma} f(z) dz \right| = 0.$$ 

**Solution:** Note that $|\int_{\gamma} f(z) dz| \leq \int_{\gamma} |f(z)| dz$. Also by the triangle inequality, we obtain $|z^3 + 1| + 1 \geq |z^3|$. Therefore we get $|z^3 + 1| \geq |z^3| - 1$ and this implies the inequality $\frac{1}{|z^3 + 1|} \leq \frac{1}{|z^3| - 1}$. 

Using the M-L inequality, we obtain
\[ \int_{\gamma} |f(z)| \, dz = \int_{\gamma} \frac{1}{|z^3 + 1|} \, dz \leq \int_{\gamma} \frac{1}{|z^3| - 1} \, dz \leq \frac{1}{R^3 - 1} (\pi R). \]

Combining with the inequality \( |\int_{\gamma} f(z) \, dz| \leq \int_{\gamma} |f(z)| \, dz \) and taking the limit gives the desired result.

**Problem 3 (15 points)** Define \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \) and \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \).

**Show that for all** \( z, w \in \mathbb{C} \),
\[ \cos(z + w) = \cos z \cos w - \sin z \sin w. \]

**Solution:** Using the definitions of the cosine function,
\[ \cos(z + w) = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2}. \]

On the other hand, we have
\[ \cos z \cos w = \frac{e^{iz} + e^{-iz}}{2} \cdot \frac{e^{iw} + e^{-iw}}{2} = \frac{e^{i(z+w)} + e^{i(-z+w)} + e^{i(-z-w)} + e^{i(z-w)}}{4} \]
and
\[ \sin z \sin w = \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iw} - e^{-iw}}{2i} = -\frac{e^{i(z+w)} - e^{i(z-w)} - e^{i(-z+w)} + e^{i(-z-w)}}{4}. \]

Hence \( \cos z \cos w - \sin z \sin w = \frac{2e^{i(z+w)} + 2e^{-i(z+w)}}{4} = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \), as desired.

**Problem 4 (20 points):**
(a) Determine the radius of convergence of the power series
\[ f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}. \]
(b) Show that \[ z^2 f''(z) + z f'(z) = 4z^2 f(z). \]

**Solution:** We use ratio test.

\[
\lim_{n \to \infty} \left| \frac{z^{2n+2} ((n)!)^2}{((n + 1)!)^2 z^{2n}} \right| = \lim_{n \to \infty} \left| \frac{z^2}{(n + 1)^2} \right| = 0
\]

and hence the power series converges for all \( z \).

Now, we show that the series satisfies the differential equation. Note that \( f'(z) = \sum_{n=1}^{\infty} \frac{2nz^{2n-1}}{(n!)^2} \) and \( f''(z) = \sum_{n=1}^{\infty} \frac{(2n)(2n-1)z^{2n-2}}{(n!)^2} \). Hence we obtain

\[
z^2 f''(z) + zf'(z) = \sum_{n=1}^{\infty} \frac{2nz^{2n}}{(n!)^2} + \sum_{n=1}^{\infty} \frac{2nz^{2n}}{(n!)^2} = \sum_{n=1}^{\infty} \frac{4z^{2n}}{(n!)^2} = 4z^2 \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2} = 4z^2 f(z).
\]

**PROBLEM 5 (20 points):** Consider the function \( f : \mathbb{C} \to \mathbb{C} \) defined by

\[
f(x + iy) = \begin{cases} 
0 & \text{if } x = y = 0 \\
\frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} & \text{otherwise}
\end{cases}
\]

(a) (2 points) Express \( f(x + iy) \) as \( u(x, y) + iv(x, y) \). (Write out the functions \( u(x, y) \) and \( v(x, y) \) explicitly!)

**Solution:** Note that

\[
u(x, y) = \begin{cases} 
0 & \text{if } x = y = 0 \\
\frac{x^3 - y^3}{x^2 + y^2} & \text{otherwise}
\end{cases}
\]

and

\[
v(x, y) = \begin{cases} 
0 & \text{if } x = y = 0 \\
\frac{x^3 + y^3}{x^2 + y^2} & \text{otherwise}
\end{cases}
\]
(b) (3 points) Show that \(u_x(0, 0), u_y(0, 0), v_x(0, 0), v_y(0, 0)\) exist.

**Solution:** We calculate the following limits.

\[
\begin{align*}
    u_x(0, 0) &= \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{h^3}{h^2} - 0 = 1. \\
    u_y(0, 0) &= \lim_{h \to 0} \frac{u(0, h) - u(0, 0)}{h} = \lim_{h \to 0} \frac{-h^3}{h^2} - 0 = -1. \\
    v_x(0, 0) &= \lim_{h \to 0} \frac{v(h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{h^3}{h^2} - 0 = 1. \\
    v_y(0, 0) &= \lim_{h \to 0} \frac{v(0, h) - v(0, 0)}{h} = \lim_{h \to 0} \frac{h^3}{h^2} - 0 = 1. 
\end{align*}
\]

(c) (3 points) Show that \(f\) satisfies the Cauchy-Riemann equations at the point \((x, y) = (0, 0)\).

**Solution:** In part b, we found that \(u_x(0, 0) = 1, u_y(0, 0) = -1, v_x(0, 0) = 1\) and \(v_y(0, 0) = 1\). Since \(u_x(0, 0) = v_y(0, 0)\) and \(u_y(0, 0) = -v_x(0, 0)\), the function \(f\) satisfies the Cauchy-Riemann equations at the point \((0, 0)\).

(d) (4 points) Show that \(f\) is not differentiable at the origin.

**Solution:** For \(h = w + iw\), we obtain

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{i2w^3}{2iw^2} = \lim_{h \to 0} \frac{iw}{w + iw} = \frac{i}{1 + i}.
\]
On the other hand, if $h$ is a real number, we obtain

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h + ih}{h} = 1 + i.$$ 

Thus the limit does not exist and hence $f$ is not differentiable at the origin.

(e) (3 points) Is $f$ differentiable at $z = i$? If so calculate $f'(i)$.

Solution:

$$u_x(0, 1) = \lim_{h \to 0} \frac{u(h, 1) - u(0, 1)}{h} = \lim_{h \to 0} \frac{h^3 - 1}{h^2 + 1} + \frac{1}{h} = \lim_{h \to 0} \frac{h^3 - 1 + h^2 + 1}{h(h^2 + 1)} = 0.$$ 

$$v_y(0, 1) = \lim_{h \to 0} \frac{v(0, 1 + h) - v(0, 1)}{h} = \lim_{h \to 0} \frac{(1 + h)^3 - 1}{(1 + h)^2 - 1} = \lim_{h \to 0} \frac{1 + h - 1}{h} = 1.$$ 

Since $u_x(0, 1) \neq v_y(0, 1)$, the function $f$ is not differentiable at $z = i$, since it does not satisfy the Cauchy-Riemann equations at $z = i$.

PROBLEM 6 (15 points): If $z_1, z_2, z_3$ are the vertices of an equilateral triangle in $\mathbb{C}$, then show that $z_2 + z_3 - z_1 = z_1 z_2 + z_2 z_3 + z_3 z_1$. Hint: If we rotate one side of an equilateral triangle by an angle of $\pi/3$ in the appropriate direction we obtain another side.

Solution:

Observe that $(z_3 - z_2)e^{i\pi/3} = (z_1 - z_2)$ and $(z_3 - z_1)e^{i\pi/3} = (z_3 - z_2)$. Then we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_3 - z_2}{z_3 - z_1}$$

which gives the desired result after cross multiplication.