PROBLEM 1 (45 points): Write the definition of a
(a) group (b) ring (c) ring with unity (d) commutative ring (e) division ring (f) field (g) zero divisor in a
ring (h) integral domain (h) characteristic of a ring

SOLUTION: Every student should know these!

PROBLEM 2 (20 points): Give an example (without proofs) of a
(a) noncommutative ring without unity (b) integral domain which is not a field (c) commutative ring with
unity which is not an integral domain (d) commutative ring without unity

SOLUTION: a) $M_2(2\mathbb{Z})$, b) $\mathbb{Z}$, c) $\mathbb{Z}_4$, d) $2\mathbb{Z}$

PROBLEM 3a (10 points): Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

SOLUTION: Suppose there is a ring isomorphism $f: 2\mathbb{Z} \rightarrow 3\mathbb{Z}$. Since $f$ is an additive group isomorphism, we know by group theory that either $f(2n) = 3n$ or $f(2n) = -3n$. Suppose that $f(2) = 3.3 = 9$, which is a contradiction. Similarly, $f(2n) = -3n$ also leads to a contradiction. Hence $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic as rings.

PROBLEM 3b (10 points): Show that the characteristic of an integral domain $D$ must be either 0 or a prime $p$.

SOLUTION: If the characteristic of $D$ is 0, then we are done. Suppose the characteristic of $D$ is $mn$ for $m > 1$ and $n > 1$. The distributive laws show that $(m.1)(n.1) = (mn).1 = 0$. Since we are in an integral domain, we must have either $m.1 = 0$ or $n.1 = 0$, but if $m.1 = 0$ then the characteristic is at most $m$ and if $n.1 = 0$ then the characteristic is at most $n$. Thus, the characteristic of $D$ can not be a composite positive integer.

PROBLEM 4 (15 points): An element of a ring $R$ is idempotent if $a^2 = a$.

a) Show that the set of all idempotent elements of a commutative ring is closed under multiplication.

SOLUTION: Let $a$ and $b$ be two idempotent elements of $R$. Then, $(ab)^2 = a^2b^2 = ab$ (Note that the first equality comes from commutativity and the second comes from $a,b$ being idempotent.) Hence, the set of all idempotent elements of a commutative ring is closed under multiplication.

b) Find all idempotents in the ring $\mathbb{Z}_4 \times \mathbb{Z}_6$.

SOLUTION: The idempotent elements in $\mathbb{Z}_4$ are clearly 0 and 1 and those in $\mathbb{Z}_6$ are 0,1,3 and 4.

We are looking for elements $(a,b)$ such that $(a,b)^2 = (a^2,b^2) = (a,b)$ in $\mathbb{Z}_4 \times \mathbb{Z}_6$. Hence all idempotent elements are $(0,0), (0,1), (0,3), (0,4), (1,0), (1,1), (1,3), (1,4)$.

c) Show that a division ring contains exactly two idempotent elements.

SOLUTION: Let $a$ be an idempotent element in a division ring. Then $a^2 = a$, so $a^2 - a = 0$ which implies that $a(a - 1) = 0$. Since a division ring has no zero divisors either $a = 0$ or $a = 1$. Hence, the idempotent elements of a division ring are exactly 0 and 1.