NONCOMPLEX SMOOTH 4-MANIFOLDS WITH GENUS-2 LEFSCHETZ FIBRATIONS

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ABSTRACT. We construct noncomplex smooth 4-manifolds which admit genus-2 Lefschetz fibrations over $S^2$. The fibrations are necessarily hyperelliptic, and the resulting 4-manifolds are not even homotopy equivalent to complex surfaces. Furthermore, these examples show that fiber sums of holomorphic Lefschetz fibrations do not necessarily admit complex structures.

In the following we will prove the following theorem.

**Theorem 1.** There are infinitely many (pairwise nonhomeomorphic) 4-manifolds which admit genus-2 Lefschetz fibrations but do not carry complex structure with either orientation.

Matsumoto [M] showed that $S^2 \times T^2 \# 4CP^2$ admits a genus-2 Lefschetz fibration over $S^2$ with global monodromy $(\beta_1, \ldots, \beta_4)^2$, where $\beta_1, \ldots, \beta_4$ are the curves indicated by Figure 1. (For definitions and details regarding Lefschetz fibrations see [M], [GS].)

![Figure 1](image)

Let $B_n$ denote the smooth 4-manifold which admits a genus-2 Lefschetz fibration over $S^2$ with global monodromy

$$((\beta_1, \ldots, \beta_4)^2, (h^n(\beta_1), \ldots, h^n(\beta_4))^2)$$

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where \( h = D(a_2) \) is a positive Dehn twist about the curve \( a_2 \) indicated in Figure 2.

**Theorem 2.** For the 4-manifold \( B_n \) given above we have \( \pi_1(B_n) = \mathbb{Z} \oplus \mathbb{Z}_n. \)

**Proof.** Standard theory of Lefschetz fibrations gives that

\[
\pi_1(B_n) = \pi_1(\Sigma_2)/ \langle \beta_1, \ldots, \beta_4, h^n(\beta_1), \ldots, h^n(\beta_4) \rangle.
\]

Let \( \{a_1, b_1, a_2, b_2\} \) be the standard generators for \( \pi_1(\Sigma_2) \) (Figure 2).

![Figure 2](image)

Then we observe that

- \( \beta_1 = b_1b_2 \),
- \( \beta_2 = a_1b_1a_1^{-1}b_1^{-1} = a_2b_2a_2^{-1}b_2^{-1} \),
- \( \beta_3 = b_2a_2b_2^{-1}a_1 \),
- \( \beta_4 = b_2a_2a_1b_1 \),
- \( h^n(\beta_1) = b_1b_2a_2^n \),
- \( h^n(\beta_2) = \beta_2 \),
- \( h^n(\beta_3) = \beta_3 \),
- \( h^n(\beta_4) = b_2a_2^{n+1}a_1b_1 \).

Hence

\[
\pi_1(B_n) = \langle a_1, b_1, a_2, b_2 \mid b_1b_2, [a_1, b_1], [a_2, b_2], b_2a_2b_2^{-1}a_1, b_2a_2a_1b_1, b_1b_2a_2^n, b_2a_2^{n+1}a_1b_1 \rangle
\]

\[
= \langle a_2, b_2 \mid [a_2, b_2], a_2^n \rangle = \mathbb{Z} \oplus \mathbb{Z}_n, \text{ and this concludes the proof.}
\]

The above definition of \( B_n \) provides a handlebody decomposition for it [GS] and shows, in particular, that the Euler characteristic \( \chi(B_n) \) is equal to 12. Since \( B_n \) is the fiber sum of two copies of \( S^2 \times T^2 \# 4\mathbb{C}P^2 \), we get that the signature \( \sigma(B_n) = -8 \). Consequently, \( b_2(B_n) = 12 \) and \( b_2^+(B_n) = 2 \), \( b_2^-(B_n) = 10 \). Let \( M_n \) denote the \( n \)-fold cover of \( B_n \) with \( \pi_1(M_n) \cong \mathbb{Z} \). Easy computation shows that \( b_2^+(M_n) = 2n \) and \( b_2^-(M_n) = 10n \).
**Theorem 3.** $B_n$ does not admit a complex structure.

*Proof.* Assume that $B_n$ admits a complex structure and let $M'_n$ denote the minimal model of $M_n$. By the Enriques-Kodaira classification of complex surfaces [BPV], (since $b_1(M'_n) = 1$) $M'_n$ is either a surface of class VII (in which case $b_2^+(M'_n) = 0$), a secondary Kodaira surface (in which case $b_2(M'_n) = 0$) or a (minimal) properly elliptic surface. Since $b_2^+(M'_n) = b_2^+(M_n) = 2n$, the first two possibilities are ruled out.

Suppose now that $M'_n$ admits an elliptic fibration over a Riemann surface. If the Euler characteristic of $M'_n$ is 0, then (following form the fact that $b_1(M'_n) = b_3(M'_n) = 1$) we get that $b_2(M'_n) = 0$, which leads to the above contradiction. Suppose finally that $M'_n$ is a minimal elliptic surface with positive Euler characteristic. Since $b_1(M'_n) = 1$, it can only be fibered over $S^2$ (see for example [FM]). In that case (according to [G], for example) its fundamental group is

$$\pi_1(M'_n) = \langle x_1, \ldots, x_k \mid x_1^{b_1} = 1, \ i = 1, \ldots, k; \ x_1 \cdots x_k = 1 \rangle.$$ 

This cannot be isomorphic to $\mathbb{Z}$, since if $\pi_1(M_n) \cong \mathbb{Z} = \langle a \rangle$, then $x_1 = a^{m_1}$ for some $m_1 \in \mathbb{Z}$, so $a$ has finite order, which is a contradiction. Consequently the assumption that $B_n$ is complex leads us to a contradiction, hence the theorem is proved.

□

**Remark.** The above proof, in fact, shows that $B_n$ is not even homotopy equivalent to a complex surface — our arguments used only homotopic invariants (the fundamental group, $b_2$ and $b_2^+$) of the 4-manifold $B_n$. Note that the same idea shows that $\overline{B}_n$ (the manifold $B_n$ with the opposite orientation) carries no complex structure: The arguments involving the fundamental group, $b_2$ and the Euler characteristic only, apply without change. Since $b_2^+(M_n) = b_2^+(M_n) = 0$, the arguments using that fact that $b_2^+ \neq 0$ apply as well.

*Proof of Theorem 1.* By the definition of the 4-manifolds $B_n$ we get infinitely many manifolds admitting genus-2 (consequently hyperelliptic) Lefschetz fibrations which are — by Theorem 2. — nonhomeomorphic. As Theorem 3. and the above remark show, the manifolds $B_n$ do not carry complex structures with either orientation, hence the proof of the Theorem 1. is complete. □

**Remark.** We would like to point out that similar examples have been found by Fintushel and Stern [FS] — they used Seiberg-Witten theory to prove that their (simply connected) genus-2 Lefschetz fibrations are noncomplex.

Note that $B_n$ is given as the fiber sum of two copies of $S^2 \times T^2 \# 4\overline{CP^2}$, hence provides an example of the phenomenon that the fiber sum of holomorphic Lefschetz fibrations is not necessarily complex.
Acknowledgement. Examples of genus-2 Lefschetz fibrations with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_n$ were also constructed (as fiber sums) independently by Ivan Smith [S].

REFERENCES


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